On Transcendence of Sturmian and Morphic Words over Algebraic Bases

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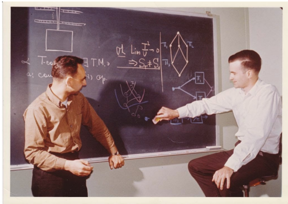
Hartmanis-Stearns Conjecture

Hartmanis-Stearns Conjecture (1965)

A number whose base-b expansion is computed by a **linear-time Turing machine** (prints first n digits in $O(n)$ time) is either **rational or transcendental**.



"The Hartmanis-Stearns conjecture implies that there is no linear-time algorithm for integer multiplication."

J. Borwein and P. B. Borwein, 1988

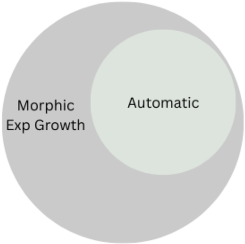
Cobham Conjectures

Cobham's First Conjecture (1968)

A number whose base-b expansion is generated by a **finite automaton** is either **rational or transcendental**.

Cobham's Second Conjecture (1968)

A number whose base-b expansion is generated by a **morphism of exponential growth** is either **rational or transcendental**.



Morphisms capture a very simple kind of linear time computation

Thue-Morse Word:

- Fixed point of the *2-uniform* morphism

$$\sigma : 0 \mapsto 01, \quad 1 \mapsto 10$$

Fibonacci Word:

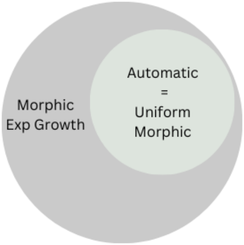
- Fixed point of morphism

$$\sigma : 0 \mapsto 01, \quad 1 \mapsto 0$$

of *exponential growth*

Characteristic word of cubic numbers:

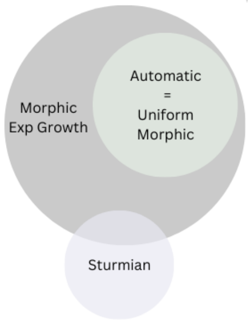
- Coding of morphism of *polynomial growth*



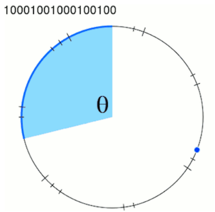
The Fibonacci word

$$F_\infty = 0100101001001010010100\ldots$$

has subword complexity $p(n) = n + 1$ for all n and is thereby **Sturmian**.



Given $\theta \in [0, 1)$, consider rotation map $R_\theta(x) = (x + \theta) \bmod 1$.



The θ-**coding** of $x \in [0, 1)$ is the sequence $(x_n)_{n=0}^\infty$, where

$$x_n := \begin{cases} 1 & \text{if } R_\theta^n(x) \in [0, \theta) \\ 0 & \text{otherwise} \end{cases}$$

Sturmian word with **slope** θ

Pisot Substitutions

- A primitive irreducible substitution is **Pisot** if the Perron-Frobenius eigenvalue of its incidence matrix is a Pisot number.

Example

The **Tribonacci word** is the limit of the sequence defined by

$$T_n = T_{n-1} T_{n-2} T_{n-3} \quad T_0 = 0,\, T_1 = 01,\, T_2 = 0102$$

$$\lim_{n \to \infty} T_n = T_\infty = 0102010010201010201001020102010 0\dots$$

Also generated by the morphism $\sigma(0) = 01$, $\sigma(1) = 02$, $\sigma(2) = 0$.

Central Question

Given an infinite word $u = u_1 u_2 u_3 \ldots$ *and integer base* $b \geq 2$*, is the number*

$$S_b(u) := \sum_{n=0}^{\infty} \frac{u_n}{b^n}$$

rational, irrational algebraic or transcendental?

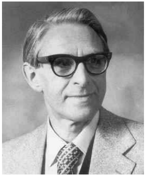
How do you prove a number is transcendental?

Find rational approximations that are “too good”.

Theorem (Roth 1955)

Let α be real and algebraic, then for all $\varepsilon > 0$ there are finitely many coprime integers p, q such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\varepsilon}}\,.$$



We use *p-adic Subspace Theorem*: a higher-dimensional generalisation of Roth's Theorem, proven by Schlickewei in 1977

The Fibonacci word yields a transcendental number

We look for periodicity

$$F_\infty = 01001010010010100101001001010\ldots$$

- Prefix of length 3

$$F_\infty = \boxed{010}\boxed{010}1001001010010100100\ldots$$

- Prefix of length 5

$$F_\infty = \boxed{01001}\boxed{01001}\boxed{0}0101001010010000\ldots$$

- Prefix of length 8

$$F_\infty = \boxed{01001010}\boxed{01001010}\boxed{010}100100101\ldots$$

Use p-**adic Roth theorem** to deduce transcendence

Transcendence of Sturmian Words

Theorem (Danilov 1972)

