A convenient differential category

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Abstract

In this paper, we show that the category of Mackey-complete, separated, topological convex bornological vector spaces and bounded linear maps is a differential category. Such spaces were introduced by Frölicher and Kriegl, where they were called *convenient vector spaces*.

While much of the structure necessary to demonstrate this observation is already contained in Frölicher and Kriegl's book, we here give a new interpretation of the category of convenient vector spaces as a model of the *differential linear logic* of Ehrhard and Regnier.

Rather than base our proof on the abstract categorical structure presented by Frölicher and Kriegl, we prefer to focus on the bornological structure of convenient vector spaces. We believe bornological structures will ultimately yield a wide variety of models of differential logics.

1 Introduction

The question of how to differentiate functions into and out of function spaces has a significant history. The importance of such structures is fundamental in the classical theory of *variational calculus*, for example [15]. It is also a notoriously difficult question. This can be seen by considering the category of smooth manifolds and smooth functions beween them. While products evidently exist in this category, there is no way to make the set of functions between two manifolds into a manifold. This is most concisely expressed by saying that the category of smooth manifolds is not cartesian closed.

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Category theory provides a particularly appropriate framework for the analysis of function spaces through the notion of cartesian closed category. In the categorical approach to modelling logics, one typically starts with a logic presented as a sequent calculus. One then arranges equivalence classes of derivations into a category. If the equivalence relation is chosen wisely, the resulting category will be a free category with structure. For example, the conjunction-implication fragment of intuitionistic logic yields the free cartesian closed category. The times-implication fragment of intuitionistic linear logic yields the free symmetric monoidal closed category. In both these cases, the implication connective is modelled as a function space, i.e. the right adjoint to product. So any attempt to model the *differential linear logic* (discussed below) of Ehrhard and Regnier should be a category whose morphisms are smooth maps for some notion of smoothness. Then to model logical implication, the category must also be closed. The possibility of differential linear logic was suggested by semantic examples, discovered by Ehrhard in [9, 10]. A sequent calculus, a typed λ-calculus and extension of the usual theory of proof nets was presented in [13, 14]. Differential linear logic is an extension of linear logic [17] to include an additional inference rule to capture differentiation as a formal derivation. Subsequent analysis of differential linear logic and applications in concurrency can be found in [11, 12].

A significant question raised by the work of Ehrhard and Regnier is to write down the appropriate notion of categorical model of differential linear logic. This was undertaken by Blute, Cockett and Seely in [3]. There a notion of *differential category* is defined. As usual, the syntax of the logic naturally forms an example, and several other examples are given. In a followup [4], the coKleisli category of a differential category is considered directly and the notion of *cartesian differential category* is introduced. In [5], differentiation is considered via a universal property similar to Kähler differentiation in commutative algebra [19, 27]. Kähler categories are introduced as symmetric monoidal categories with an algebra modality and a universal derivation. It is then demonstrated that every codifferential category with a minor structural property is Kähler. In the paper [7], the authors introduce the notion of a *differential λ-category* which provides appropriate axiomatic structure to model the differential λ-calculus, and the resource calculus, a non-lazy axiomatisation of Boudols λ-calculus with multiplicities. The notion of differential λ-category is an extension of cartesian differential categories, taking into account closed structure. In the papers [9, 10], the author introduces what are now seen as the first two models of differential linear logic. They are the categories of *finiteness spaces* and *Köthe spaces.*

In this paper, we focus on the category of convenient vector spaces and bounded linear maps, and demonstrate that it provides an ideal framework for modelling differential linear logic. The idea behind this structure is inspired largely by a theorem of Boman [6], discussed in [16, 23, 1, 30].

Theorem 1.1 (Boman, 1967)

- *Let E be a Banach space and $c\colon \mathbb{R} \to E$. Then c is smooth with respect to the normed structure of E if and only if $\ell \circ c\colon \mathbb{R} \to \mathbb{R}$ is smooth in the usual sense for all linear, continuous maps $\ell\colon E \to \mathbb{R}$.*
- *Let E and F be Banach spaces and $f\colon E \to F$ be a function. Then f is smooth if and only if it takes smooth curves on E to smooth curves in F.*

This remarkable theorem suggests the possibility of defining a notion of smooth map between topological (or bornological) spaces without any notion of norm. One needs only to define a reasonable notion of smooth curve into a space and then define general smoothness to be the preservation of smooth curves.

The two monographs [16, 23] as well as numerous papers indicates how successful this idea is. In the more abstract approach to convenient vector spaces, one considers the monoid $\mathcal{M}$ of smooth maps from the reals to the reals. Roughly, an arbitrary set X is equipped with an $\mathcal{M}$-*structure* if equipped with a class of functions $f\colon \mathbb{R} \to X$, called a set of *smooth curves*. (This set must satisfy a closure condition.) We call such a structure an $\mathcal{M}$-*space*. Then a *smooth functional* on X is a function $g\colon X \to \mathbb{R}$ such that its composite with every smooth curve is in $\mathcal{M}$. This approach to defining smoothness also appears in [24]. A *smooth function* between $\mathcal{M}$-spaces is a function taking smooth curves to smooth curves. Equivalently, precomposition takes smooth functionals to smooth functionals. If one further requires that the set X be a (real) vector space, and that the vector space operations are compatible with the $\mathcal{M}$-structure, we have the notion of a *smooth vector space*. Finally, adding a separation axiom and a bornological notion of completeness called *Mackey completeness*, one has a *convenient vector space*.

While this abstract approach to describing convenient vector spaces has its advantages, there is another equivalent description, which uses ideas from functional analysis. There is an adjunction between convex bornological vector spaces and locally convex vector spaces (definitions and constructions explained below). Convenient vector spaces can be seen as the fixed points of the composite of the two functors. This allows one to use classical tools from functional analysis in their consideration. It also suggests the appropriate notion of linear map. It turns out that the notion of bounded linear maps, when a convenient vector space is viewed as bornological, is equivalent to the notion of continuous linear map, when viewed as locally convex.

The category of convenient vector spaces and bounded (or continuous) linear maps has a number of remarkable properties. It is symmetric monoidal closed, complete and cocomplete. But most significantly, it is equipped with a comonad, for which the resulting coKleisli category is the category of smooth maps, in the sense defined above. This is of course reminiscent of the structure of linear logic, which provides a decomposition of intuitionistic logical implication as a linear implication composed with a comonad. The classic (and motivating) example is that the category of coherence spaces and stable maps can be obtained as the coKleisli category for a monad on the category of coherence spaces and linear maps, see [18].

