Totality, towards completeness

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The barycentric boolean calculus

**Definition**

For all \( n \in \mathbb{N} \), we define inductively the terms of \( \Lambda_{\mathcal{B}n} \) by

\[
R, S \ ::= \sum_{i=1}^{m} a_i \ s_i \quad \text{with} \quad \sum_{i=1}^{m} a_i = 1, \text{and}
\]

\[
s, s_i \ ::= \lambda x_1 \ldots x_n \cdot x_i \mid T \mid F \mid \text{if } s \text{ then } S \text{ else } R.
\]

Notice, that every barycentric boolean term is of type \( \mathcal{B}^n \Rightarrow \mathcal{B} \)

\[
\sum a_i (\lambda \bar{x} . s_i) \simeq \lambda \bar{x} . \sum a_i s_i
\]

\[
T \simeq \lambda \bar{x} . T, \quad F \simeq \lambda \bar{x} . F,
\]

if \((\lambda \bar{x} s)\) then \((\lambda \bar{x} S)\) else \((\lambda \bar{x} R)\) \(\simeq\) \(\lambda \bar{x} \) if \(s\) then \(S\) else \(R\)
Semantics of $\Lambda_{\mathcal{B}n}$

Every $S \in \Lambda_{\mathcal{B}n}$, is interpreted by a pair

$$(\llbracket S \rrbracket_t, \llbracket S \rrbracket_f) \in \mathbb{k}[X_1, \ldots, X_{2n}] \times \mathbb{k}[X_1, \ldots, X_{2n}]$$

inductively defined by

$$\llbracket \sum a_i s_i \rrbracket = \sum a_i \llbracket s_i \rrbracket,$$
$$\llbracket T \rrbracket = (1, 0), \quad \llbracket F \rrbracket = (0, 1),$$
$$\llbracket \lambda x_1 \ldots x_n \cdot x_i \rrbracket = (X_{2i-1}, X_{2i}),$$
$$\llbracket \text{if } P \text{ then } Q \text{ else } R \rrbracket = \left( \llbracket P \rrbracket_t \llbracket Q \rrbracket_t + \llbracket P \rrbracket_f \llbracket R \rrbracket_t, \right.$$
$$\left. \llbracket P \rrbracket_t \llbracket Q \rrbracket_f + \llbracket P \rrbracket_f \llbracket R \rrbracket_f \right).$$
Reduction

The reduction

\[
\text{if } (a \text{ T} + b \text{ F}) \text{ then R else S } \rightarrow a \text{ R } + b \text{ S}
\]

Proposition (Soundeness)
Let \( S \in \Lambda_B \). If \( S \rightarrow T \), then \([S] = [T]\).

Theorem (Computational adequacy)
Let \( S \in \Lambda_B \). If \([S] = (a, b)\), then \( S \rightarrow a \text{ T } + b \text{ F} \).
Boolean polynomials and completeness

**Definition**

Boolean polynomials are the pairs of polynomials \((P, Q)\) such that there is \(S \in \Lambda_{B_n}\) such that \([S] = (P, Q)\).

Boolean polynomials can be algebraically characterized.

**Proposition**

Let \(S \in \Lambda_{B_n}\) and \((x_i) \in \kappa^{2n}\).

\[
(\forall i, \ x_{2i-1} + x_{2i} = 1) \Rightarrow [S]_t (x_i) + [S]_f (x_i) = 1.
\]

Reciprocally,

**Theorem (Completeness)**

For every \(P, Q \in \kappa [X_1, \ldots, X_{2n}]\) such that \(P + Q - 1\) vanishes on the common zeros of \(X_{2i-1} + X_{2i} - 1\), there is \(S \in \Lambda_{B_n}\) with \([S] = (P, Q)\).
Proof of completeness (1)

Some notations:

\[ \neg S = \text{if } S \text{ then } F \text{ else } T, \]
\[ S^+ = \text{if } S \text{ then } T \text{ else } T, \]
\[ S^- = \text{if } S \text{ then } F \text{ else } F, \]
\[ \Pi_i = \lambda x_1, \ldots, x_n \cdot x_i. \]