Let $\boldsymbol{u}$ be the **Fibonacci word***. Then for all integers $b \geq 2$ the number*

$$S_b(\boldsymbol{u}) := \sum_{n=0}^{\infty} \frac{u_n}{b^n}$$

is **transcendental***.*

Theorem (Ferenczi and Mauduit 1997)

Let $b \geq 2$ be an integer and let $u \in \{0, 1, \ldots, b-1\}^\omega$ be a **Sturmian** *or* **Arnoux-Rauzy word***. Then $S_b(u) := \sum_{n=0}^{\infty} \frac{u_n}{b^n}$ is* **transcendental***.*

Non-commutative Diophantine Approximation

Definition (Adamczewski and Bugeaud 2007)

The **Diophantine exponent** of $\boldsymbol{u}$ is the supremum of all real ρ such that $\boldsymbol{u}$ has arbitrarily long prefixes of the form UV^α, for $\alpha \geq 1$, satisfying

$$\frac{|UV^\alpha|}{|UV|} \geq \rho$$

- We have $1 \leq \mathrm{Dio}(\boldsymbol{u}) \leq \infty$ for all $\boldsymbol{u}$
- Eventually periodic words have infinite Diophantine exponent.

Theorem (Adamczewski and Bugeaud 2007)

For an integer $b \geq 2$ and sequence $\boldsymbol{u} \in \{0, \ldots, b-1\}$, if $\mathrm{Dio}(\boldsymbol{u}) > 1$ then $S_b(\boldsymbol{u})$ is either **rational** *or* **transcendental**.

Resolving Cobham's Conjectures

Ferenczi and Mauduit 97

Sturmian words have Diophantine exponent > 2.

Adamczewski, Bugeaud, Luca 04

Automatic words have Diophantine exponent > 1. (Resolves Cobham's first conjecture.)

Adamczewski, Cassaigne, Le Gonidec 20

Words generated by morphisms of exponential growth have Diophantine exponent > 1. (Resolves Cobham's second conjecture.)

From Combinatorics to Arithmetic

1. Assume $\alpha = \sum_{n=0}^{\infty} \frac{u_n}{b^n}$ is algebraic

2. Ferenzci and Mauduit's condition yields a sequence of good rational approximations to α, yielding infinitely many points in $\mathbb{Z}^2$ on which linear form $L(x_1, x_2) = \alpha x_1 - x_2$ is "small"

3. Apply Ridout's p-adic version of the Thue-Siegel-Roth Theorem to obtain a **contradiction**.

4. Weaker condition $\mathrm{Dio}(\boldsymbol{u}) > 1$ yields infinite sequence of points in $\mathbb{Z}^3$ on which linear form $L(x_1, x_2, x_3) = \alpha x_1 - \alpha x_2 - x_3$ is "small"

5. Apply Subspace Theorem to conclude that α is **rational**

Algebraic Bases

We consider the sum

$$S_\beta(\boldsymbol{u}) := \sum_{n=0}^{\infty} \frac{u_n}{\beta^n} ,$$

where the base β is an algebraic number with $|\beta| > 1$.

Algebraic Bases

Theorem (Adamczewski and Bugeaud 2007b)

Let β be an algebraic integer with $|\beta| > 1$. If $\mathrm{Dio}(\boldsymbol{u}) > \frac{\log M(\beta)}{\log|\beta|}$. Then $S_\beta(\boldsymbol{u})$ either lies in $\mathbb{Q}(\beta)$ or is transcendental.

Theorem (B. Adamczewski and C. Faverjon 2018)

For any algebraic number β with $|\beta| > 1$ and ***automatic*** *word $\boldsymbol{u} = u_0 u_1 \dots$ the number $S_\beta(\boldsymbol{u})$ either lies* **in** $\mathbb{Q}(\beta)$ **or is transcendental.**

Theorem (Loxton and van der Poorten 1977)

Let β be an algebraic integer with $|\beta| > 1$. For any non-constant polynomial $f(x) \in \mathbb{Z}[x]$ and irrational θ the sum $\sum_{n=0}^{\infty} f(\lfloor n\theta \rfloor)\beta^{-n}$ is transcendental.

Implies transcendence of $S_\beta(\boldsymbol{u})$ for **standard Sturmian words** $\boldsymbol{u}$.

Our Results – Sturmian Words

Theorem (LOW 24)

Let β be algebraic with $|\beta| > 1$. If u is **Sturmian** *then $S_\beta(u)$ is* **transcendental**.