After describing the category of convenient spaces, we demonstrate that it is a model of intuitionistic linear logic, and that the coKleisli category corresponding to the model of the exponential modality is the category of smooth maps. We construct a differential operator on smooth maps, and show that it is a model of the differential inference rule of differential linear logic. The concrete approach taken here makes it much more convenient[1] to describe differentiation.

[1]We promise that this is the only time we make this pun.

We note that essentially all of the structure we describe here can be found scattered in the literature [16, 20, 22, 23, 33], but we think that presenting everything from the bornological perspective sheds new light on both the categorical and logical structures. For most of the results, we give at least sketches of proofs so that the paper is as self-contained as possible. We also give detailed references, so that the reader who wishes can explore further. We hope this paper demonstrates the remarkable nature of this subject initiated by Frölicher, which predates not only differential linear logic, but linear logic, itself.

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2 Differential linear logic and differential categories

We assume that the reader is familiar with the sequent calculus of linear logic [17]. *Differential linear logic* [13, 14] is an extension of the traditional sequent calculus to include an inference rule representing differentiation. The rule is as follows:

$$\frac{\,!\,A \vdash B}{A, \,!\,A \vdash B}$$

The semantic interpretation is that, if an arrow $f\colon\ !\,\mathbb{R}^n \to \mathbb{R}^m$ is a smooth map from $\mathbb{R}^n$ to $\mathbb{R}^m$, then its differential would be of the form $df\colon\ !\,\mathbb{R}^n \to \mathbb{R}^n \multimap \mathbb{R}^m$, where $\mathbb{R}^n \multimap \mathbb{R}^m$ denotes the linear maps. So the differential of f is a smooth function that takes a vector v in $\mathbb{R}^n$ and calculates the directional derivative of f in the direction of v. One treats this inference rule as any other sequent calculus inference rule, and verifies cut-elimination for this extension of linear logic.

A stronger sequent calculus is given by assuming the duals of some of the usual structural rules of linear logic:

$$\frac{\Gamma \vdash A}{\Gamma \vdash \,!\,A}\ \textit{codereliction}$$

$$\frac{\Gamma \vdash \,!\,A \qquad \Delta \vdash \,!\,A}{\Gamma, \Delta \vdash \,!\,A}\ \textit{cocontraction}$$

$$\frac{}{\vdash \,!\,A}\ \textit{coweakening}$$

The *differential proof nets* of [14] provide a graph-theoretic syntax for specifying proofs in the logic, and the differential λ-calculus of [13] is an extension of the usual simply-typed λ-calculus to include this rule. (One should think of this λ-calculus as living in the cartesian-closed coKleisli category of the comonad $\,!\,$.)

The appropriate categorical structure to model differential linear logic is the *differential category*, which is introduced in [3]. One begins with the usual notion of categorical model of linear logic, also known as a *Seely model*. So we assume a symmetric, monoidal closed category.[2] We also assume the existence of a comonad, denoted $(\,!\,, \rho, \epsilon)$, satisfying a standard set of properties. See [28] for an excellent overview of the topic. One of the fundamental properties, necessary for modeling the contraction rule, is that each object of the form $!\,A$ is canonically equipped with a symmetric coalgebra structure. Thus the functor $!$ induces what [3] refers to as a *coalgebra modality*.

To model the remaining differential structure, it suffices to have a *deriving transformation*, i.e. a natural transformation of the form:

$$d_A \colon A \otimes\, !\,A \longrightarrow\, !\,A$$

satisfying evident equations. These equations correspond to the standard rules of calculus:

- The derivative of a constant is 0.
- Leibniz rule.
- The derivative of a linear function is a constant.
- Chain rule.

Each of these must be expressed coalgebraically. We will give the equations for a slightly different theory below.

As in the syntax, there is an equivalent presentation. In the case where one has biproducts, then the functor $!$ actually determines a *bialgebra modality*. So, in addition to the usual coalgebra structure:

$$\Delta \colon\, !\,A \to\, !\,A \otimes\, !\,A, \qquad e \colon\, !\,A \to I,$$

we also have compatible algebra structure:

$$\nabla \colon\, !\,A \otimes\, !\,A \to\, !\,A, \qquad \nu \colon I \to\, !\,A.$$

The algebra structure gives us the interpretations of the logical rules coweakening and cocontraction. The only remaining structure is the map corresponding to codereliction. Categorically, codereliction is expressed as a natural transformation:

$$\mathsf{coder}_A \colon A \longrightarrow\, !\,A,$$

satisfying certain equations which are analogues of the above listed equations:

[dC.1] $\mathsf{coder}; e = 0,$

[dC.2] $\mathsf{coder}; \Delta = \mathsf{coder} \otimes \nu + \nu \otimes \mathsf{coder},$

[dC.3] $\mathsf{coder}; \epsilon = 1,$

[dC.4] $(\mathsf{coder} \otimes 1); \nabla; \rho = (\mathsf{coder} \otimes \Delta); ((\nabla; \mathsf{coder}) \otimes \rho)); \nabla.$

[2] The $*$-autonomous structure of classical linear logic will not concern us here.

3 Convenient vector spaces

In this section, we present the category of convenient spaces. They can be seen either as topological or bornological vector spaces, with the two structures satisfying a compatibility.

Let us first recall the more traditional notion of *locally convex space*.

Definition 3.1 A *locally convex space* is a topological vector space such that 0 has a neighborhood basis of convex sets. A morphism between locally convex spaces is simply a linear, continuous map. We thus obtain a category denoted LCS.

A locally convex space can equivalently be defined as having a topology determined by a family of seminorms. See [33] for details.

3.1 Bornology

For the significance of this structure and an analysis of convergence properties, see [20]. A set is bornological if, roughly speaking, it is equipped with a notion of boundedness.

Definition 3.2 A set X is *bornological* if equipped with a *bornology*, i.e. a set of subsets $\mathcal{B}_X$ called bounded such that

- All singletons are in $\mathcal{B}_X$.
- $\mathcal{B}_X$ is downward closed with respect to inclusion.
- $\mathcal{B}_X$ is closed under finite unions.

A map between bornological spaces is *bornological* if it takes bounded sets to bounded sets. The resulting category will be denoted Born.

Theorem 3.3 *The category* Born *is cartesian closed.*

Proof. (**Sketch**) The product bornology is defined to be the coarsest bornology such that the projections are bornological. So a subset of $X \times Y$ is bounded if and only if its two projections are bornological.