Lemma (Basic pairs)

The pairs of polynomials \((X_{2i}, X_{2i-1}), (X_{2i-1} + X_{2i}, 0), (1 - X_{2i-1}, X_{2i-1}) \) and \((1 - X_{2i}, X_{2i}) \) are booleans.

\[ (X_{2-1}, X_{2i}) = \llbracket \Pi_i \rrbracket, \]
\[ (X_{2i}, X_{2i-1}) = X_{2i-1} \cdot (1, 0) + X_{2i} \cdot (0, 1) \]
\[ = \llbracket \text{if } \Pi_i \text{ then } T \text{ else } F \rrbracket \]
\[ = \llbracket \neg \Pi_i \rrbracket. \]
Proof of completeness (2)

**Lemma (Affine pairs)**

For every polynomial $P \in \mathbb{k}[X_1, \ldots, X_n]$, the pair of polynomials $(1 - P, P)$ is boolean.

Let $d$ be the degree of $P$.

If $d = 0$, then $(1 - P, P) = (1 - a, a) = \llbracket (1 - a) T + a F \rrbracket$.

If $d > 0$ and $X^\mu = \prod X_i^{\mu_i}$ with $\mu_1 \geq 1$, then

$$(1 - X^\mu, X^\mu) = (1 - X_1) \cdot (1, 0) + X_1 \cdot \left(1 - X_1^{\mu_1 - 1} \prod_{i \neq 1} X_i^{\mu_i}, X_1^{\mu_1 - 1} \prod_{i \neq 1} X_i^{\mu_i}\right)$$

$$= \llbracket \text{if } \Xi_1 \text{ then } T \text{ else } \Xi_{d-1} \rrbracket = \llbracket \Xi_\mu \rrbracket.$$

If $P = \sum a_\mu \prod X_i^{\mu_i}$, then

$$(1 - P, P) = (1 - \sum a_\mu) (1, 0) + (\sum a_\mu) (1 - X^\mu, X^\mu)$$

$$= \llbracket (1 - \sum a_\mu) T + (\sum a_\mu) \Xi_\mu \rrbracket.$$
Proof of completeness (3)

**Lemma (Spanning polynomials)**

Let \( P \in \mathbb{k}[X_1, \ldots, X_{2n}] \). If \( P \) vanishes on the zeros common to \( X_{2i-1} + X_{2i} - 1 \), then there are \( Q_i \in \mathbb{k}[X_1, \ldots, X_{2n}] \) such that \( P = \sum_{i=1}^{n} Q_i(X_{2i-1} + X_{2i} - 1) \).

Change of variables

\[
\begin{align*}
Y_i &= X_{2i-1} + X_{2i} - 1 \\
Y_{i+n} &= X_{2i}
\end{align*}
\] \( \Rightarrow P_Y(0, \ldots, 0, y_{n+1}, \ldots, y_{2n}) = 0. \)

Since \( \mathbb{k}[Y_2, \ldots, Y_{2n}][Y_1] \) is an euclidean ring, there are \( Q \in \mathbb{k}[Y_1, \ldots, Y_{2n}], R \in \mathbb{k}[Y, \ldots, Y_{2n}] \) such that

\[ P_Y = Q_1 Y_1 + \]

\( \forall (y_i) \in \mathbb{k}^n, R_n(y_{n+1}, \ldots, y_{2n}) = 0, \) hence if \( \mathbb{k} \) is infinite

\[ P_Y = \sum_{i=1}^{n} Q_i Y_i. \]
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**Lemma (Spanning polynomials)**

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**Change of variables**

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\end{align*}
\]

\( \Rightarrow P_Y(0, \ldots, 0, y_{n+1}, \ldots, y_{2n}) = 0. \)

Since \( \mathbb{k}[Y_2, \ldots, Y_{2n}][Y_1] \) is an euclidean ring, there are \( Q_1 \in \mathbb{k}[Y_1, \ldots, Y_{2n}], R_1 \in \mathbb{k}[Y_2, \ldots, Y_{2n}] \) such that

\[
P_Y = Q_1 Y_1 + R_1
\]

