Permeable Sets

G. Leobacher, A. Steinicke,
(with T. Rajala and J. M. Thuswaldner; with Z. Buczolich)

Roscoff, Sept. 2024

A Quizz on Permeable Sets

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Definition

Definition ((Null-)permeability)

Let $\mathbb{R}^d$ be equipped with some norm $\|\cdot\|$.

- *A set $\Theta \subset \mathbb{R}^d$ is null permeable if for any two points $x, y \in \mathbb{R}^d$ and any $\delta > 0$, x and y can be connected by a path γ that is disjoint from $\Theta \setminus \{x, y\}$ and has length at most $\|x - y\| + \delta$.*
- *A set $\Theta \subset \mathbb{R}^d$ is permeable if for any two points $x, y \in \mathbb{R}^d$ and any $\delta > 0$, x and y can be connected by a path γ with $\overline{\Theta \cap \gamma}$ countable and with length at most $\|x - y\| + \delta$.*

Question 1

Which of the following sets is permeable in $\mathbb{R}^2$?

1. $\mathbb{Q}^2$
2. $\mathbb{Q} \times (\mathbb{R} \setminus \mathbb{Q})$
3. $(\mathbb{R} \setminus \mathbb{Q})^2$

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Dependence on norm

Indeed, we need to be careful, which norm we use:

Example (The set $(\mathbb{R} \setminus \mathbb{Q})^2$)

- $(\mathbb{R} \setminus \mathbb{Q})^2$ *is null permeable in* $(\mathbb{R}^2, \|\cdot\|_1)$.
- $(\mathbb{R} \setminus \mathbb{Q})^2$ *impermeable in* $(\mathbb{R}^2, \|\cdot\|_2)$.

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However, we have the following theorem:

Theorem (Invariance of norms, LSRT 2024)

Let $\|\cdot\|$ be any norm on $\mathbb{R}^d$ such that the boundary of its unit ball is strictly convex. Then $\Theta \subset \mathbb{R}^d$ is null permeable in $(\mathbb{R}^d, \|\cdot\|)$ if and only if it is null permeable in $(\mathbb{R}^d, \|\cdot\|_2)$. The same equivalence is true for permeability.

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We will work exclusively with the Euclidean norm.

Products

Theorem (LSRT 2024)

Let $A \subset \mathbb{R}^j$, $B \subset \mathbb{R}^k$. If A is a Lebesgue nullset and B has dense complement, then $A \times B \subset \mathbb{R}^{j+k}$ is null permeable

Theorem (LSRT 2024)

Let $A \subset \mathbb{R}$, $B \subset \mathbb{R}$. If A and B have positive Lebesgue measure, then $A \times B \subset \mathbb{R}^2$ is impermeable

Therefore the answer to Question 1 is:

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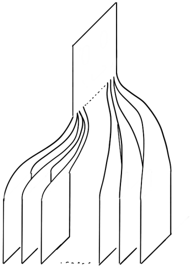
Question 2

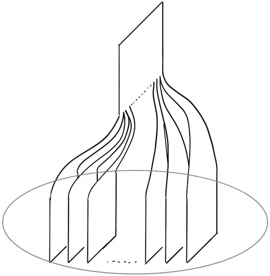
Let $\Theta \subset \mathbb{R}^d$ be a $d-1$-dimensional submanifold. Which of the following conditions implies permeability?

1. *Θ is C^∞*
2. *Θ is Lipschitz and closed (i.e. $\overline{\Theta} = \Theta$)*
3. *Θ is the graph of a Hölder-continuous function*

Manifolds



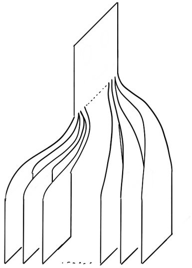
Manifolds



A a countable set where $\overline{A}$ contains a Cantor set

Connect $A \times (0,1)^2 \subset \mathbb{R}^3$ smoothly

Manifolds



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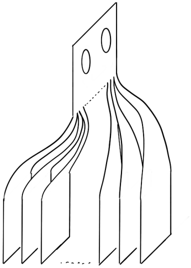