One then defines $X \Rightarrow Y$ as the set of bornological functions. A subset $B \subseteq X \Rightarrow Y$ is bounded if and only if $B(A)$ is bounded in Y, for all A bounded in X. $\square$

As this bornology will arise in a number of different contexts, we will denote $X \Rightarrow Y$ by $\mathsf{Born}(X, Y)$. We note that the same construction works for products of arbitrary cardinality.

Definition 3.4 A *convex bornological vector space* is a vector space E equipped with a bornology such that

1. $\mathcal{B}$ is closed under the convex hull operation.
2. If $B \in \mathcal{B}$, then $-B \in \mathcal{B}$ and $2B \in \mathcal{B}$.

The last condition ensures that addition and scalar multiplication are bornological maps, when the reals are given the usual bornology. A map of convex bornological vector spaces is a linear map such that the direct image of a bounded set is bounded. We thus get a category that we denote CBS.

3.2 Topology vs. bornology

As described in [16, 22, 20], the topology and bornology of a convenient vector space are related by an adjunction, which we now describe.

Theorem 3.5 *There exists a pair of adjoint functors* $\beta\colon \mathsf{LCS} \to \mathsf{CBS}$ *and* $\gamma\colon \mathsf{CBS} \to \mathsf{LCS}$*, with* β *right adjoint to* γ*.*

Proof. Let E be a locally convex space. Say that $B \subseteq E$ is bounded if it is absorbed by every neighborhood of 0, that is to say if U is a neighborhood of 0, then there exists a positive real number λ such that $B \subseteq \lambda U$. This is called the *von Neumann bornology* associated to E. We will denote the corresponding convex bornological space by βE.

On the other hand, let E be a convex bornological space. Define a topology on E by saying that its associated topology is the finest locally convex topology compatible with the original bornology. More concretely, one says that the bornivorous disks form a neighborhood basis at 0. A *disk* is a subset A which is both convex and satisfies that $\lambda A \subseteq A$, for all λ with $|\lambda| \leq 1$. A disk A is said to be *bornivorous* when for every bounded subsets B of E, there is $\lambda \neq 0$ such that $\lambda B \subseteq A$. $\square$

Definition 3.6 A convex bornological space E is *topological* if $E = \beta\gamma E$. A locally convex space E is *bornological* if $E = \gamma\beta E$.

Let tCBS denote the full subcategory of topological convex bornological vector spaces and bornological linear maps, and let bLCS denote the category of bornological locally convex spaces and continuous linear maps. We note immediately:

Corollary 3.7 *The categories* tCBS *and* bLCS *are isomorphic.*

There is a third equivalent category. By using the von Neumann bornology associated to a locally convex space, one can discuss bornological maps between locally convex spaces. Thus we can consider the category of locally convex spaces and bornological linear maps. Denote this category by LCSb.

Lemma 3.8 (See Theorem 2.4.3.(i),(iv) of [16]) *If* V *and* W *are bornological locally convex spaces, then the notions of bornological linear map and continuous linear map coincide. The category* LCSb *is equivalent to* bLCS *under the functorial operation of bornologification.*

Note that this is only an equivalence, since many different locally convex spaces can yield the same bornological locally convex space.

We also have the following extremely useful characterization.

Lemma 3.9 (See Lemma 2.1.23, [16]) *Let* E *be a convex bornological space.* E *is topological if and only if:*

A subset is bounded if and only if it is sent to a bounded subset of $\mathbb{R}$ by any bornological linear functional.

Thus tCBS's are precisely those for which the bornology is maximal with respect to the family of bornological seminorms or to the family of bornological linear functionals. A consequence of this lemma is that to specify a tCBS, it suffices to specify its dual space, that is the set of bornological linear functionals. We will repeatedly take advantage of this in what follows.

This can be made more precise by considering the category of *dualized vector spaces*. These will be very familiar to linear logicians in that they are similar to the *Chu space* and double gluing constructions [2, 21, 31].

A dualized vector space is a (discrete) vector space V with a linear subspace V' of the dual space V^*. There is an evident notion of map between dualized vector spaces, giving a category DVS. Given (V, V') a dualized vector space, one can associate to it a bornological vector space by saying that $U \subseteq V$ is bounded if and only if it is *scalarly bounded*, i.e. $\ell(U)$ is bounded in the reals, for all ℓ in V'. It is straightforward to check that this in fact determines a functor $\sigma\colon \mathsf{DVS} \to \mathsf{CBS}$. The previous lemma states that the topological convex bornological spaces are precisely the image of this functor.

Theorem 3.10 (See Theorem 2.4.3.(i) of [16]) *The construction of the previous paragraph provides an adjunction between the categories* DVS *and* CBS*.*

3.3 Separation and completion

The tCBS's that we are interested in have the desirable further properties of *separation* and *completion*. We begin with the easiest of the two notions.

Definition 3.11 A bornological space is *separated* if E' separates points, that is if $x \neq 0 \in E$ then there is $l \in E'$ such that $l(x) \neq 0$.

One can verify a number of equivalent definitions as done in [16], page 53. For example, E is separated if and only if the singleton $\{0\}$ is the only bounded linear subspace.

We now introduce the notion of a completion with respect to a bornology. However, notice that bornological completeness is a different and weaker notion than topological completeness, so we give details.

Definition 3.12 Let E be a bornological space. A net $(x_\gamma)_{\gamma \in \Gamma}$ is *Mackey-Cauchy* if there exists a bounded subset B and a net $(\mu_{\gamma,\gamma'})_{\gamma,\gamma' \in \Gamma,\Gamma'}$ of real numbers converging to 0 such that

$$x_\gamma - x_{\gamma'} \in \mu_{\gamma,\gamma'} B.$$

Contrary to what generally happens in locally convex spaces, here the convergence of Cauchy nets is equivalent to the convergence of Cauchy sequences, and both are equivalent to a property of an associated Banach space. Given any hausdorff, locally convex space E and any closed bounded disk B, one can associate a normed subspace $E_B \subseteq E$ by

$$E_B = \bigcup_{n \in \mathsf{N}} nB.$$

This is a normed space under the Minkowski functional $p(x) = \inf\{\lambda > 0 | x \in \lambda B\}$.

Lemma 3.13 (See Lemma 2.2 of [23]) *Let E be a bornological space. Then the following are equivalent:*

- *Every Mackey-Cauchy net converges.*
- *Every Mackey-Cauchy sequence converges.*
- *For every closed bounded disk B, the space E_B is a Banach space.*

Definition 3.14

- A bornological space is *Mackey-complete* if either of the above conditions hold.
- A *convenient vector space* is a Mackey-complete, separated, topological convex bornological vector space.
- The category of convenient vector spaces and bornological linear maps is denoted Con.

