Quantitative Semantics for Probabilistic Programming

joint work with T. Ehrhard and M. Pagani

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A denotational semantics for probabilistic higher-order functional computation,

(based on quantitative semantics of Linear Logic)

Discrete setting:

Probabilistic Coherent Spaces are fully abstract for a programming language with natural numbers as base types suitable to encode discrete probabilistic programs.

Continuous setting:

A CCC of measurable spaces and stable maps that soundly denotes a programming language with reals as base types suitable to encode continuous probabilistic programs.
1. **Discrete Probability**
   - Syntax: Discrete Probabilistic PCF
   - Semantics: $\text{Pcoh}$ (Probabilistic Coherent Spaces)
   - Results: Probabilistic Adequacy & Full Abstraction
   - Discrete Probabilistic Call By Push Value

2. **Continuous Probability**
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<th>Quantitative Semantics</th>
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Bibliography

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<td>Plotkin</td>
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### General Framework

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#### How to interpret a program $M : \mathcal{N} \Rightarrow \mathcal{N}$

**Type:**

- $\mathbb{N}_\bot$ flat domain,
- $\mathcal{V}(\mathbb{N}_\bot)$ proba. distr. over $\mathbb{N}_\bot$,
- $|\mathbb{Nat}| = \mathbb{N}$
- $P(\mathbb{Nat})$ subproba. dist. over $\mathbb{N}$

**Prog:**

- $[M] : \mathbb{N}_\bot \rightarrow \mathcal{V}(\mathbb{N}_\bot)$,
- $[\text{let } n=x \text{ in } M] : \mathcal{V}(\mathbb{N}_\bot) \rightarrow \mathcal{V}(\mathbb{N}_\bot)$
- $x \mapsto \left( \sum_n [M]_{n,q}x_n \right)_q$
- $x \mapsto \left( \sum_{\mu} [M]_{\mu,q} \prod_{i=1}^k x_{n_i} \right)_q$

**Prog:**

- $[M] : P(\mathbb{Nat}) \rightarrow P(\mathbb{Nat})$
- $x \mapsto \left( \sum_{\mu=[n_1,\ldots,n_k]} [M]_{\mu,q} \prod_{i=1}^k x_{n_i} \right)_q$
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Problematic in domain

Finding a full subcategory of continuous dcpos that is: **Cartesian Closed** and **closed** under the proba. monad $\mathcal{V}$.

Full Abs.: PCOH/pPCF

\[
\begin{align*}
\text{Red}(C[M], n) & \quad \forall n, \forall C[] \\
\text{Red}(C[N], n) & \quad \text{iff} \\
[M] & = [N].
\end{align*}
\]
1. **Discrete Probability**
   - Syntax: **Discrete** Probabilistic PCF
   - Semantics: **Pcoh** (Probabilistic Coherent Spaces)
   - Results: Probabilistic **Adequacy** & **Full Abstraction**
   - Discrete Probabilistic Call By Push Value

2. **Continuous Probability**
# Syntax of PPCF

## Syntax of PPCF:

<table>
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<th>Types:</th>
<th>$A, B ::= N \mid A \rightarrow B$</th>
</tr>
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<tbody>
<tr>
<td>Terms:</td>
<td>$M, N, L ::= x \mid \lambda x^A.M \mid (M)N \mid \forall M \mid$</td>
</tr>
<tr>
<td></td>
<td>$\text{coin} \mid n \mid \text{succ}(M) \mid \text{ifz}(L, M, N)$</td>
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## Operational Semantics:

$\text{Red}(M, N)$ is the **probability** that $M$ reduces to $N$ in a step.

$\text{Red}((\lambda x^A.M)N, M[N/x]) = 1$, as $(\lambda x^A.M)N \xrightarrow{1} M[N/x]$

$\text{Red}(\text{coin}, 0) = \text{Red}(\text{coin}, 1) = \frac{1}{2}$, as $\text{coin} \xrightarrow{1/2} 0$

If $\vdash M : N$, then $\text{Red}^\infty(M, \_)$ is the discrete distribution over $\mathbb{N}$ of all normal forms computed by $M$. 

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C. Tasson  
Introduction  Discrete (Pcoh)  Continuous (Cstab_m)  Conclusion  
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1 Discrete Probability

- Syntax: Discrete Probabilistic PCF
- Semantics: **Pcoh** (Probabilistic Coherent Spaces)
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2 Continuous Probability
Types as **Probabilistic Coherent Spaces**: \((|X|, P(X))\)

### Proba. Space

- **\(|X|\)**: the web, a (potentially infinite) set of final states
- **\(P(X)\)**: a set of vectors \(\subseteq (\mathbb{R}^+)^{|X|}\) such that

  **closure**: \(P(X)^\perp\perp = P(X)\) with
  \[
  \forall u, v \in (\mathbb{R}^+)^{|X|}, \langle u, v \rangle = \sum_{a \in |X|} u_a v_a
  \]

  \[
  \forall P \subseteq (\mathbb{R}^+)^{|X|}, P^\perp = \{ v \in (\mathbb{R}^+)^{|X|}; \forall u \in P, \langle u, v \rangle \leq 1 \}
  \]

  **bounded covering**: \(\forall a \in |X|,
  \exists v \in P(X); v_a \neq 0\) and \(\exists p > 0, ; \forall v \in P(X), v_a \leq p.\)

### Proposition: Proba. spaces as Domains

\((|X|, P(X))\) is a **Proba. space iff** \(P(X)\) is bounded covering, **Scott Closed** (downwards-closed and dcpo) and **Convex**.
Types as **Probabilistic Coherent Spaces**: \((|X|, P(X))\)

**Example:**

| Type          | \(P(X) \subseteq (\mathbb{R}^+)^{|X|}\) |
|---------------|----------------------------------------|
| \(|1| = \{\ast\}\) | \(P(1) = [0, 1]\) |
| \(|\text{Bool}| = \{t, f\}\) | \(P(\text{Bool}) = \{(x_t, x_f) \mid x_t + x_f \leq 1\}\) |
| \(|\text{Nat}| = \{0, 1, 2, \ldots\}\) | \(P(\text{Nat}) = \{x \in [0, 1]^\mathbb{N} \mid \sum_n x_n \leq 1\}\) |
| \(|\text{Bool} \rightarrow 1| = \{[t^n, f^m] \mid n, m \in \mathbb{N}\}\), \(P(\text{Bool} \rightarrow 1) = \{Q \in (\mathbb{R}^+)^{\text{Bool} \rightarrow 1} \mid \forall x_t + x_f \leq 1, \sum_{n,m=0}^\infty Q[t^n,f^m] x_t^n x_f^m \leq 1\}\) |