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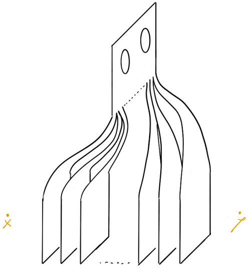

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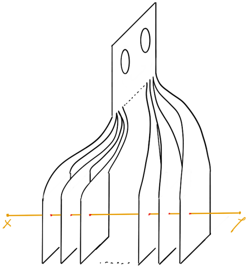

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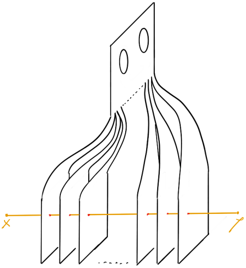

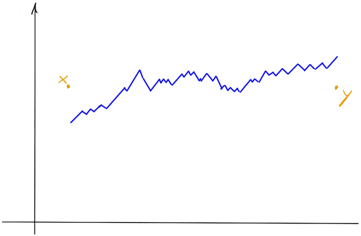

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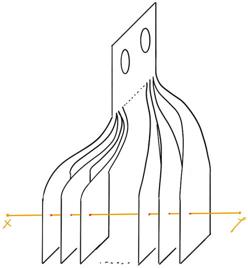

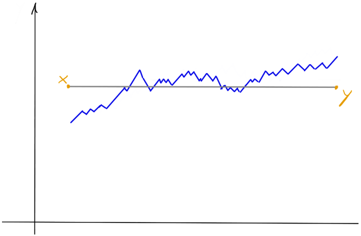

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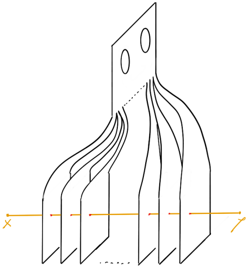

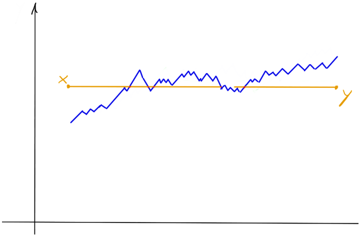

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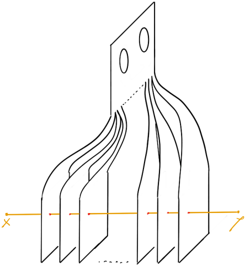

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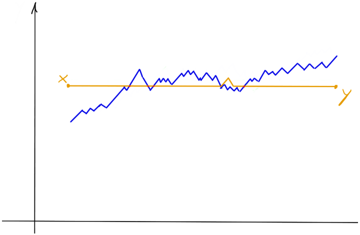
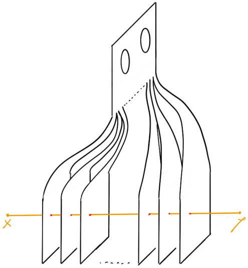
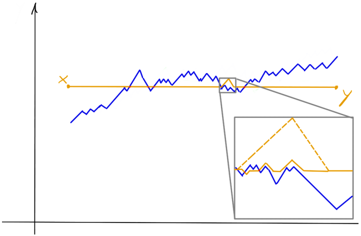


Avoid the graph with a Lipschitz detour.

Manifolds





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Avoid the graph with a Lipschitz detour.
'Zoom in on finer scale' if necessary

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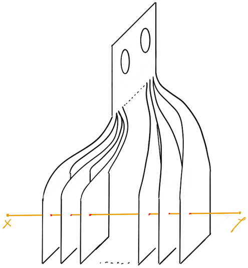

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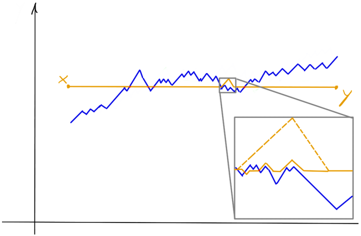

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Does not work for Hölder

Sets of large measure

Definition ((Null-)permeability)

Let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^d$.