We note that Kriegl and Michor in [23] denote the concept of Mackey completenesss as c^∞*-completeness* and define a convenient vector space as a c^∞-complete locally convex space. If one takes the bornological maps between these as morphisms, then the result is an equivalent category. We note that the category of convenient vector spaces is closed under several crucial operations. The following is easy to check:

Theorem 3.15 (See Theorem 2.6.5, [16], and Theorem 2.15 of [23])

- *Assuming that E_j is convenient for all $j \in J$, then $\prod_{j\in J} E_j$ is convenient with respect to the product bornology, with J an arbitrary indexing set.*
- *If E is convenient, then so is $\mathsf{Born}(X, E)$ where X is an arbitrary bornological set.*

The remainder of the section is devoted to an extension theorem that allows one to derive a convenient space from any tCBS.

Let us first denote by stCBS the full subcategory of separated topological convex bornological spaces. It is reflective in tCBS. More precisely, if E is any object in tCBS, then define $\omega_1(E)$ to be the image of E under the map

$$E \longrightarrow \prod_{E'} \mathbb{R} \text{ defined by } v \mapsto (f(v))_f$$

One can verify that the kernel of this map is the closure of $\{0\}$ (with respect to the topology induced by Mackey-Cauchy convergence).

Lemma 3.16 (See Lemma 2.5.6 of [16]) *Modding out by this kernel induces an adjoint to the inclusion functor*

$$\omega_1 \colon \mathsf{stCBS} \longrightarrow \mathsf{tCBS}$$

Let E be an object in stCBS. We define its completion $\omega_2(E)$ by again considering the map

$$E \longrightarrow \prod_{E'} \mathbb{R} \text{ defined by } v \mapsto (f(v))_f.$$

Since E is separated, this map is now an embedding. $\omega_2(E)$ is the closure with respect to Mackey-convergence of the image of this embedding. We get the following result:

Lemma 3.17 (See Lemma 2.6.5 of [16]) *The functor* $\omega_2\colon \mathsf{stCBS} \to \mathsf{Con}$ *is left adjoint to the inclusion functor.*

Proof. (**Sketch**) Let $g\colon E \to F$ be a bornological linear map from an stCBS E to a convenient vector space F. Define the bornological linear map $\bar{g}$ of convenient spaces by

$$\begin{array}{rccc} \bar{g}\colon & \prod_{E'} \mathbb{R} & \to & \prod_{F'} \mathbb{R} \\ & (\lambda_f)_{f \in E'} & \mapsto & (\lambda_{f \circ g})_{f \in F'}. \end{array}$$

Check that $\bar{g}(\omega_2(E)) \subseteq \omega_2(F) = F$. Then the extension $\omega_2(g)$ of g to $\omega_2(E)$ is the restriction of $\bar{g}$. □

Theorem 3.18 *Combining the previous two adjunctions, we obtain a functor*

$$\omega\colon \mathsf{tCBS} \to \mathsf{Con}$$

which is left adjoint to the inclusion.

4 Monoidal structure

We wish now to show that the category tCBS is symmetric monoidal closed. It is the category of *Mackey-complete, separated* tCBS that we will be primarily interested in, but it is of interest that the symmetric monoidal closed structure exists already at this level. Let E and F be tCBS. Using Lemma 3.9, we define a bornology on its algebraic tensor product by specifying its dual space. Define

$$(E \otimes F)' = \{h\colon E \otimes F \to \mathbb{R} \mid \hat{h}\colon E \times F \to \mathbb{R} \text{ is bornological}\}$$

where $\hat{h}$ refers to the associated bilinear map, and to be bornological means with respect to the product bornology. A subset of $E \times F$ is bounded if and only if its two projections are bounded. Evidently, the tensor unit will be the base field $I = \mathbb{R}$.

Now, let $L(E, F)$ denote the space of bornological linear maps. $L(E, F)$ obtains a bornology as a subset of $\mathsf{Born}(E, F)$. We now wish to prove that it is a tCBS. Recall that the dual space is defined by:

$$L(E, F)' = \{h\colon L(E, F) \to \mathbb{R} \mid \text{ If } U \text{ is bounded, then } h(U) \text{ is bounded}\}.$$

Lemma 4.1 *A subset $U \subseteq L(E,F)$ is bounded if and only if it is scalarly bounded with respect to the above dual space.*

Proof. $\Rightarrow$ Trivial.

$\Leftarrow$ Assume $U \subseteq L(E,F)$ is scalarly bounded. Suppose for contradiction that U is not bounded. So, there is some bounded subset A of E such that $U(A)$ is not bounded in F. Since F is a tCBS, there is a bornological linear function $l : F \to \mathbb{R}$ such that $l(U(A))$ is not bounded in $\mathbb{R}$.

Now, we can choose a pair of sequences $(f_n)_{n\in\mathbb{N}} \subseteq U$ and $(a_n)_{n\in\mathbb{N}} \subseteq A$ such that $(l(f_n(a_n)))_{n\in\mathbb{N}}$ is not bounded in $\mathbb{R}$. Without loss of generality, we can assume that $l(f_n(a_n)) \geq 4^n$ for all $n \in \mathbb{N}$. Set $u = (\frac{1}{2^n})_{n\in\mathbb{N}}$ and note that, because it is summable, it defines a bornological linear functional:

$$\begin{array}{rccl} \langle u, \cdot\,\rangle \colon & \ell^\infty & \longrightarrow & \mathbb{R} \\ & v & \mapsto & \displaystyle\sum_{n=0}^{\infty} u(n)v(n) \end{array}$$

on the set ℓ^∞ of bounded sequences on $\mathbb{R}$. Consider

$$D = \{((l \circ f_p)(a_n))_{n\in\mathbb{N}}\,;\, p \in \mathbb{N}\}$$

which is an unbounded subset of ℓ^∞. Indeed, a set of sequences is bounded if and only if the union of all elements is bounded in the reals. Recall that $f_p \in L(E,F)$ for all $p \in \mathbb{N}$ and $(a_n)_{n\in\mathbb{N}}$ takes its values in the bounded subset A of E. Further, for all natural number p, we have

$$\langle u, ((l \circ f_p)(a_n))_{n\in\mathbb{N}}\rangle \geq u(p)\, l(f_p(a_p)) \geq p,$$

hence $\langle u, D\rangle$ is not bounded in $\mathbb{R}$.