**Proposition: Proba. spaces as Domains**

\((|X|, P(X))\) is a **Proba. space iff** \(P(X)\) is bounded covering, **Scott Closed** (downwards-closed and dcpo) and **Convex**.
A model of Linear Logic

**Pcoh : Linear Category**
- Objects: Proba. Spaces
- Morphisms: Linear Functions

**Call by Name**
\[ A \rightarrow B =! A \rightarrow B \]

**Pcoh\!: Kleisli Category**
- Objects: Proba. Spaces
- Morphisms: Analytic Functions

- **Smcc** \((1, \otimes, \rightarrow)\)
- **biproduct**

- **Comonad** \((!, \text{der}, \text{dig})\)

- **Com. Comonoid**
  \((!A, 1, \otimes)\)

- **CCC**

- **(PCF+coin)**
**Pcoh**(*X*, *Y*)

Matrices \( Q \in (\mathbb{R}^+)^{|X| \times |Y|} \) such that:

\[
\forall x \in P(X), \quad Q \cdot x = \left( \sum_{a \in |X|} Q_{a,b} x_a \right)_b \in P(Y)
\]

**Example**

**Pcoh**(Nat, Nat): Stochastic Matrices \( Q \in (\mathbb{R}^+)^{N \times N} \).

\[
\forall x \in (\mathbb{R}^+)^N ; \quad \sum_{n \in N} x_n \leq 1, \quad \sum_{m,n \in N} Q_{m,n} x_n \leq 1
\]
Exponential

\[ |!X| = M_{\text{fin}} (|X|) \text{ the set of finite multisets} \]
\[ P (!X) = \{ x^1 ; x \in P (X) \} \] \[ \text{where } x_{[a_1, \ldots, a_k]}^1 = \prod_{i=1}^k x_{a_i} \]

Example

Let \( B\text{coin} = (p, 1 - p) \in P (\text{Bool}) = \{ (p, q) ; p + q \leq 1 \} \).

\[ B\text{coin}^1[\_] = 1, \quad B\text{coin}^1[t, t] = p^2, \quad B\text{coin}^1[t, f] = p(1 - p), \ldots \]

Theorem (2017: Crubillé - Ehrhard - Pagani - T.)

This exponential computes the free commutative comonoid.
**Exponential**

\[ |!X| = \mathcal{M}_{\text{fin}}(|X|) \text{ the set of finite multisets} \]

\[ P(!X) = \{ x^l ; x \in P(X) \}^{\perp \perp} \text{ where } x^l_{[a_1,\ldots,a_k]} = \prod_{i=1}^{k} x_{a_i} \]

---

**Commutative Comonoid**

Cocontr.: \(!X \xrightarrow{c_{!X}} !X \otimes !X\)

Coweak.: \(!X \xrightarrow{w_{!X}} 1\)

---

**Comonad**

Comult.: \(\text{dig}_{!X} : !!X \rightarrow !X\)

Counit: \(\text{der}_{!X} : !X \rightarrow X\)

---

**Theorem (2017: Crubillé - Ehrhard - Pagani - T.)**

This exponential computes the free commutative comonoid.
\[ P_{coh!}(X, Y) = P_{coh}(!X, Y) \]

Matrices \( Q \in (\mathbb{R}^+)^{\mathcal{M}_{\text{fin}}(|X|) \times |Y|} \) such that

\[ \forall U \in P(!X), \quad Q \cdot U = \left( \sum_{m \in \mathcal{M}_{\text{fin}}(|X|)} Q_{m,b} U_m \right)_{b} \in P(Y) \]

Non-Linear Morphisms are **analytic** and **Scott Continuous**.

\[ P_{coh!}({\text{Bool}}, 1) = \{ Q \in (\mathbb{R}^+)^{|\text{Bool} \rightarrow 1|} \text{ s.t. } Q[t^n, f^m] \leq \frac{(n+m)^{n+m}}{n^n m^m} \} \]

let rec f x =
  if x then if x then f x
  else ()
else if x then ()
else f x

\[ \sum_{n,m=0}^{\infty} \frac{(n+m)!}{n! m!} x_t^{2n+1} x_f^{2m+1} \]
Non Linear Category

\[ \text{Pcoh}_!(X, Y) = \text{Pcoh}(!X, Y) \]

**Density**

Matrices \( Q \in (\mathbb{R}^+)^{\mathcal{M}_{\text{fin}}(|X|) \times |Y|} \) such that if \( x_m^! = \prod_{a \in m} x_a^{m(a)} \)

\[ \forall x \in P(X), \quad Q(x) = \left( \sum_{m \in \mathcal{M}_{\text{fin}}(|X|)} Q_{m,b} x_m^! \right)_b \in P(Y) \]

Non-Linear Morphisms are **analytic** and **Scott Continuous**.

\[ \text{Pcoh}_!(\text{Bool}, 1) = \{ Q \in (\mathbb{R}^+)^{|\text{Bool} \to 1|} \mid \text{s.t. } Q[t^n, f^m] \leq \frac{(n+m)^{n+m}}{n! m!} \} \]

let rec f x =
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denotes

\[ \sum_{n,m=0}^{\infty} \left( \frac{(n + m)!}{n! m!} \right) x_t^{2n+1} x_f^{2m+1} \]
Non Linear Category

\[ P_{\text{coh}}(X, Y) = P_{\text{coh}}(\mathcal{!}X, Y) \]

**Density**

Matrices \( Q \in (\mathbb{R}^+)^{\mathcal{M}_{\text{fin}}(|X|) \times |Y|} \) such that if \( x_m^! = \prod_{a \in m} x_{a}^{m(a)} \)

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Non-Linear Morphisms are **analytic** and **Scott Continuous**.