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Question 3

Let $\Theta \subset [0, 1]^d$. Which of the following conditions implies (im)permeability?

1. *Θ is a Lebesgue-nullset*
2. *Θ has positive measure*
3. *Θ has measure 1*
4. *Θ is a simple curve and has positive measure*

A “large” null permeable set

$(A)_\varepsilon$: open ε-neighborhood of $A \subset \mathbb{R}^d$.

Example 1 (Null permeable with large Lebesgue measure)

- Let $(q_i)_{i\in\mathbb{N}}$ be a sequence of all points in $[0,1]^d$ with rational coordinates.
- $\overline{xy}$ denotes the line segment from x to y.
- $\Theta = [0,1]^d \setminus \bigcup_{1\le i<j} \overline{q_i q_j}$.

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- Θ null permeable with $\lambda(\Theta) = 1$.
- Θ_n closed and null permeable with $\lambda(\Theta_n) \to 1$.

An impermeable Osgood curve

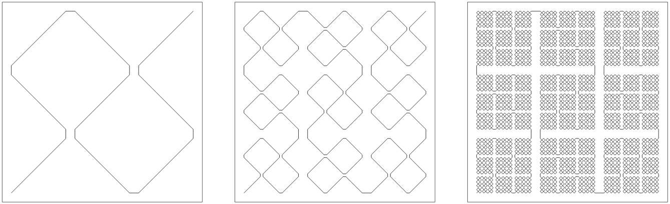

Figure: 1, 2, and 4 iterations of an impermeable Osgood curve

A permeable Osgood curve

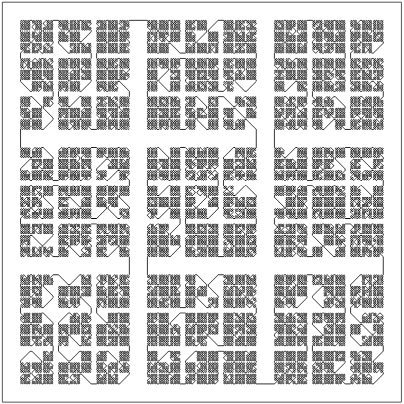

Figure: Five iterations of a permeable Osgood curve

Sets of low dimension

Definition ((Null-)permeability)

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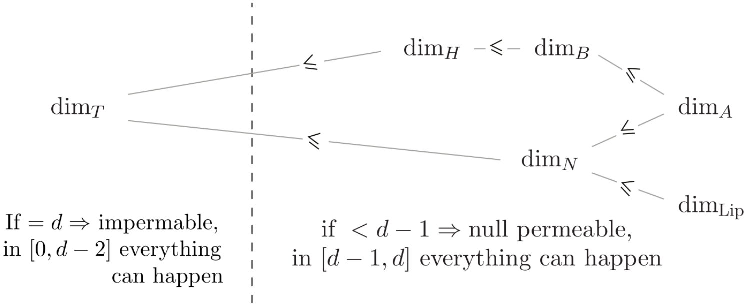
Question 4

Let $\Theta \subset \mathbb{R}^d$. Which of the following conditions implies null-permeability?

1. $\dim(\Theta) = 0$
2. $\dim(\Theta) < d - 1$
3. $\dim(\Theta) < d$

Synopsis



Relations between different notions of dimension and permeability

Fractal examples

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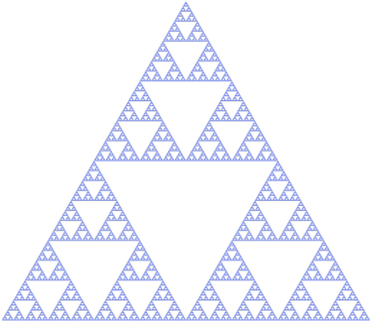
Question 5

Which of the following sets is permeable?