This is shown in Bugeaud & Laurent 2023 using Mahler method

Theorem (LOW 24)

Let β be algebraic with $|\beta| > 1$. Let $u_1, \ldots, u_k$ be **Sturmian words**, *all having the same slope and such that no word is a tail of another. Then $\{1, S_\beta(u_1), \ldots, S_\beta(u_k)\}$ is* **linearly independent over** $\overline{\mathbb{Q}}$.

Theorem

Let β be an algebraic integer with $|\beta| > 1$. For any non-constant polynomial $f(x) \in \mathbb{Z}[x]$, $\alpha \in \mathbb{R}$ and irrational θ the sum $\sum_{n=0}^{\infty} f(\lfloor n\theta + \alpha \rfloor)\beta^{-n}$ is transcendental.

Our Results – Morphic Words

Theorem (KLOSW 24)

Let β be algebraic with $|\beta| > 1$. Then $S_\beta(u)$ is **transcendental** *where u is the* **Tribonacci word**.

Theorem

*Let u be the d-***bonacci word.** *Then for any algebraic number β with $|\beta| > 1$ the number $S_\beta(u)$ is* **transcendental**.

Theorem

Let u be the fixed point of a Pisot morphism on a binary alphabet. Then for any algebraic number β with $|\beta| > 1$ the number $S_\beta(u)$ is **transcendental or belongs to** $\mathbb{Q}(\beta)$.

Periodicity Modulo Errors

Let $(r_n)_{n=0}^{\infty}$ be Fibonacci sequence and write $F_\infty^{(n)}$ for tail of Fibonacci word after dropping first r_n letters.

$$F_\infty := \boxed{010010}\,\mathbf{10}\,\boxed{01001010010}\,\mathbf{10}\,\boxed{010010}\,\mathbf{10}\ldots$$
$$F_\infty^{(5)} := \boxed{010010}\,\mathbf{01}\,\boxed{01001010010}\,\mathbf{01}\,\boxed{010010}\,\mathbf{01}\ldots$$

$$F_\infty := \boxed{01001010010}\,\mathbf{01}\,\boxed{0100101001001010010}\,\mathbf{01}\,\boxed{01001010010}\,\mathbf{01}\ldots$$
$$F_\infty^{(6)} := \boxed{01001010010}\,\mathbf{10}\,\boxed{0100101001001010010}\,\mathbf{10}\,\boxed{01001010010}\,\mathbf{10}\ldots$$

- Mismatches come in consecutive symmetric pairs
- Gaps between these pairs "expand with n"
- Mismatches contribute $\pm\beta^n(\beta - 1)$ in base β

Periodicity in Tribonacci Word

Let $(r_n)_{n=0}^{\infty}$ be Tribonacci word and write $T_{\infty}^{(n)}$ for the word obtained by dropping the first r_n letters

$$T_{\infty} := \boxed{01020100102010}\,\mathbf{\color{red}{102}}\,\boxed{0}\,\mathbf{\color{red}{10}}\,\boxed{0}\,\mathbf{\color{red}{102}}\,\boxed{01020100102010}\,\mathbf{\color{red}{102010}}\,\boxed{0\ldots}$$
$$T_{\infty}^{(13)} := \boxed{01020100102010}\,\mathbf{\color{red}{201}}\,\boxed{0}\,\mathbf{\color{red}{01}}\,\boxed{0}\,\mathbf{\color{red}{201}}\,\boxed{01020100102010}\,\mathbf{\color{red}{010201}}\,\boxed{0\ldots}$$

There is now a finite alphabet of palindromic mismatches

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

- Expanding gaps between *groups of mismatches*
- Non-vanishing condition on groups of mismatches

Combinatorial Criterion – Echoing Words

Definition

A word $\boldsymbol{u}$ is **echoing** if for all $\rho > 0$ and $\varepsilon > 0$ there exist unbounded sequence $\langle r_n \rangle_{n=0}^{\infty}$ of **quasi-periods** and $d \geq 2$ such that, writing $s_n := \rho r_n$, we have

1. **Few mismatches**: the set of mismatches between prefix $u_0 \ldots u_{s_n}$ and factor $u_{r_n} \ldots u_{r_n+s_n}$ is a contained in a union of at most d intervals of total length at most εs_n.
2. **Expanding gaps** the gaps between intervals expand with n.
3. **Non-vanishing** There is some interval in which the contributions of the mismatches is non-zero in base β.

Linear form in application of Subspace Theorem depends has $d + 3$ terms.

Sturmian Words are Echoing

Theorem

Sturmian words are echoing.

Proof.