Now, define the linear function

$$\begin{array}{rccl} \Psi \colon & L(E,F) & \longrightarrow & \ell^\infty \\ & f & \mapsto & (l(f(a_n)))_{n\in\mathbb{N}}. \end{array}$$

As l is bornological, it is easy to check that Ψ takes a bounded set V in $L(E,F)$ to a bounded set in ℓ^∞.

Consider the composite

$$h \colon L(E,F) \xrightarrow{\Psi} \ell^\infty \xrightarrow{\langle u, \cdot\rangle} \mathbb{R}.$$

It takes bounded sets in $L(E,F)$ to bounded sets of reals and hence is in the dual space $L(E,F)'$. As U is assumed to be scalarly bounded, $h(U)$ has to be bounded. However, it contains the set of functions f_p for $p \in \mathbb{N}$ whose image under the map h is $\langle u, D\rangle$ and so is not bounded. Thus we have a contradiction. $\square$

It follows from the cartesian closedness of Born that there is an isomorphism

$$L(E_1; E_2, F) \cong L(E_1, L(E_2, F))$$

where $L(E_1; E_2, F)$ is the space of multilinear maps. Now, the algebraic tensor product, equipped with the above bornology, classifies multilinear maps. Therefore, the above structure makes tCBS a symmetric monoidal closed category.

We finally lifts the symmetric monoidal closed structure of tCBS to Con by defining the tensor product of convenient vector spaces as the Mackey completion of the tensor product in tCBS. The result then follows from two standard observations (Section 3.8 of [16]):

- If F is complete, then so is $L(E, F)$.
- If E and F are separated, then so is $E \otimes F$.

Hence, we have proved that:

Theorem 4.2 *The category* Con *is symmetric monoidal closed.*

5 Smooth maps and differentiation

5.1 Smooth curves

The notion of a smooth curve into a locally convex space E is straightforward. One simply has a curve $c\colon \mathbb{R} \to E$ and defines its derivative by:

$$c'(t) = \lim_{s \to 0} \frac{c(t+s) - c(t)}{s}.$$

Note that this limit is simply the limit in the underlying topological space of E. Then, we define a curve to be *smooth* if all iterated derivatives exist. We denote the set of smooth curves in E by $\mathcal{C}_E$.

In order to endow $\mathcal{C}_E$ with a convenient structure, we introduce the notion of *different quotients* which is the key idea behind the theory of finite difference methods, described in [26]. Let $\mathbb{R}^{<i>} \subseteq \mathbb{R}^{i+1}$ consist of those $i+1$-tuples with no two elements equal. It inherits its bornological structure from $\mathbb{R}^{i+1}$. Given any function $f\colon \mathbb{R} \to E$ with E a vector space, we recursively define maps

$$\delta^i f\colon \mathbb{R}^{<i>} \to E$$

by saying $\delta^0 f = f$, and then the prescription:

$$\delta^i f(t_0, t_1, \ldots, t_i) = \frac{i}{t_0 - t_i} \left[\, \delta^{i-1} f(t_0, t_1, \ldots, t_{i-1}) - \delta^{i-1} f(t_1, \ldots, t_i) \,\right]$$

For example,

$$\delta^1 f(t_0, t_1) = \frac{1}{t_0 - t_1} \left[\, f(t_0) - f(t_1) \,\right].$$

Notice that the extension of this map along the missing diagonal would be the derivative of f. There are similar interpretations of the higher-order formulas. So these difference formulas provide approximations to derivatives.

Lemma 5.1 (See 1.3.22 of [16]) *Let $c\colon \mathbb{R} \to E$ be a function. Then c is a smooth curve if and only if for all natural numbers i, $\delta^i c$ is a bornological map.*

Proof. (**Sketch**) If c is smooth, we will show that there is a smooth extension of $\delta^i c\colon \mathbb{R}^{<i>} \to E$ to $\bar{\delta}^i c\colon \mathbb{R}^{i+1} \to E$. This is done by induction. The 0'th case is by definition, and assuming that $\bar{\delta}^j c$ exists, define

$$\bar{\delta}^{j+1}c\colon \mathbb{R}^{j+2} \to E \qquad \text{by} \qquad \bar{\delta}^{j+1}c(t,t',\vec{x}) = \int_0^1 \partial_t \bar{\delta}^j c(t+s(t'-t),\vec{x})ds,$$

where ∂_t denotes the partial derivative with respect to t.

Since $\bar{\delta}^i c$ is smooth, its first derivative is well defined. Therefore, its first difference quotient is a Mackey-Cauchy net at 0 from which we deduce that $\delta^{i+1}c = \delta^1(\bar{\delta}^i c)|_{\mathbb{R}^{<i+1>}}$ is bornological.

On the other hand, suppose that for all natural numbers i, $\delta^i c$ is a bornological map. We prove by induction that for all i, the i'th derivatives $c^{(i)}$ is well defined and its difference quotients are bornological. The 0'th case is straightforward. Assume the difference quotients of $c^{(j)}$ are bornological. In particular, $\delta^2 c^{(j)}$ is bornological, hence $c^{(j+1)} = (c^{(j)})'$ exists as the limit of the Mackey-Cauchy net of $\delta^1 c^{(j)}$. By calculation, one sees that $\delta^i c^{(j+1)}(\vec{x})$ is a linear combination:

$$\delta^i c^{(j+1)}(\vec{x}) = \frac{1}{i+1}\sum_{k=0}^{i} \delta^{i+1}c^{(j)}(\vec{x},t_i)$$

for arbitrary t_i. Since, for all natural number i, $\delta^{i+1}c^{(j)}$ is bornological, we get that $\delta^i c^{(j+1)}$ is bornological too.

$\square$

Theorem 5.2 *Suppose E is convenient. Then:*

> *If $c\colon \mathbb{R} \to E$ is a curve such that $\ell \circ c$ is smooth for every bornological linear map $\ell\colon F \to \mathbb{R}$, then c is itself smooth.*

Proof. By the preceding lemma, c is smooth if and only if $\delta^i c$ is bornological, for all i. Since E is a tCBS, this is equivalent to saying that for all $\ell \in E'$, we have $\ell \circ \delta^i c$ is bornological and since ℓ is linear, $\ell \circ \delta^i c = \delta^i(\ell \circ c)$. We conclude by using again the preceding lemma. $\square$

If X and Y are bornological sets, recall that $\mathsf{Born}(X, Y)$ denote the bornological space of bornological functions from X to Y with bornology as already described.