\( \forall (y_i) \in \mathbb{k}^n, R_n(y_{n+1}, \ldots, y_{2n}) = 0, \) hence if \( \mathbb{k} \) is infinite

\[
P_Y = \sum_{i=1}^{n} Q_i Y_i.
\]
Lemma (Spanning polynomials)

Let $P \in \mathbb{k}[X_1, \ldots, X_{2n}]$. If $P$ vanishes on the zeros common to $X_{2i-1} + X_{2i} - 1$, then there are $Q_i \in \mathbb{k}[X_1, \ldots, X_{2n}]$ such that $P = \sum_{i=1}^{n} Q_i(X_{2i-1} + X_{2i} - 1)$.

Change of variables

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\begin{align*}
Y_i &= X_{2i-1} + X_{2i} - 1 \\
Y_{i+n} &= X_{2i}
\end{align*}
\] \quad \Rightarrow \quad P_Y(0, \ldots, 0, y_{n+1}, \ldots, y_{2n}) = 0.

Since $\mathbb{k}[Y_2, \ldots, Y_{2n}][Y_1]$ is an euclidean ring, there are $Q_i \in \mathbb{k}[Y_1, \ldots, Y_{2n}]$, $R_2 \in \mathbb{k}[Y_{i+1}, \ldots, Y_{2n}]$ such that

$P_Y = Q_1 Y_1 + Q_2 Y_2 + R_2$

$\forall (y_i) \in \mathbb{k}^n$, $R_n(y_{n+1}, \ldots, y_{2n}) = 0$, hence if $\mathbb{k}$ is infinite

$P_Y = \sum_{i=1}^{n} Q_i Y_i$. 
Proof of completeness (3)

**Lemma (Spanning polynomials)**

Let \( P \in \mathbb{K}[X_1, \ldots, X_{2n}] \). If \( P \) vanishes on the zeros common to \( X_{2i-1} + X_{2i} - 1 \), then there are \( Q_i \in \mathbb{K}[X_1, \ldots, X_{2n}] \) such that \( P = \sum_{i=1}^{n} Q_i(X_{2i-1} + X_{2i} - 1) \).

Change of variables

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Since \( \mathbb{K}[Y_2, \ldots, Y_{2n}][Y_1] \) is an euclidean ring, there are \( Q_i \in \mathbb{K}[Y_1, \ldots, Y_{2n}], R_n \in \mathbb{K}[Y_{n+1}, \ldots, Y_{2n}] \) such that

\[
P_Y = Q_1 Y_1 + Q_2 Y_2 + \cdots + Q_n Y_n + R_n
\]

\( \forall (y_i) \in \mathbb{K}^n, R_n(y_{n+1}, \ldots, y_{2n}) = 0, \) hence if \( \mathbb{K} \) is infinite

\[
P_Y = \sum_{i=1}^{n} Q_i Y_i.
\]
Proof of completeness (the end)

**Theorem (Completeness)**

For every \( P, Q \in \mathbb{k}[X_1, \ldots, X_{2n}] \) such that \( P + Q - 1 \) vanishes on the common zeros of \( X_{2i-1} + X_{2i} - 1 \), there is \( t \in \Lambda_B \) with \( \llbracket t \rrbracket = (P, Q) \).
Proof of completeness (the end)

**Theorem (Completeness)**

For every $P, Q \in \mathbb{k} [X_1, \ldots, X_{2n}]$ such that $P + Q - 1$ vanishes on the common zeros of $X_{2i-1} + X_{2i} - 1$, there is $t \in \Lambda_B$ with $\llbracket t \rrbracket = (P, Q)$.

**Spanning:**

$$P + Q - 1 = \sum_{i=1}^{n} Q_i (X_{2i-1} + X_{2i} - 1).$$

$$\begin{align*}
(P, Q) &= \sum_{i=1}^{n} [(1 - Q_i) \cdot (1, 0) + Q_i \cdot (X_{2i-1} + X_{2i}, 0)] \\
&\quad + (1 - Q, Q) - n(1, 0).
\end{align*}$$

**Basic pairs:** $\llbracket \Pi_i^+ \rrbracket = (X_{2i-1} + X_{2i}, 0)$,

**Affine pairs:** $\llbracket Q \rrbracket = (1 - Q, Q)$.

$$\begin{align*}
(P, Q) &= \llbracket \sum_{i=1}^{n} (\text{if } Q_i \text{ then } \top \text{ else } \Pi_i^+) \rrbracket \\
&\quad + Q - n \top,
\end{align*}$$
Where does it come from?