1. *Sierpiński gasket*
2. *Sierpiński carpet*
3. *Sierpiński tetrahedron*
4. *Menger sponge*
5. *"(Fat) Cantor dust"*

Fractal examples



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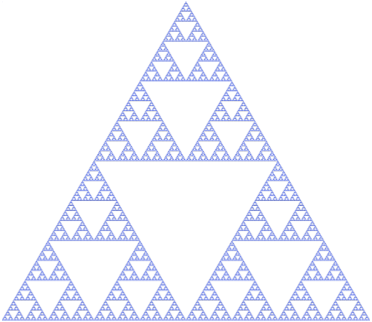
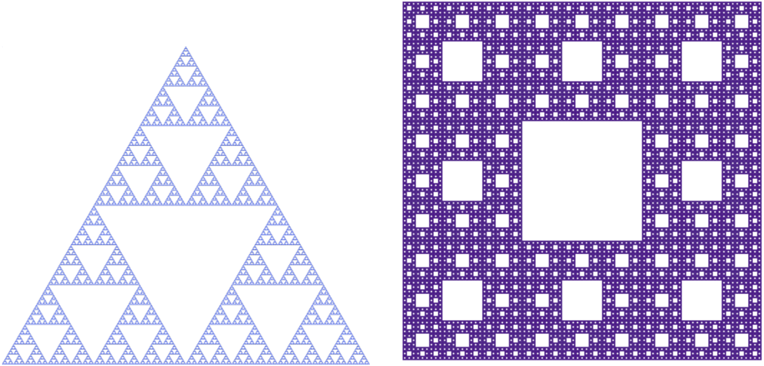


The Sierpiński gasket is permeable

Fractal examples



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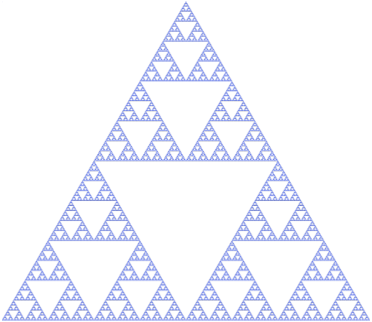
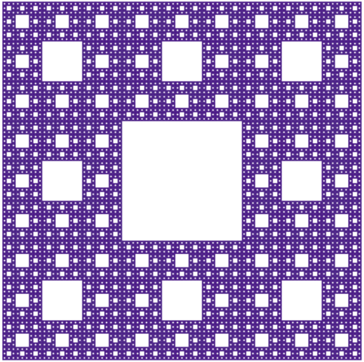
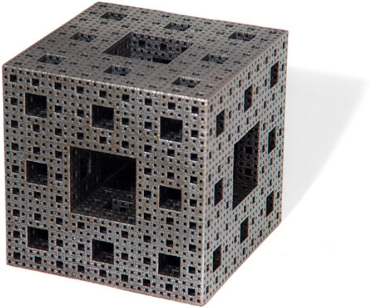




The Sierpiński gasket is permeable, the Sierpiński carpet is not (pictures taken from Wikipedia)

Fractal examples



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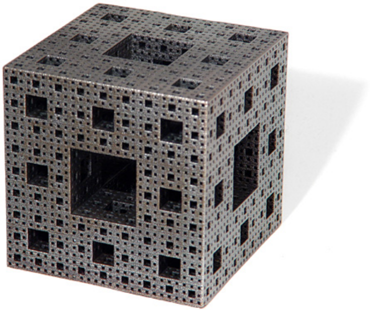
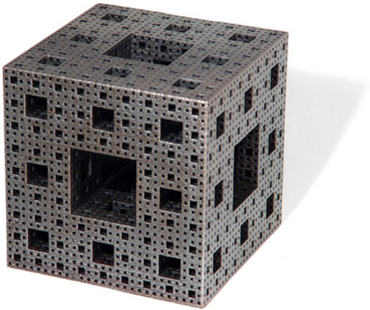
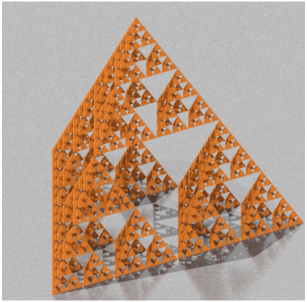