By Lemma 5.1, the above described difference quotients define an infinite family of maps:

$$\delta^i \colon \mathcal{C}_E \to \mathsf{Born}(\mathbb{R}^{<i>}, E)$$

Definition 5.3 Say that $U \subseteq \mathcal{C}_E$ is *bounded* if and only if its image $\delta^i(U)$ is bounded for every natural number i.

Theorem 5.4 *This structure makes $\mathcal{C}_E$ a convenient vector space.*

Proof. Let us first prove that $\mathcal{C}_E$ is a tCBS. A subset U of $\mathcal{C}_E$ is bounded if and only if its image under every difference quotient δ^i is bounded in $\mathsf{Born}(\mathbb{R}^{<i>}, E)$. Since E is a tCBS, this is equivalent to saying that the image of U under every difference quotient is scalarly bounded in $\mathsf{Born}(\mathbb{R}^{<i>}, E)$, by Lemma 3.9. We conclude by noting that any linear functional commutes with the difference quotient.

Separation follows from the separation of $\prod_{i=0}^{\infty} \mathsf{Born}(\mathbb{R}^{<i>}, E)$. Let us prove that $\mathcal{C}_E$ is complete. Let (c_n) be a Mackey-Cauchy sequence of smooth curves into E. By definition of the bornology of $\mathcal{C}_E$, we infer that $(\delta^i c_n)$ is a Mackey-Cauchy sequence of $\mathsf{Born}(\mathbb{R}^{<i>}, E)$ and hence converges. For $i = 0$, we get that (c_n) converges and we denote its limit by c_∞. We then prove by induction that $\delta^i c_\infty$ is the limit of $(\delta^i c_n)$ and that it is bornological. We conclude that c_∞ is smooth by Lemma 5.1. $\square$

5.2 Smooth maps

We are then left with the question of how to define smoothness of a function between two locally convex spaces. A motivation for the following definition is Boman's theorem, as discussed in the Introduction.

Definition 5.5 A function $f \colon E \to F$ is *smooth* if $f(\mathcal{C}_E) \subseteq \mathcal{C}_F$. Let $\mathcal{C}^\infty(E, F)$ denote the set of smooth functions from E to F.

We note the obvious fact that $\mathcal{C}_E = \mathcal{C}^\infty(\mathbb{R}, E)$, as seen by considering $id \colon \mathbb{R} \to \mathbb{R}$ as a smooth curve. We have the following characterization for when a linear map is smooth.

Lemma 5.6 *A bornological linear map between convenient spaces is smooth.*

Proof. Let $f : E \to F$ be a linear bornological map and $c \colon \mathbb{R} \to E$ a smooth curve. By Lemma 5.1, $\delta^i c$ is bornological for all i. Since f is bornological, so is $f \circ \delta^i c$. Now, f is linear and thus commutes with the difference quotients $f \circ \delta^i c = \delta^i (f \circ c)$. Again by Lemma 5.1, $f \circ c$ is smooth. $\square$

Let $\mathcal{C}^\infty$ denote the category of convenient vector spaces and smooth maps. One of the crucial results of [16] and [23] is that $\mathcal{C}^\infty$ is a cartesian closed category. In fact, this category is the coKleisli category of a model of intuitionistic linear logic, from which the above follows. But this is hardly an enlightening proof! We first give a convenient vector space structure on $\mathcal{C}^\infty(E,F)$.

Now, let E and F be convenient vector spaces. If $c\colon \mathbb{R} \to E$ is a smooth curve, we get a map $c^*\colon \mathcal{C}^\infty(E,F) \to \mathcal{C}_F$ by precomposing.

Definition 5.7 *Say that $U \subseteq \mathcal{C}^\infty(E,F)$ is bounded if and only if its image $c^*(U)$ is bounded for every smooth map $c : E \to \mathbb{R}$.*

The space $\mathcal{C}^\infty(E,F)$ has a natural interpretation as a projective limit:

Lemma 5.8 (See [23], p. 30) *The space $\mathcal{C}^\infty(E,F)$ is the projective limit of spaces $\mathcal{C}_F$, one for each $c \in \mathcal{C}_E$. Equivalently, it consists of the Mackey-closed linear subspace of*

$$\mathcal{C}^\infty(E,F) \subseteq \prod_{c\in\mathcal{C}_E} \mathcal{C}_F$$

consisting of all $\{f_c\}_{c\in\mathcal{C}_E}$ such that $f_{c\circ g} = f_c \circ g$ for every $g \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$.

Proof. Let us prove that the following linear function is a bijection:

$$\begin{array}{rcl}\Phi : \mathcal{C}^\infty(E,F) & \to & \prod_{c\in\mathcal{C}_E} \mathcal{C}_F \\ f & \mapsto & \{c^*(f)\,;\, c \in \mathcal{C}_E\}.\end{array}$$

Let $f \neq 0$ be a smooth function from E to F. There is $x \in E$ such that $f(x) \neq 0$. Let const_x be the constant curve equal to x. Then $\Phi(f)_{\mathrm{const}_x} \neq 0$, hence Φ is one to one.

Now, consider $\{f_c\,;\, c \in \mathcal{C}_E\}$ such that $f_{c\circ g} = f_c \circ g$ for all $g \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$. Set $f : x \mapsto f_{\mathrm{const}_x}(0)$, then for all $c \in \mathcal{C}_E$, $c^*(f) = f_c$. We have shown that Φ is onto.

To conclude, the bornological structure of $\mathcal{C}^\infty(E,F)$ has been defined to be the bornological structure induced by $\prod_{c\in\mathcal{C}_E} \mathcal{C}_F$. $\square$

We have shown that $\mathcal{C}^\infty(E,F)$ is equivalent to a Mackey-closed subspace of a convenient vector space. So we have:

Corollary 5.9 *The above structure makes $\mathcal{C}^\infty(E,F)$ a convenient vector space.*

As another consequence of the above Lemma, we get a characterization of smooth curves in $\mathcal{C}^\infty(E,F)$:

Corollary 5.10 *A curve $f : \mathbb{R} \to \mathcal{C}^\infty(E,F)$ is smooth if and only if $c^*(f) : \mathbb{R} \to F$ is smooth for all smooth curves c.*

We are now able to describe the cartesian closed structure.

Theorem 5.11 (See Theorem 3.12 of [23]) *The category $\mathcal{C}^\infty$ is cartesian closed.*

Proof. We need to show that a map $f\colon E_1 \times E_2 \to F$ is smooth if and only if its transpose $\hat{f}\colon E_1 \to \mathcal{C}^\infty(E_2, F)$ is smooth.