**Thesis subject**

To define a linear space model of linear logic.
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- Interest? Lots of intuitions of linear logic come from linear algebra.
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To define a linear space model of linear logic.

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Thesis subject
To define a linear space model of linear logic.

- Interest? Lots of intuitions of linear logic come from linear algebra.
- Difficulty? Because of exponential, infinite dimension appears, hence problem of reflexivity solved with topology.
- Other attempts?
  - [Blute96] Linear Laüchli semantics,
  - [Girard99] Coherent Banach spaces,
  - [Ehrhard02] On Köthe sequence spaces and LL,
  - [Ehrhard05] Finiteness spaces.
Where does it come from?

**Thesis subject**

To define a linear space model of linear logic.

- **Interest?** Lots of intuitions of linear logic come from linear algebra.
- **Difficulty?** Because of exponential, infinite dimension appears, hence problem of reflexivity solved with topology.
- **My attempt:** Linearly topologized spaces (Lefschetz),
  - a generalization of finiteness spaces,
  - a natural notion of totality.

The boolean polynomials corresponds to the totality space associated to \(!B \rightarrow B\).
### Linear Logic

\[
A, B := \\
0 \quad | \quad A \oplus B \quad | \quad A \& B \\
1 \quad | \quad A \otimes B \quad | \quad A \nRightarrow B \\
A \perp \quad | \quad !A \quad | \quad ?A.
\]

### Reflexivity

\[
A \perp \perp = A.
\]

### Linear implication

\[
A \nrightarrow B = A \perp \nRightarrow B.
\]

### Intuitionistic implication

\[
A \Rightarrow B = !A \nrightarrow B.
\]

### Finiteness space

A is interpreted by a linear space \( \mathbb{k} \langle A \rangle \).

\( \pi \vdash A \) is interpreted by a vector \([\pi] \in \mathbb{k} \langle A \rangle \).

### Totality space

A is interpreted by an affine subspace \( T(A) \) of \( \mathbb{k} \langle A \rangle \).

\( \pi \vdash A \) is interpreted by a vector \([\pi] \in T(A) \).
Relational Finiteness Spaces

Let \( \mathcal{I} \) be countable, for each \( \mathcal{F} \subseteq \mathcal{P}(\mathcal{I}) \), let us denote

\[
\mathcal{F}^\perp = \{ u' \subseteq \mathcal{I} | \forall u \in \mathcal{F}, u \cap u' \text{ finite} \}.
\]

**Definition**

A relational finiteness space is a pair \( A = (|A|, \mathcal{F}(A)) \) where the web \( |A| \) is countable and the collection \( \mathcal{F}(A) \) of finitary subsets satisfies \( (\mathcal{F}(A))^\perp = \mathcal{F}(A) \).

**Example**

**Booleans.**

\( \mathcal{B} = (\mathbb{B}, \mathcal{P}(\mathbb{B})) \) with

\[
\begin{align*}
\mathbb{B} &= \{ \text{T, F} \} \\
\mathcal{P}(\mathbb{B}) &= \{ \emptyset, \{ \text{T} \}, \{ \text{F} \}, \{ \text{T, F} \} \}.
\end{align*}
\]

**Integers.**

\( \mathcal{N} = (\mathbb{N}, \mathcal{P}_{\text{fin}}(\mathbb{N})) \) and \( \mathcal{N}^\perp = (\mathbb{N}, \mathcal{P}(\mathbb{N})) \).
Linear Finiteness Spaces

For every $x \in \mathbb{k}^{|A|}$, the support of $x$ is $|x| = \{a \in |A||x_a \neq 0\}$.