**Pcoh!(\text{Bool}, 1) = \{ Q \in (\mathbb{R}^+)^{|\text{Bool} \rightarrow 1|} \; s.t. \; Q[t^n, f^m] \leq \frac{(n+m)^{n+m}}{n^n m^m} \}**

```
let rec f x =
    if x then if x then f x
      else ()
    else if x then ()
      else f x
```

\[ \sum_{n,m=0}^{\infty} \frac{(n + m)!}{n! m!} x_t^{2n+1} x_f^{2m+1} \]

**pb of DEFINABILITY**

C. Tasson

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   - Discrete Probabilistic Call By Push Value

2. **Continuous Probability**
Probabilistic Full Abstraction

Theorem (2014: Ehrhard - Pagani - T.)

\[
\begin{align*}
P_{coh} & \iff \text{Adequacy} \\
\llbracket M \rrbracket = \llbracket N \rrbracket & \iff \text{Red}^{\infty}(C[M], n) \equiv \forall C[I] \forall n \text{ Red}^{\infty}(C[N], n) \\
& \iff \text{Full Abstraction} \\
p_{PCF} & \iff M \sim_0 N
\end{align*}
\]

Adequacy Lemma (2011: Danos - Ehrhard):

If \( \vdash M : N \), then \( \forall n \in \mathbb{N}, \llbracket M \rrbracket_n = \text{Red}^{\infty}(M, n) \).

Adequacy proof:

If \( \llbracket M \rrbracket = \llbracket N \rrbracket \) then, \( \text{Red}^{\infty}((C)M, n) = \text{Red}^{\infty}((C)N, n) \)

1. Apply Adequacy Lemma:
\[
\text{Red}^{\infty}((C)M, n) = \llbracket (C)M \rrbracket.
\]

2. Apply Compositionality:
\[
\llbracket (C)M \rrbracket = \sum_{\mu} \llbracket C \rrbracket_{\mu} \prod_{\alpha \in \mu} \llbracket M \rrbracket^{\mu(\alpha)} = \sum_{\mu} \llbracket C \rrbracket_{\mu} \prod_{\alpha \in \mu} \llbracket N \rrbracket^{\mu(\alpha)} = \llbracket (C)N \rrbracket
\]
Probabilistic Full Abstraction

Theorem (2014: Ehrhard - Pagani - T.)

\[
P_{coh} \quad \text{Adequacy} \quad \text{Full Abstraction}
\]

\[
\begin{align*}
[M] &= [N] \\
\iff & \\
\text{Red}^{\infty}(C[M], n) & \equiv \forall C \forall n \text{Red}^{\infty}(C[N], n)
\end{align*}
\]

Adequacy Lemma (2011: Danos - Ehrhard):

If \( \vdash M : N \), then \( \forall n \in \mathbb{N}, [M]_n = \text{Red}^{\infty}(M, n) \).

Full Abstraction proof:

- Find testing terms that depend only on points of the web.
- Use regularity of analytic functions.
Discrete Probability

- Syntax: *Discrete* Probabilistic PCF
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Continuous Probability
How to encode a LasVegas Algorithm?

**Input:** A 0/1 array of length \( n \geq 2 \) s.t. \( \frac{1}{2} \) cells are 0.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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\( f : 0, 2, 5 \mapsto 0 \)

\( 1, 3, 4 \mapsto 1 \)

**Output:** Find the index of a cell containing 0.

**Caml:**

```
let rec LasVegas (f: nat -> nat) (n:nat) =
  let k = random n in
  if (f k = 0) then k
  else LasVegas f n
```

**pPCF:**

\[ \mathcal{Y} \left( \lambda_{\text{LasVegas}}^{\text{nat} \rightarrow \text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat}} \right. \lambda_{f}^{\text{nat} \Rightarrow \text{nat}} \lambda_{n}^{\text{nat}} \]

\[ (\lambda_{k}^{\text{nat}} \text{ifz } f \ k \ \text{then } k \ \text{else } \text{LasVegas } f \ n) \ (\text{rand} \ n) \]
How to encode a LasVegas Algorithm?

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$f : \begin{align*}
0, 2, 5 & \mapsto 0 \\
1, 3, 4 & \mapsto 1
\end{align*}$

**Output:** Find the index of a cell containing $0$.

**Caml:**

```caml
let rec LasVegas (f: nat -> nat) (n:nat) =
  let k = random n in
  if (f k = 0) then k
  else LasVegas f n
```

**pPCF:**

```
Y \left( \lambda \text{LasVegas} (\text{nat} \rightarrow \text{nat}) \Rightarrow \text{nat} \Rightarrow \text{nat}
\lambda f \text{nat} \Rightarrow \text{nat}
\lambda n \text{nat}
(\lambda k \text{nat} \text{ifz} f k \text{then} k
\text{else} \text{LasVegas} f n) (\text{rand} n) \right)
```
Semantics gives the answer

Storage Operator

```
let k = rand n in if k = 0 then k else 42
```

Integer in Pcoh: \([\mathcal{N}] = \text{Nat} = (\mathbb{N}, P(\text{Nat}) = \{ (\lambda_n) \mid \sum_n \lambda_n \leq 1 \})\)

Equipped with a structure of comonoid in the linear Pcoh:

- Cocontraction: \(c^\mathcal{N} : \mathcal{N} \rightarrow \mathcal{N} \otimes \mathcal{N}\)
- Coweakening: \(w^\mathcal{N} : \mathcal{N} \rightarrow 1\)

Bibliography

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- 1999 Levy, Call by Push Value, a subsuming paradigm.
- 2000 Nour, On Storage operator.
- 2016 Curien, Fiore, Munch-Maccagnoni, A Theory of Effects and Resources.
What sem. object to encode Storage Operator.

The Eilenberg Moore Category: \( \mathbf{Pcoh} \)

Coalgebras \( P = (P, h_P) \) with \( P \in \mathbf{Pcoh} \) and \( h_P \in \mathbf{Pcoh}(P, !P) \):

\[
P \xrightarrow{h_P} !P \\
\text{Id} \\
P \xrightarrow{\text{der}_P} P
\]

\[
P \xrightarrow{h_P} !P \\
h_P \\
!P \xrightarrow{!h_P} !!P
\]

Coalgebras have a comonoid structure: values can be stored.