Recall that, by definition, $\hat{f}$ is smooth if and only if $\hat{f} \circ c_1 \colon \mathbb{R} \to \mathcal{C}^\infty(E_2, F)$ is smooth for all $c_1 \colon \mathbb{R} \to E_1$ smooth. In turn, by the characterization of smooth curves into $\mathcal{C}^\infty(E_2, F)$ described above, this holds if and only if

$$(c_2^* \circ \hat{f} \circ c_1)\colon \mathbb{R} \to \mathcal{C}^\infty(\mathbb{R}, F)$$

is smooth, for all smooth curves $c_2 \colon \mathbb{R} \to E_2$.

But this map $(c_2^* \circ \hat{f} \circ c_1)$ is equal to $f \circ (c_1 \times c_2)$. Thus the question is reduced to the one-dimensional version of Boman's theorem, proved for example on page 29 of [23]. □

As usual, having a cartesian closed category gives us an enormous amount of structure to work with, as will be seen in what follows.

5.3 Differentiating smooth maps

If these functions are genuinely to be thought of as smooth, then we should be able to differentiate them. That is the content of the following:

Theorem 5.12 (See [23], p. 33) *Let E and F be convenient vector spaces. The differentiation operator*

$$d\colon \mathcal{C}^\infty(E, F) \to \mathcal{C}^\infty(E, L(E, F))$$

defined as

$$df(x)(v) = \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}$$

is linear and bounded. In particular, this limit exists and is linear in the variable v.

Proof. We have an evident map of the form:

$$\begin{array}{rcl} \mathcal{C}^\infty(E, F) \times E \times E & \to & \mathcal{C}^\infty(\mathbb{R}, F) \\ (f, x, v) & \mapsto & [c\colon s \mapsto f(x + sv)]. \end{array}$$

If we denote the corresponding curve in F by c, then

$$df(x)(v) = c'(0).$$

Moreover, if f, x and v are smooth curves respectively into $\mathcal{C}^\infty(E, F)$, and the two occurrences of E, then the map of two variables $c : (s, t) \mapsto f(t)(x(t) + sv(t))$ is smooth and its derivative $t \mapsto df(t)(x(t))(v(t))$ with respect to the first variable s is also smooth in t. Therefore, the above defined function is smooth. By cartesian closedness, this gives our desired smooth function

$$d : \mathcal{C}^\infty(E, F) \to \mathcal{C}^\infty(E, \mathcal{C}^\infty(E, F)).$$

It remains to prove that for all $f \in \mathcal{C}^\infty(E, F)$ and $x \in E$, $df(x)$ is linear. It is the standard calculus proof of the linearity of differentiation. To conclude recall that, thanks to Lemma 5.6, every linear smooth function is bornological, hence $df(x) \in L(E, F)$. □

6 Exponential structure

The most difficult aspect of linear logic [17] from a semantic point of view is the intepretation of the exponential fragment. As noted previously, one must have a comonad with a great deal of extra structure. Moreover, in Theorem 5.1.1 of [16], it is shown that in this setting, the comonad precisely demonstrates the relationship between linear maps and smooth maps which was envisioned by the differential λ-calculus.

We begin by noting that if E is a convenient vector space and $x \in E$, there is a canonical morphism of the form $\delta_x \colon \mathcal{C}^\infty(E, \mathbb{R}) \to \mathbb{R}$, defined by $\delta_x(f) = f(x)$. This is of course the *Dirac delta distribution*.

Lemma 6.1 *The Dirac distribution $\delta \colon E \to \mathcal{C}^\infty(E, \mathbb{R})'$ is smooth.*

Proof. First, it is easy to see that δ_x is linear for every $x \in E$. Let us check it is bornological. Let U be a bounded subset of $\mathcal{C}^\infty(E, \mathbb{R})$, that is $c^*(U)$ is bounded in $\mathbb{R}$ for every smooth curve $c \in \mathcal{C}_E$. In particular, $\delta_x = \mathrm{const}_x^*(U)$ is bounded, and we are done.

Now, let c and f be smooth curves into E and $\mathcal{C}^\infty(E, \mathbb{R})$ respectively. The map $t \mapsto \delta_{c(t)} f(t) = f(t)(c(t))$ is smooth. Thus, by cartesian closedness, δ is smooth. $\Box$

Definition 6.2 The *exponential modality* $!\,E$ is the Mackey-closure of the set $\delta(E)$ in $\mathcal{C}^\infty(E, \mathbb{R})'$.

We will demonstrate that this determines a comonad on Con.

Theorem 6.3 *We have the following canonical adjunction:*

$$\mathcal{C}^\infty(E, F) \cong L(\,!\,E, F)$$

Proof. We establish the bijection, leaving the straightforward calculation of naturality to the reader. So let $\varphi \colon\, !\,E \to F$ be a bornological linear map. Define a smooth map from E to F by $\hat{\varphi}(e) = \varphi(\delta_e)$.

Conversely, suppose $f \colon E \to F$ is a smooth map. Define a linear map $\tilde{f} \colon\, !\,E \to F$ by $\tilde{f}(\delta_e) = f(e)$. Let us show that $\tilde{f}$ is bornological. Let $U \subseteq \delta(E)$ bounded as a subspace of $!\,E$, that is as a subset of $\mathcal{C}^\infty(E, \mathbb{R})'$. The image $\tilde{f}(U)$ is equal to $U(\{f\})$ which is bounded as the image of a singleton set. By Theorem 3.18, we can then extend f to the Mackey completion of $\delta(E)$ through the functor ω. We get a function $\bar{f} :\, !\,E \to F$.

It is clear that this determines a bijection and hence an adjunction. $\Box$

We now describe the structure that comes out of this adjunction:

- The counit is the linear map $\epsilon \colon\, !\,E \to E$, defined by $\eta(\delta_x) = x$, and then extending linearly and applying the adjunction ω.
- The unit is a smooth map $\iota \colon E \to\, !\,E$, defined by $\iota(x) = \delta_x$.

- The associated comonad has comultiplication $\rho\colon\ !E \to\ !\,!E$ given by $\rho(\delta_x) = \delta_{\delta_x}$.

Lemma 6.4 *The fundamental isomorphism is satisfied:*

$$!(E \times F) \cong\ !E \otimes\ !F$$

Proof. The trick, as usual, is to verify that $!(E \times F)$ satisfies the universal property of the tensor product.