**Definition**

The *linear finiteness space* associated to $A = (|A|, \mathcal{F}(A))$ is

$$\mathbb{k}\langle A \rangle = \{x \in \mathbb{k}^{|A|} ||x| \in \mathcal{F}(A)\}.$$ 

The *linearized topology* is generated by the neighborhoods of 0

$$V_J = \{x \in \mathbb{k}\langle A \rangle ||x| \cap J = \emptyset\}, \quad \text{with} \quad J \in \mathcal{F}(A)^\perp.$$ 

**Example**

*Booleans.* $\mathbb{k}\langle B \rangle = \mathbb{k}^2$.

*Integers.* $\mathbb{k}\langle N \rangle = \mathbb{k}(\omega)$ and $\mathbb{k}\langle N^\perp \rangle = \mathbb{k}^\omega$. 
Finiteness Spaces

A Linear Logic Model

\[
\begin{align*}
A & \perp & \rightsquigarrow & \text{k}\langle A \rangle' \\
0 & & \rightsquigarrow & \{0\} \\
A \& B & & \rightsquigarrow & \text{k}\langle A \rangle \oplus \text{k}\langle B \rangle \\
A \oplus B & & \rightsquigarrow & \text{k}\langle A \rangle \oplus \text{k}\langle B \rangle \\
1 & & \rightsquigarrow & \text{k} \\
A \rightarrow B & & \rightsquigarrow & \text{L}_c(A, B) \\
A \otimes B & & \rightsquigarrow & \text{k}\langle A \rangle \otimes \text{k}\langle B \rangle \\
!A & & \rightsquigarrow & \text{k}\langle !A \rangle
\end{align*}
\]
Finiteness Spaces

A Linear Logic Model

\[ A \perp \rightsimeq k\langle A \rangle' \quad \Rightarrow \text{Reflexivity} \]

\[ 0 \rightsimeq \{0\} \]
\[ A \& B \quad \{ A \oplus B \} \rightsimeq k\langle A \rangle \oplus k\langle B \rangle \]

\[ 1 \rightsimeq k \]
\[ A \multimap B \rightsimeq \mathcal{L}_c(A, B) \]
\[ A \otimes B \rightsimeq k\langle A \rangle \otimes k\langle B \rangle \]

\[ !A \rightsimeq k\langle !A \rangle \quad \Rightarrow \text{Infinite dimension} \]
Exponentials

The relational finiteness space associated with $!A$ is

$$|!A| = \mathcal{M}_{\text{fin}}(|A|),$$

$$\mathcal{F}(!A) = \left\{ M \subseteq \mathcal{M}_{\text{fin}}(|A|) \mid \bigcup_{\mu \in M} |\mu| \in \mathcal{F}(A) \right\}.$$
Exponentials

The relational finiteness space associated with $!A$ is

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Example

$$|\mathcal{B}| = \{T, F\} \quad \mathcal{F}(\mathcal{B}) = \mathcal{P}(\{T, F\})$$
Exponentials

The relational finiteness space associated with $!A$ is

$$
|!A| = \mathcal{M}_{\text{fin}}(|A|),
$$

$$
\mathcal{F}(!A) = \left\{ M \subseteq \mathcal{M}_{\text{fin}}(|A|) \mid \bigcup_{\mu \in M} |\mu| \in \mathcal{F}(A) \right\}.
$$

Example

$$
|B| = \{\text{T}, \text{F}\}, \quad \mathcal{F}(B) = \mathcal{P}(\{\text{T}, \text{F}\})
$$

$$
|?B^\perp| = |!B| = \mathcal{M}_{\text{fin}}(\text{T}, \text{F}) \cong \mathbb{N}^2
$$

$$
\mathcal{F}(!B) = \left\{ M \subseteq \mathcal{M}_{\text{fin}}(\text{T}, \text{F}) \mid \bigcup_{\mu \in M} |\mu| \in \mathcal{F}(B) \right\} = \mathcal{P}(\mathbb{N}^2)
$$

$$
\mathcal{F}(?B^\perp) = \left\{ M \subseteq \mathbb{N}^2 \mid \forall M' \subseteq \mathbb{N}^2, \ M \cap M' \text{ fin.} \right\} = \mathcal{P}_{\text{fin}}(\mathbb{N}^2)
$$
Exponentials