Types interpreted as coalgebras:

\( !X \) by def. of the exp. \( \quad \otimes, \oplus \) and \( Y \) preserve coalgebras.

Example

Stream: \( S_\varphi = \varphi \otimes !S_\varphi \) \quad List: \( \lambda_0 = 1 \oplus (\varphi \otimes \lambda_0) \)
### Probabilistic Call By Push Value

#### Types:

| (Value) | $A ::= UB | A_1 \oplus A_2 | 1 | A_1 \otimes A_2 | \alpha | \text{Fix} \alpha \cdot A |

Example of natural numbers: $\mathcal{N} ::= \text{Fix} \alpha \cdot 1 \oplus \alpha$

| (Computation) | $B ::= FA | A \rightarrow B$ |

#### Terms:

| (Value) | $V ::= x | \text{thunk}(M) | \text{inj}_i V | () | (V, W)$ |

| (Computation) | $M ::= \text{return}(V) | \text{force}(M)$ |

| $\lambda x^A M | \langle M \rangle V | Y M$ |

| coin | case($M, x_1 \cdot N_1, x_2 \cdot N_2$) |

| $n | \text{succ}(V) | \text{let}(x, V, M) | \text{ifz}(V, M, N)$ |
### Probabilistic Call By Push Value

**Types:**

\[ !B \]

<table>
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<tr>
<th>(Value)</th>
<th>( A ::= )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( UB )</td>
<td>( A_1 \oplus A_2 )</td>
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Example of natural numbers:

\[ \mathcal{N} ::= \text{Fix} \alpha \cdot 1 \oplus \alpha \]

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**Terms:**

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<td>( x )</td>
<td>( \text{thunk}(M) )</td>
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<th>( M ::= )</th>
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<td>( \text{return}(V) )</td>
<td>( \text{force}(M) )</td>
</tr>
<tr>
<td>( \lambda x^A M )</td>
<td>( \langle M \rangle V )</td>
</tr>
<tr>
<td>( \text{coin} )</td>
<td>( \text{case}(M, x_1 \cdot N_1, x_2 \cdot N_2) )</td>
</tr>
<tr>
<td>( n )</td>
<td>( \text{_succ}(V) )</td>
</tr>
</tbody>
</table>
### Probabilistic Call By Push Value

**Types:**

\[ !B \]

\[
A ::= \quad UB \mid A_1 \oplus A_2 \mid 1 \mid A_1 \otimes A_2 \mid \alpha \mid \text{Fix} \alpha \cdot A
\]

**Example of natural numbers:**

\[ \mathcal{N} ::= \text{Fix} \alpha \cdot 1 \oplus \alpha \]

**Terms:**

**Value**

\[ V ::= x \mid \text{thunk}(M) \mid \text{in}_{i}V \mid () \mid (V, W) \]

**Computation**

\[ M ::= \quad \text{return}(V) \mid \text{force}(M) \]

\[ \quad \lambda x^A M \mid \langle M \rangle V \mid \text{Y} M \]

\[ \mid \text{coin} \mid \text{case}(M, x_1 \cdot N_1, x_2 \cdot N_2) \]

\[ \mid n \mid \text{succ}(V) \mid \text{let}(x, V, M) \mid \text{ifz}(V, M, N) \]
## Probabilistic Call By Push Value

### Types:

- \(!B\)

### (Value)

- \(A ::= UB | A_1 \oplus A_2 | 1 | A_1 \otimes A_2 | \alpha | \text{Fix} \alpha \cdot A\)

### Example of natural numbers:

- \(\mathcal{N} ::= \text{Fix} \alpha \cdot 1 \oplus \alpha\)

### (Computation)

- \(B ::= FA | A \rightarrow B\)

### Forget:

- \(A\)

### Terms:

#### (Value)

- \(V ::= x | \text{thunk}(M) | \text{in}_i V | () | (V, W)\)

#### (Computation)

- \(M ::= \text{return}(V) | \text{force}(M)\)
  - \(\lambda x^A M | \langle M \rangle V | \text{YM}\)
  - \(\text{coin} | \text{case}(M, x_1 \cdot N_1, x_2 \cdot N_2)\)
  - \(n | \text{succ}(V) | \text{let}(x, V, M) | \text{ifz}(V, M, N)\)
Probabilistic Call By Push Value

**Types:**

$$A ::= UB | A_1 \oplus A_2 | 1 | A_1 \otimes A_2 | \alpha | \text{Fix } \alpha \cdot A$$

Example of natural numbers:

$$\mathcal{N} ::= \text{Fix } \alpha \cdot 1 \oplus \alpha$$

**Computation**

$$B ::= FA | A \rightarrow B$$

Forget: $$A$$

---

**Terms:**

$$M!$$

$$\text{der}(M)$$

$$V ::= x | \text{thunk}(M) | \text{in}_i V | () | (V, W)$$

**Computation**

$$M ::= \text{return}(V) | \text{force}(M)$$

$$| \lambda x^A M | \langle M \rangle V | YM$$

$$| \text{coin} | \text{case}(M, x_1 \cdot N_1, x_2 \cdot N_2)$$

$$| n | \text{succ}(V) | \text{let}(x, V, M) | \text{ifz}(V, M, N)$$

C. Tasson

Introduction  Discrete ($P_{coh}$)  Continuous ($C_{stab_m}$)  Conclusion 19/41
Dense coalgebra

\[ P = (P, h_P) \] such that coalgebraic points characterize morphisms:
\[ \forall Y \in \text{Pcoh} \text{ and } \forall t, t' \in \text{Pcoh}(P, Y), \]
if \[ \forall v \in \text{Pcoh}^!(1, P), \ t v = t' v, \] then \[ \forall u \in \text{Pcoh}(1, P), \ t u = t' u. \]

Already known for \( !X \) as: if \[ \forall x \in \text{Pcoh}(1, X), \ t x^! = t' x^! \] then \( t = t' \).