First we note that there is a bilinear map $m\colon\ !E \times\ !F \to\ !(E \times F)$. Consider the smooth map $\iota_{E\times F}\colon E \times F \to\ !(E \times F)$. By cartesian closedness, we get a smooth map $E \to \mathcal{C}^\infty(F,\ !(E\times F))$, which extends to a linear map $!E \to \mathcal{C}^\infty(F,\ !(E\times F)) \cong L(!F,\ !(E\times F))$. The transpose is the desired bilinear map. It satisfies $m \circ (\iota_E \times \iota_F) = \iota_{E\times F}$.

We check that m satisfies the appropriate universality. Assume $f\colon\ !E \times\ !F \to G$ is a bornological bilinear map. Let us show that f is smooth. Let $(c_1, c_2)\colon \mathbb{R} \to\ !E \times\ !F$ be a smooth curve. We want to show that $t \mapsto f(c_1(t), c_2(t))$ is a smooth curve into G. Thanks to Theorem 5.2, it is sufficient to show that for every linear bornological functional l over G, the real function $l \circ f \circ (c_1, c_2)\colon \mathbb{R} \to \mathbb{R}$ is smooth.

Now, notice that, from simple calculations, we get

$$\delta^1(l \circ f \circ (c_1, c_2)) = l \circ f \circ (\delta^1(c_1), c_2) + l \circ f \circ (c_1, \delta^1(c_2))$$

and hence $\delta^1(l \circ f \circ (c_1, c_2))$ is bornological. More generally, every difference quotient of $l \circ f \circ (c_1, c_2)$ is bornological. From Lemma 5.1, we get that it is smooth. Then, in turn, $f \circ (\iota \times \iota)$ is smooth. By Theorem 6.3, f lifts to a linear map $\bar{f}\colon\ !(E \times F) \to G$. By definition, $\bar{f} \circ \delta_{(x_1,x_2)} = f(x_1, x_2)$. Hence f factors through m and $\bar{f}$.

Therefore, the universal property is satisfied by $!(E \times F)$ which is hence isomorphic to $!E \otimes\ !F$. $\square$

We conclude:

Theorem 6.5 *The category* Con *is a model of intuitionistic linear logic.*

It is straightforward to see that this category has finite biproducts, as required in a differential category. The bialgebra structure is as follows:

- $\Delta\colon\ !A \to\ !A \otimes\ !A$ is $\Delta(\delta_x) = \delta_x \otimes \delta_x$, and then extending linearly and using the functor ω to extend to the completion.
- $e\colon\ !A \to I$ is $e(\delta_x) = 1$.
- $\nabla\colon\ !A \otimes\ !A \to\ !A$ is $\nabla(\delta_x \otimes \delta_y) = \delta_{x+y}$.
- $\nu\colon I \to\ !A$ is $\nu(1) = \delta_0$.

Thus by results of [13, 14, 3], it remains to establish a *codereliction map* of the form:

$$coder\colon E \to \mathbf{!}\,E$$

The map ι defined above almost meets the requirement of codereliction but fails to be linear. Fortunately, its differential at 0 will supply us with the desired map.

Theorem 6.6 *The category* Con *is a differential category, with codereliction given by*

$$coder(v) = d\iota(v \otimes \delta_0) = \lim_{t\to 0} \frac{\delta_{tv} - \delta_0}{t}$$

Proof. It remains only to verify the equations of the definition. These all follow from the fact that the derivative in this category really is just a derivative in the usual sense. We verify two of the equations:

Consider [dC.2], which is the Leibniz rule. Using the fact that limits in the tensor product are calculated componentwise, we calculate as follows: The lefthand side gives

$$v \mapsto \Delta\left(\lim_{t\to 0} \frac{\delta_{tv} - \delta_0}{t}\right) = \lim_{t\to 0} \frac{\Delta(\delta_{tv}) - \Delta(\delta_0)}{t} = \lim_{t\to 0} \frac{\delta_{tv}\otimes\delta_{tv} - \delta_0\otimes\delta_0}{t}.$$

Thanks to a standard calculation, this is

$$\lim_{t\to 0}\left(\frac{\delta_{tv} - \delta_0}{t} \otimes \delta_{tv} + \delta_0 \otimes \frac{\delta_{tv} - \delta_0}{t}\right)$$

which is equal to the righthand side by exchange of limit with sum and tensor.

Now consider [dC.3]. Let $v \in E$. We must show $(coder;\epsilon)v = v$. This comes from the calculation:

$$v \mapsto \lim_{t\to 0} \frac{\delta_{tv} - \delta_0}{t} \mapsto \epsilon\left(\lim_{t\to 0} \frac{\delta_{tv} - \delta_0}{t}\right) = \lim_{t\to 0} \frac{\epsilon(\delta_{tv} - \delta_0)}{t} = \lim_{t\to 0} \frac{tv - 0}{t} = v.$$

$\square$

7 Conclusion

Fundamental to understanding the structure of convenient vector spaces is the duality between bornology and topology in the definition of tCBS. Another place where there is such duality is the notion of a finiteness space, introduced in [10]. But there, the duality is between bornology and the *linear topology* of Lefschetz [25]. The advantage of the present setting is that the topology takes place in the more familiar world of locally convex spaces. However, it remains an interesting question to work out a similar structure in the Lefschetz setting. This program was initiated in the thesis of the third author [32].

Evidently, a next fundamental question is the logical/syntactic structure of integration. One would like an *integral linear logic*, which would again treat integration as an inference rule. It should not be a surprise at this point that convenient vector spaces are extremely well-behaved with respect to integration. The category Con will likely provide an excellent indicator of the appropriate structure.

One can also ask about other classes of functions beside the smooth ones. Chapter 3 of [23] is devoted to the calculus of holomorphic and real-analytic functions on convenient vector spaces. It is an important question as to whether there is an analogous comonad to be found, inducing the category of holomorphic maps as its coKleisli category. Then one can investigate whether the corresponding logic is in any way changed.

Of course, once one has a good notion of structured vector spaces, it is always a good question to ask whether one can build manifolds from such spaces. Manifolds based on convenient vector spaces is the subject of the latter half of [23], and it seems an excellent idea to view these structures from the logical perspective developed here.

Finally, we find categories of bornological spaces to also be worthy of further study from the categorical/logical perspective. We note in particular the *analytic cyclic cohomology* of Meyer [29]. This is a cohomology theory based on convex, bornological vector spaces, and provides an approach to analyzing the *entire cyclic cohomology* of Connes [8].

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