The linear finiteness space associated with !A is

\[ \mathbb{k} \langle !A \rangle = \left\{ z \in \mathbb{k}^{\mathcal{M}_{\text{fin}}(|A|)} \mid \bigcup_{\mu \in |z|} |\mu| \in \mathcal{F}(A) \right\}. \]

Example

\[ |B| = \{T, F\} \quad \mathcal{F}(B) = \mathcal{P}(\{T, F\}) \]

\[ |?B\perp| = |!B| = \mathcal{M}_{\text{fin}}(T, F) \simeq \mathbb{N}^2 \]

\[ \mathcal{F}(!B) = \{ M \subseteq \mathcal{M}_{\text{fin}}(T, F) \mid \bigcup_{\mu \in M} |\mu| \in \mathcal{F}(B) \} = \mathcal{P}(\mathbb{N}^2) \]

\[ \mathcal{F}(?B\perp) = \{ M \subseteq \mathbb{N}^2 \mid \forall M' \subseteq \mathbb{N}^2, M \cap M' \text{ fin.} \} = \mathcal{P}_{\text{fin}}(\mathbb{N}^2) \]

\[ \mathbb{k} \langle !B \rangle = \left\{ z \in \mathbb{k}^{|\mathbb{N}^2|} \mid |z| \in \mathcal{P}(\mathbb{N}^2) \right\} = \mathbb{k} (X_t, X_f) \]

\[ \mathbb{k} \langle ?B\perp \rangle = \left\{ z \in \mathbb{k}^{|\mathbb{N}^2|} \mid |z| \in \mathcal{P}_{\text{fin}}(\mathbb{N}^2) \right\} = \mathbb{k} [X_t, X_f] \]
Finiteness Spaces

**Theorem (Taylor expansion)**

For every \( f \in \mathcal{L}_c(\mathbb{k}\langle A \rangle, \mathbb{k}\langle B \rangle) \), there is an analytic function \( \phi \) such that \( \forall x \in \mathbb{k}\langle A \rangle, \phi(x) \in \mathbb{k}\langle B \rangle \).

\[
\forall b \in |B|, \quad \phi_b(x) = \sum_{\mu} f_{\mu, b} x^{\mu} \quad \text{with} \quad x^{\mu} = \prod_a x^{\mu(a)}.
\]

**Example**

\[
\mathbb{k}\langle !B \rightarrow 1 \rangle = \mathbb{k}\langle ?B^\perp \rangle = \mathbb{k}[X_t, X_f],
\]

\[
\mathbb{k}\langle !B \rightarrow B \rangle = \mathbb{k}\langle !B \rightarrow 1 \oplus 1 \rangle = \mathbb{k}\langle !B \rightarrow 1 \rangle^2 = \mathbb{k}[X_t, X_f] \times \mathbb{k}[X_t, X_f].
\]
What is totality?

A way to refine the semantics of a calculus and a hope to have completeness.

Let \( A \) be a finiteness space \( A = (|A|, \mathcal{F}(A)) \). The associate linear space is \( \mathbb{k} \langle A \rangle = \{ x \in \mathbb{k}^{|A|} \mid |x| \in \mathcal{F}(A) \} \).

**Definition**

A totality candidate is an affine subspace \( \mathcal{T} \) of \( \mathbb{k} \langle A \rangle \) such that \( \mathcal{T}^{\bullet \bullet} = \mathcal{T} \) with

\[
\mathcal{T}^{\bullet} = \{ x' \in \mathbb{k} \langle A \rangle' \mid \forall x \in \mathcal{T}, \langle x', x \rangle = 1 \}.
\]

A totality space is a pair \((A, \mathcal{T}(A))\) with \( \mathcal{T}(A)^{\bullet \bullet} = \mathcal{T}(A) \).
A model of linear logic

A refinement of finiteness spaces.