The Eilenberg Moore category \( \text{Pcoh}^! \)

**Value Types** are interpreted as **dense** coalgebras

**Values** are morphisms of coalgebras

The Linear category \( \text{Pcoh} \)

**Computation Types** are interpreted in \( \text{Pcoh} \)

**Computations** are linear morphisms in \( \text{Pcoh} \)
**Theorem (2016: Ehrhard - T.)**

<table>
<thead>
<tr>
<th>$\text{Pcoh}$</th>
<th>Adequacy</th>
<th>$\text{pCBPV}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[M] = [N]$</td>
<td>$\implies$</td>
<td>$M \simeq_\circ N$</td>
</tr>
</tbody>
</table>

Adequacy Lemma Proof:

- **Handle values** separately
- **Logical relations**: fixpoint of types (hidden step indexing, biorthogonality closure, fixpoints of pairs of logical relations)
- **Density**: Morphisms on positive types are characterized by their action on coalgebraic points.
Theorem (2016: Ehrhard - T.)

\[
\begin{align*}
Pcoh: & \quad [M] = [N] \\
\text{Adequacy: } & \quad \iff \text{Full Abstraction} \\
pCBPV: & \quad M \simeq N
\end{align*}
\]

Full Abstraction Proof:

1. By contradiction: \( \exists \alpha \in |\sigma|, \ [M]_\alpha \neq [N]_\alpha \)
2. Find testing context: \( T_\alpha \) such that \( [\langle T_\alpha \rangle M!] \neq [\langle T_\alpha \rangle N!] \) (context only depends on \( \alpha \))
3. Prove definability: \( T_\alpha \in pCBPV \) using coin and regularity of analytic functions and density.
4. Apply Adequacy Lemma:
\[
\text{Red}(\langle T_\alpha \rangle M! \to ()) \neq \text{Red}(\langle T_\alpha \rangle N! \to ()).
\]
A denotational semantics for probabilistic higher-order functional computation, (based on quantitative semantics of Linear Logic)

Discrete setting:

Probabilistic Coherent Spaces are fully abstract for a programming language with natural numbers as base types suitable to encode discrete probabilistic programs.

Continuous setting:

A CCC of measurable spaces and stable maps that soundly denotes a programming language with reals as base types suitable to encode continuous probabilistic programs.
Discrete Probability

Continuous Probability

- Syntax: \textbf{Real} Probabilistic PCF
- Semantics: \textbf{Cstab}_m (Cones and Stable measurable functions)
- Results: \textbf{Adequacy}
From Discrete to Continuous syntax

PPCF

**Types:** \( A, B ::= \mathcal{N} \mid A \to B \)**

**Terms:** \( M, N, L ::= \)

\( x \mid \lambda x^A.M \mid (M)N \mid \mathbf{Y}M \mid \)

\( n \mid \mathbf{succ}(M) \mid \)

\( \mathbf{ifz}(L, M, N) \mid \)

\( \mathbf{coin} \mid \mathbf{let}(x, M, N) \)

**Operational Semantics:**

\( \text{Red}(\mathbf{coin}, 0) = \text{Red}(\mathbf{coin}, 1) = \frac{1}{2} \)

If \( \vdash M : \mathcal{N} \), \( \text{Red}^\infty(M, _) \) is the discrete distribution over \( \mathbb{N} \) computed by \( M \).
From Discrete to Continuous syntax

**PPCF**

**Types:** \( A, B ::= \mathcal{N} \mid A \rightarrow B \)

**Terms:** \( M, N, L ::= \)

- \( x \)
- \( \lambda x^A.M \)
- \( (M)N \)
- \( \gamma M \)
- \( n \mid \text{succ}(M) \)
- \( \text{ifz}(L, M, N) \)
- \( \text{coin} \mid \text{let}(x, M, N) \)

**Operational Semantics:**
\[
\text{Red}(\text{coin}, 0) = \text{Red}(\text{coin}, 1) = \frac{1}{2}
\]

If \( \vdash M : \mathcal{N} \), \( \text{Red}^\infty(M, _) \) is the discrete distribution over \( \mathbb{N} \) computed by \( M \).

**Real PPCF**

**Types:** \( A, B ::= \mathcal{R} \mid A \rightarrow B \)

**Terms:** \( M, N, L ::= \)

- \( x \)
- \( \lambda x^A.M \)
- \( (M)N \)
- \( \gamma M \)
- \( r \mid f(M_1, \ldots, M_n) \)
- \( \text{ifz}(L, M, N) \)
- \( \text{sample} \mid \text{let}(x, M, N) \)

**Operational Semantics:**
\[
\text{Red}(\text{sample}, U) = \lambda_{[0,1]}(U)
\]

If \( \vdash M : \mathcal{R} \), \( \text{Red}^\infty(M, _) \) is the continuous distribution over \( \mathbb{R} \) computed by \( M \).
The probability to observe $U$ after at most one reduction step applied to $M$ is $\text{Red}(M, U)$.

$\text{Red} : \Lambda^{\Gamma \vdash A} \times \Sigma_{\Lambda^{\Gamma \vdash A}} \rightarrow \mathbb{R}^+$ is a Kernel, i.e:
- for all $M \in \Lambda^{\Gamma \vdash A}$, $\text{Red}(M, \_)$ is a measure;
- for all $U \in \Sigma_{\Lambda^{\Gamma \vdash A}}$, $\text{Red}(\_, U)$ is a measurable function.

$\text{Red}^\infty(M, U)$ is the probability to observe $U$ after any steps.
The probability to observe $U$ after at most one reduction step applied to $M$ is $\text{Red}(M, U)$

$\Lambda^{\Gamma \vdash A}$: the set of terms $M$ s.t. $\Gamma \vdash M : A$.