- Analysis of simple continued-fraction expansion of slope
- Quasi-periods are denominators of convergents
- Mismatches are always consecutive pairs

□

Alphabet:

$$\left\{ \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}, \begin{bmatrix}0&1\\1&0\end{bmatrix}, \begin{bmatrix}0&2\\2&0\end{bmatrix}, \begin{bmatrix}1&0&2\\2&0&1\end{bmatrix}, \begin{bmatrix}0&1&0&2\\2&0&1&0\end{bmatrix}, \ldots \right\}$$

Matching Morphism:

$$\begin{bmatrix}0\\0\end{bmatrix} \mapsto \begin{bmatrix}0\\0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} \qquad \begin{bmatrix}1\\1\end{bmatrix} \mapsto \begin{bmatrix}0\\0\end{bmatrix}\begin{bmatrix}2\\2\end{bmatrix} \qquad \begin{bmatrix}2\\2\end{bmatrix} \mapsto \begin{bmatrix}0\\0\end{bmatrix}$$

$$\begin{bmatrix}0&1\\1&0\end{bmatrix} \mapsto \begin{bmatrix}0\\0\end{bmatrix}\begin{bmatrix}1&0&2\\2&0&1\end{bmatrix} \qquad \begin{bmatrix}1&0&2\\2&0&1\end{bmatrix} \mapsto \begin{bmatrix}0\\0\end{bmatrix}\begin{bmatrix}2&0&1&0\\0&1&0&2\end{bmatrix}$$

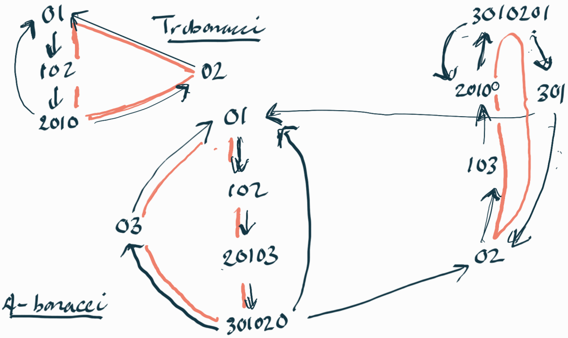
$$\begin{bmatrix}2&0&1&0\\0&1&0&2\end{bmatrix} \mapsto \begin{bmatrix}0\\0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}0&2\\2&0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}$$

Initial shift:

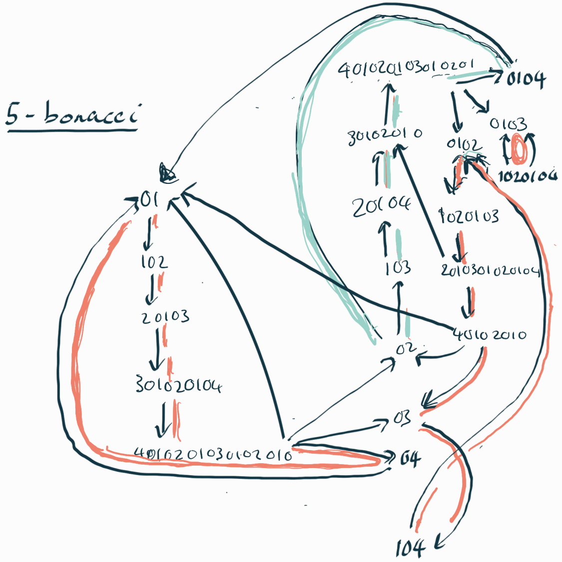
$$\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}0&2\\2&0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}0&2\\2&0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}0&2\\2&0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix}\cdots$$

Non-Vanishing Condition

Incidence graph of matching morphism has unique cycle cover:



Non-Vanishing Condition



Summary of the Results

Let β be algebraic with $|\beta| > 1$:

Theorem

If u is Sturmian then $S_\beta(u)$ is transcendental.

Theorem

For any non-constant polynomial $f(x) \in \mathbb{Z}[x]$, $\alpha \in \mathbb{R}$ and irrational θ the sum $\sum_{n=0}^{\infty} f(\lfloor n\theta + \alpha \rfloor)\beta^{-n}$ is transcendental.

Theorem

If $\boldsymbol{u}$ is a d-Bonacci word then $S_\beta(u)$ is transcendental.

Theorem

Let $\boldsymbol{u}$ be a Pisot morphic word on a binary alphabet. Then $S_\beta(u)$ is either transcendental or belongs to $\mathbb{Q}(\beta)$.

Conclusion and Future Work

- Let $\boldsymbol{u}$ be generated by a Pisot morphism. Assuming the Pisot conjecture, the balanced-pair algorithm terminates with coincidence. One obtains that $S_\beta(\boldsymbol{u})$ is either transcendental or lies in $\mathbb{Q}(\beta)$.
- Use non-vanishing condition to exclude the possibility that $S_\beta(\boldsymbol{u}) \in \mathbb{Q}(\beta)$. We are currently investigating when this holds.