The Menger sponge is null permeable

Fractal examples





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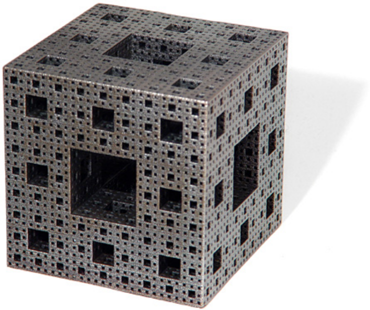

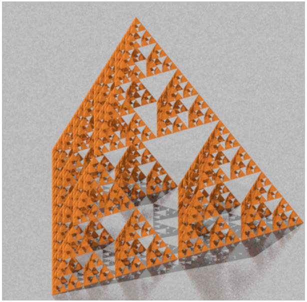

The Menger sponge is null permeable and so is the Sierpiński tetrahedron (pictures taken from Wikipedia)

Results on self-similar sets

Theorem 2

[LRST 2024] Let $K \subset \mathbb{R}^2$ be the attractor of a self-similar IFS $\{f_1, \ldots, f_m\}$ satisfying $\# f_i(K) \cap f_j(K) < \infty$ for $1 \leq i < j \leq m$. If K is connected and satisfies the finite type condition *then K is* permeable.

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Theorem 3

[LRST 2024] For $d \geq 3$ let $K \subset \mathbb{R}^d$ be the attractor of a self-similar IFS $\{f_1, \ldots, f_m\}$ satisfying $\# f_i(K) \cap f_j(K) < \infty$ for $1 \leq i < j \leq m$. Suppose further that K satisfies the finite type condition*. Then K is* null permeable.

The last result is a consequence of our result on the Nagata dimension.

Bedford-McMullen Carpets

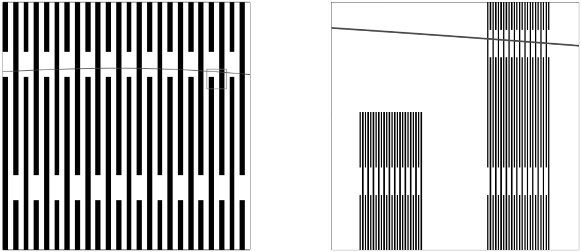

Figure: Left: A path crossing a Bedford-McMullen carpet. Right: A magnified section.

Bedford-McMullen Carpets

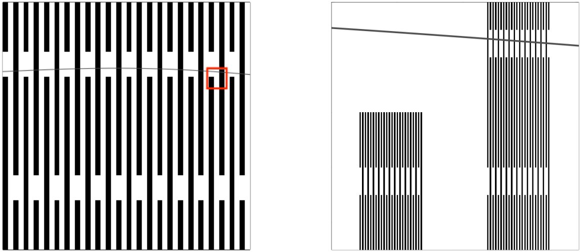

Figure: Left: A path crossing a Bedford-McMullen carpet. Right: A magnified section.

Bedford-McMullen Carpets

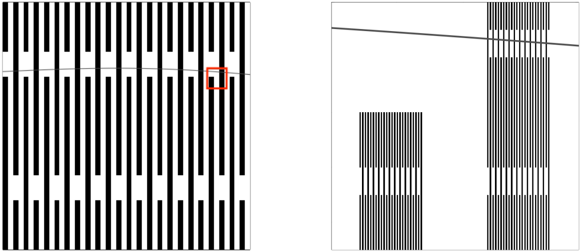

Figure: Left: A path crossing a Bedford-McMullen carpet. Right: A magnified section.

Theorem 4 (LRST 2024)

There exists an impermeable set in $\mathbb{R}^d$ which is closed, has Lebesgue measure 0 and topological dimension 0.