$\text{Red} : \Lambda^{\Gamma \vdash A} \times \Sigma^{A \vdash A} \rightarrow \mathbb{R}^+$ is a Kernel, i.e:
- for all $M \in \Lambda^{\Gamma \vdash A}$, $\text{Red}(M, \_)$ is a measure;
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$\text{Red}^\infty(M, U)$ is the probability to observe $U$ after any steps.
The probability to observe $U$ after at most one reduction step applied to $M$ is $\text{Red}(M, U)$

\[ \Lambda^{\Gamma \vdash A} : \text{the set of terms } M \text{ s.t. } \Gamma \vdash M : A. \]

\[ \Sigma_{\Lambda^{\Gamma \vdash A}} , \text{i.e. } U \text{ is measurable: } \forall n, \forall S, \{ \vec{r} \mid S\vec{r} \in U \} \text{ meas. in } \mathbb{R}^n \]

$\text{Red} : \Lambda^{\Gamma \vdash A} \times \Sigma_{\Lambda^{\Gamma \vdash A}} \rightarrow \mathbb{R}^+$ is a Kernel, i.e:

- for all $M \in \Lambda^{\Gamma \vdash A}$, $\text{Red}(M, _) \text{ is a measure}$;
- for all $U \in \Sigma_{\Lambda^{\Gamma \vdash A}}$, $\text{Red}(_, U) \text{ is a measurable function}$.

$\text{Red}^{\infty}(M, U)$ is the probability to observe $U$ after any steps.
The probability to observe $U$ after at most one reduction step applied to $M$ is $\text{Red}(M, U)$

$\Lambda^{\Gamma \vdash A}$: the set of terms $M$ s.t. $\Gamma \vdash M : A.$

$\Sigma_{\Lambda^{\Gamma \vdash A}}$, i.e. $U$ is measurable:

$\forall n, \forall S, \{\vec{r} \mid S\vec{r} \in U\}$ meas. in $\mathbb{R}^n$

$\text{Red} : \Lambda^{\Gamma \vdash A} \times \Sigma_{\Lambda^{\Gamma \vdash A}} \to \mathbb{R}^+$ is a Kernel, i.e:

- for all $M \in \Lambda^{\Gamma \vdash A}$, $\text{Red}(M, _) \text{ is a measure;}$
- for all $U \in \Sigma_{\Lambda^{\Gamma \vdash A}}$, $\text{Red}(_, U) \text{ is a measurable function.}$

Measurable sets and kernels constitute the category $\text{Kern}.$

$\text{Red}^\infty(M, U)$ is the probability to observe $U$ after any steps.
The probability to observe $U$ after at most one reduction step applied to $M$ is $\text{Red}(M, U)$.

$\Lambda^\Gamma A$: the set of terms $M$ s.t. $\Gamma \vdash M : A$.

$\Sigma^\Lambda A^\Gamma$, i.e. $U$ is measurable: $\forall n, \forall S$, $\{\vec{r} | S\vec{r} \in U\}$ meas. in $\mathbb{R}^n$.

$\text{Red}: \Lambda^\Gamma A \times \Sigma^\Lambda A^\Gamma \rightarrow \mathbb{R}^+$ is a Kernel, i.e:
- for all $M \in \Lambda^\Gamma A$, $\text{Red}(M, \_)$ is a measure;
- for all $U \in \Sigma^\Lambda A^\Gamma$, $\text{Red}(\_, U)$ is a measurable function.

Measurable sets and kernels constitute the category $\text{Kern}$.

$\text{Red}^\infty(M, U)$ is the probability to observe $U$ after any steps.

It is computed by composition and lub.
The Bernoulli distribution takes the value 1 with probability $p$ and the value 0 with probability $1 - p$.

$$\text{bernoulli } p ::= \text{let}(x, \text{sample}, x \leq p)$$

tests if sample draws a value within $[0, p]$. 

The exponential distribution is specified by its density $e^{-x}$.

$$\text{exp : } \mathcal{R} ::= \text{let}(x, \text{sample}, - \log(x))$$

by the inversion sampling method.

The standard normal distribution defined by its density $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$.

$$\text{normal ::= let}(x, \text{sample}, \text{let}(y, \text{sample}, \sqrt{-2\log(x)} \cos(2\pi y)))$$

by the Box Muller method.
Conditioning: If $U \subseteq \mathbb{R}$ measurable, then observe$(U)$ of type $\mathcal{R} \rightarrow \mathcal{R}$, taking a term $M$ and returning the renormalization of the distribution of $M$ on the only samples that satisfy $U$:

$$\text{observe}(U) = \lambda m. Y(\lambda y. \text{let}(x, m, \text{if}(x \in U, x, y)))$$

conditioning by rejection sampling.

Monte Carlo Simulation, Metropolis Hasting,...
1. Discrete Probability

2. Continuous Probability
   - Syntax: Real Probabilistic PCF
   - Semantics: Cstab\textsubscript{m} (Cones and Stable measurable functions)
   - Results: Adequacy
1981, Kozen  Memory as measurable space and programs as kernels representing the transformation of the memory. What is a measurable subset for function space?

1999, Panangaden

**Meas**, the category of measurable sets and functions
**Kern**, the category of measurable sets and kernels
They are **cartesian** but not **closed**.

2017, Heunen, Kammar, Staton, Yang  **Quasi-borel spaces**
A **CCC** based on **Meas** embedded into presheaves.
How to interpret recursive types?

2017, Keimel and Plotkin  **Kegelspitzen**
A **CCC** of dcpos equipped with a convex structure (basic operations being scott continuous) with scott continuous functions
How to restrict to measurable functions?
Semantical context

If $\vdash M : \mathcal{N}$, then $\llbracket M \rrbracket$ is a \textit{discrete} distribution over $\mathbb{N}$.

If $\vdash M : \mathcal{R}$, then $\llbracket M \rrbracket$ is a \textit{continuous} measure over $\mathbb{R}$.

- $\llbracket \mathcal{R} \rrbracket$ as $\text{Meas}(\mathbb{R})$ the set of measures over the measurable space $\mathbb{R}$.
- Fixpoint of terms.

$\text{Cstab}_m$ is a \textbf{CCC} based on Selinger’s \textit{cones} (dcpos with the order induced by addition and a convex structure).

\textbf{Objects} are cones and measurable spaces

\textbf{Morphisms} are stable and measurable functions

$\text{Pcoh}$ is a subcategory of $\text{Cstab}_m$ which is a subcategory of Kegelspitzen.
An elegant model in 3 steps

Our purpose is to be able to interpret $\mathcal{R}$ as the set of bounded measures.

1. **Complete cones** (convex dcpos with the order induced by addition) with Scott continuous functions
   However, the category is cartesian but not closed.

2. Complete cones and **Stable functions** ($\infty$-non-decreasing functions) is a CCC.
   However, not every stable function is measurable.

3. **Measurable Cones** (complete cones with measurable tests). Measurable paths pass measurable tests and Measurable functions preserve measurable paths.
   $\text{Cstab}_m$ is a CCC with measurability included!
A Cone $P$ is analogous to a real normed vector space, except that scalars are $\mathbb{R}^+$ and the norm $\|\cdot\|_P : P \to \mathbb{R}^+$ satisfies:

\[
\begin{align*}
  x + y = 0 & \Rightarrow x, y = 0, & \|x + x'\|_P & \leq \|x\|_P + \|x'\|_P, & \|\alpha x\|_P & = \alpha \|x\|_P \\
  x + y = x + y' & \Rightarrow y = y', & \|x\|_P = 0 & \Rightarrow x = 0, & \|x\|_P & \leq \|x + x'\|_P
\end{align*}
\]

The Unit Ball is the set $BP = \{x \in P \mid \|x\|_P \leq 1\}$.

Order $x \leq_P x'$ if there is a $y \in P$ such that $x' = x + y$. This unique $y$ is denoted as $y = x' - x$.

A Complete Cone is s.t. any non-decreasing $(x_n)_{n \in \mathbb{N}}$ of $BP$ has a lub and $\|\sup_{n \in \mathbb{N}} x_n\|_P = \sup_{n \in \mathbb{N}} \|x_n\|_P$.

Example of Complete Cones

- Meas($X$) with $X$ a measurable space.
- $\hat{\mathcal{X}} = \{u \in (\mathbb{R}^+)^{\mathcal{X}} \mid \exists \varepsilon > 0 \; \varepsilon u \in \mathcal{P}X\}$ if $\mathcal{X} \in \mathcal{P}_{coh}$. 

Step 2: Stable functions

The category of complete cones and Scott-continuous functions is not cartesian closed as currying fails to be non-decreasing.

A function $f : BP \to Q$ is n-non-decreasing function if:

- $n = 0$ and $f$ is non-decreasing
- $n > 0$ and $\forall u \in BP$, $\Delta f(x; u) = f(x + u) - f(x)$ is $(n - 1)$-non-decreasing in $x$.

A function is stable if it is Scott-continuous and $\infty$-non-decreasing, i.e. $n$-non-decreasing for all $n \in \mathbb{N}$.

Complete cones and stable functions constitute a CCC.

Weak Parallel Or

$\text{wpor} : [0, 1] \times [0, 1] \to [0, 1]$ given as $\text{wpor}(s, t) = s + t - st$ is Scott-continuous, but not Stable. Its currying is not Scott-continuous.
Step 3: The Measurability Problem

Type $\mathcal{R}$ is interpreted as $\llbracket \mathcal{R} \rrbracket = \text{Meas}(\mathbb{R})$,

Closed term $\vdash M : \mathcal{R}$ as a measure $\mu$ and

Term $x : \mathcal{R} \vdash N : \mathcal{R}$ as a stable $f : \text{Meas}(\mathbb{R}) \to \text{Meas}(\mathbb{R})$.

Operational semantics

$$\forall r, \text{ s.t. } M \rightarrow r, \ \text{let}(x, M, N) \rightarrow N\{r/x\}$$

By Soundness

$$\llbracket \text{let}(x, M, N) \rrbracket = \int_{\mathbb{R}} (f \circ \delta)(r) \mu(dr)$$
Step 3: The Measurability Problem

**Type** \( \mathcal{R} \) is interpreted as \( [\mathcal{R}] = \text{Meas}(\mathbb{R}) \),

**Closed term** \( \vdash M : \mathcal{R} \) as a measure \( \mu \) and

**Term** \( x : \mathcal{R} \vdash N : \mathcal{R} \) as a stable \( f : \text{Meas}(\mathbb{R}) \to \text{Meas}(\mathbb{R}) \).

Operational semantics

\[
\forall r, \text{ s.t. } M \to r, \text{ let}(x, M, N) \to N\{r/x\}
\]

By Soundness

\[
[\text{let}(x, M, N)] = \int_{\mathbb{R}} (f \circ \delta)(r) \mu(\,dr) \\
[\mathcal{N}]
\]
Step 3: The Measurability Problem

Type $\mathcal{R}$ is interpreted as $[\mathcal{R}] = \text{Meas}(\mathbb{R})$,
Closed term $\vdash M : \mathcal{R}$ as a measure $\mu$ and
Term $x : \mathcal{R} \vdash N : \mathcal{R}$ as a stable $f : \text{Meas}(\mathbb{R}) \to \text{Meas}(\mathbb{R})$.

Operational semantics

$$\forall r, \text{ s.t. } M \to r, \text{ let}(x, M, N) \to N\{r/x\}$$

By Soundness

$$[\text{let}(x, M, N)] = \int_{\mathbb{R}} (f \circ \delta)(r) \mu(dr)$$

$[N]$ Dirac measure
Step 3: The Measurability Problem

Type $\mathcal{R}$ is interpreted as $\llbracket \mathcal{R} \rrbracket = \text{Meas}(\mathbb{R})$,

Closed term $\vdash M : \mathcal{R}$ as a measure $\mu$ and

Term $x : \mathcal{R} \vdash N : \mathcal{R}$ as a stable $f : \text{Meas}(\mathbb{R}) \to \text{Meas}(\mathbb{R})$.

Operational semantics

$$\forall r, \text{ s.t. } M \rightarrow r, \ \text{let}(x, M, N) \rightarrow N\{r/x\}$$

By Soundness

$$\llbracket \text{let}(x, M, N) \rrbracket = \int_{\mathbb{R}} (f \circ \delta)(r) \mu(dr)$$

$\llbracket N \rrbracket$ Dirac measure $\llbracket M \rrbracket$
Step 3: The Measurability Problem

Type $\mathcal{R}$ is interpreted as $\llbracket\mathcal{R}\rrbracket = \text{Meas}(\mathbb{R})$, Closed term $\vdash M : \mathcal{R}$ as a measure $\mu$ and Term $x : \mathcal{R} \vdash N : \mathcal{R}$ as a stable $f : \text{Meas}(\mathbb{R}) \rightarrow \text{Meas}(\mathbb{R})$.

**Operational semantics**

$$\forall r, \text{ s.t. } M \rightarrow r, \text{ let}(x, M, N) \rightarrow N\{r/x\}$$

By **Soundness**

$$\llbracket\text{let}(x, M, N)\rrbracket = \int_{\mathbb{R}} (f \circ \delta)(r) \mu(dr)$$

Thus $f \circ \delta$ needs to be measurable.

- There are non measurable stable functions
- We need to equip every cone with a notion of measurability
Step 3: Measurability tests

Measurability tests of $\text{Meas}(\mathbb{R})$ are given by measurable sets of $\mathbb{R}$:

$$\forall U \subseteq \mathbb{R} \text{ measurable}, \ \varepsilon U \in \text{Meas}(\mathbb{R})' : \mu \mapsto \mu(U)$$

For needs of CCC, we parameterized measurable tests of a cone:

**Measurable Cone**

A cone $P$ with a collection $(M^n(P))_{n \in \mathbb{N}}$ with $M^n(P) \subseteq (P')^\mathbb{R}^n$ s.t.:

- $0 \in M^n(P)$, \quad $\ell \in M^n(P)$ and $h : \mathbb{R}^p \to \mathbb{R}^n \Rightarrow \ell \circ h \in M^p(P)$
- $\ell \in M^n(P)$ and $x \in P \Rightarrow \left\{ \begin{array}{ccc}
\mathbb{R}^n & \to & \mathbb{R}^+ \\
\vec{r} & \mapsto & \ell(\vec{r})(x)
\end{array} \right.$ measurable.
\textbf{Cstab}_m \text{ is the category of complete and measurable cones with stable and measurable functions.}

Let \( P \) and \( Q \) be measurable and complete cones:

\textbf{Measurable Test:} \( M^n(P) \subseteq (P')^\mathbb{R}^n \)

\textbf{Measurable Path:} \( \text{Path}^n(P) \subseteq P^\mathbb{R}^n \) the set of bounded \( \gamma : \mathbb{R}^n \rightarrow P \) such that \( \ell \ast \gamma : \mathbb{R}^{k+n} \rightarrow \mathbb{R}^+ \) is measurable with

\[
\ell \ast \gamma : (\vec{r}, \vec{s}) \mapsto \ell(\vec{r})(\gamma(\vec{s}))
\]

\textbf{Measurable Functions:} Stable functions \( f : P \rightarrow Q \) such that:

\[
\forall n \in \mathbb{N}, \ \forall \gamma \in \text{Path}_1^n(P), \quad f \circ \gamma \in \text{Path}^n(Q)
\]

If \( X \) is a measurable space, then \( \text{Meas}(X) \) is equipped with:

\[
M^n(X) = \{ \varepsilon_U : \mathbb{R}^n \rightarrow \text{Meas}(X)' \text{ s.t. } \varepsilon_U(\vec{r})(\mu) = \mu(U), \ \text{U meas.} \}
\]

\( \text{Path}_1^n(P) \) is the set of stochastic kernels from \( \mathbb{R}^n \) to \( X \).
Discrete Probability

Continuous Probability
- Syntax: \textbf{Real Probabilistic PCF}
- Semantics: \textbf{Cstab}_m (Cones and Stable measurable functions)
- Results: \textbf{Adequacy}
The category $\textbf{Cstab}_m$ is a CCC and a model of Real PPCF.

Interpretation of some terms:

- $\llbracket r \rrbracket = \delta_r$
- $\llbracket \text{sample} \rrbracket = \lambda_{[0,1]}$
- $\llbracket \text{let}(x, M, N) \rrbracket(U) = \int_{\mathbb{R}} [N](\delta_r)(U) [M](dr)$

Soundness

$$\llbracket M \rrbracket^{\Gamma \vdash A} = \int_{\Lambda^{\Gamma \vdash A}} [t]^{\Gamma \vdash A} \text{Red}(M, dt)$$

Adequacy

$$\llbracket M \rrbracket^{\vdash \mathcal{R}}(U) = \text{Red}^\infty(M, U)$$
Examples: Distributions

The Bernoulli distribution takes the value 1 with probability $p$ and the value 0 with probability $1 - p$.

\[
\text{bernoulli } p ::= \text{let}(x, \text{sample}, x \leq p)
\]

\[
\left[\text{bernoulli } p\right]^{\mathcal{R}} = p\delta_1 + (1 - p)\delta_0
\]

The exponential distribution is specified by its density $e^{-x}$.

\[
\text{exp} : \mathcal{R} ::= \text{let}(x, \text{sample}, -\log(x))
\]

\[
\left[\text{exp}\right]^{\mathcal{R}}(U) = \int_{\mathbb{R}^+} \chi_U(s)e^{-s}\lambda(ds)
\]

The standard normal distribution defined by its density $\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$.

\[
\text{normal} ::= \\
\text{let}(x, \text{sample}, \text{let}(y, \text{sample}, \sqrt{-2\log(x)} \cos(2\pi y)))
\]

\[
\left[\text{normal}\right]^{\mathcal{R}}(U) = \frac{1}{\sqrt{2\pi}} \int_U e^{-\frac{x^2}{2}} \lambda(dx)
\]
Conditioning: If $U \subseteq \mathbb{R}$ measurable, then $\text{observe}(U)$ of type $\mathcal{R} \rightarrow \mathcal{R}$, taking a term $M$ and returning the renormalization of the distribution of $M$ on the only samples that satisfy $U$:

$$\text{observe}(U) = \lambda m. \mathcal{Y}(\lambda y. \text{let}(x, m, \text{if}(x \in U, x, y)))$$

conditioning by rejection sampling.

Whenever $M$ represents a probability distribution, this equation gives the conditional probability:

$$\mathbb{E}[\text{observe}(U)M](V) = \frac{\mathbb{E}[M](V \cap U)}{\mathbb{E}[M](U)}$$
Pcoh and Cstab\textsubscript{m} models of probabilistic programming

- For countable data types, Pcoh is fully abstract.
- For real data types, Cstab\textsubscript{m} is a sound model that encodes probability measures used in probabilistic programming.

Further directions:
- A model of LL ?
- A model of pCBPV ?
- Full abstraction ?