Let $A \in \text{LL}$ and $\pi : A$ an affine linear logic proof.

$$\llbracket \pi \rrbracket \in \mathcal{K}(A).$$

We define by induction a totality candidate $\mathcal{T}(A)$ such that

$$\llbracket \pi \rrbracket \in \mathcal{T}(A).$$
A model of linear logic

A refinement of finiteness spaces.
Let \( A \in \text{LL} \) and \( \pi : A \) an affine linear logic proof.

\[ \llbracket \pi \rrbracket \in \mathbb{L}(A). \]

We define by induction a totality candidate \( \mathcal{T}(A) \) such that

\[ \llbracket \pi \rrbracket \in \mathcal{T}(A). \]

Some constructions

\[ A^\perp \leadsto (\mathbb{L}(A)', \mathcal{T}(A)^\bullet), \]

with \( \mathcal{T}(A)^\bullet = \{ x' \in \mathbb{L}(A)' | \forall x \in \mathcal{T}(A), \langle x', x \rangle = 1 \} \).
A model of linear logic

A refinement of finiteness spaces.
Let $A \in \text{LL}$ and $\pi : A$ an affine linear logic proof.

$$[\pi] \in \mathbb{L}(A).$$

We define by induction a totality candidate $\mathcal{T}(A)$ such that

$$[\pi] \in \mathcal{T}(A).$$

Some constructions

$A \oplus B \leadsto (\mathbb{L}(A) \oplus \mathbb{L}(B), \text{aff}(\mathcal{T}(A) \times \{0\} \cup \{0\} \times \mathcal{T}(B))).$

Example

$$\mathcal{T}(B) = \{(x_t, y_t) \in \mathbb{K}^2 | x_t + y_t = 1\},$$
$$\mathcal{T}(\bot) = \mathcal{T}(1\&1) = (1, 1).$$
A model of linear logic

A refinement of finiteness spaces.
Let $A \in \text{LL}$ and $\pi : A$ an affine linear logic proof.

$$[[\pi]] \in \mathbb{k}\langle A \rangle.$$ 

We define by induction a totality candidate $\mathcal{T}(A)$ such that

$$[[\pi]] \in \mathcal{T}(A).$$

Some constructions

$A \rightsquigarrow B \leadsto (\mathcal{L}_c(A, B), \{f \mid f(\mathcal{T}(A)) \subseteq \mathcal{T}(B)\}).$

Example

$$\mathcal{T}\langle B \rightsquigarrow B \rangle = \{f \in \mathcal{L}_c(\mathbb{k}^2, \mathbb{k}^2) \mid x_t + y_t = 1 \Rightarrow f(x_t, x_f) \in \mathcal{T}(B)\}$$

$$= \{f_t, f_f \in \mathcal{L}(\mathbb{k}^2, \mathbb{k}) \mid x_t + y_t = 1 \Rightarrow f_t(x_t, x_f) + f_f(x_t, x_f) = 1\}.$$
A model of linear logic

*A refinement of finiteness spaces.*

Let \( A \in \text{LL} \) and \( \pi : A \) an affine linear logic proof.

\[
\llbracket \pi \rrbracket \in \llbracket A \rrbracket.
\]

We define by induction a totality candidate \( \mathcal{T}(A) \) such that

\[
\llbracket \pi \rrbracket \in \mathcal{T}(A).
\]

**Some constructions**

\[
!A \leadsto (\llbracket !A \rrbracket , \overline{\text{aff}} \left\{ x^I \mid x \in \mathcal{T}(A) \right\}).
\]

**Example**

\[
\mathcal{T}(\!B) = \overline{\text{aff}} \left\{ (x_t \ T + y_f \ F)^I \mid x_t + y_f = 1 \right\}
\]

\[
= \overline{\text{aff}} \left\{ \sum_{p,q} x_t^p x_f^q \mid x_t + y_f = 1 \right\}.
\]
**Totality Spaces**

**Theorem (Taylor expansion)**

For every $f \in T \langle !A \rightarrow B \rangle$, the associated analytic function $\phi : k\langle A \rangle \Rightarrow k\langle B \rangle$ satisfies

$$x \in T\langle A \rangle \Rightarrow \phi(x) \in T\langle B \rangle.$$ 

**Example**

$$k\langle !B \rightarrow B \rangle = k[X_t, X_f] \times k[X_t, X_f],$$

$$T\langle !B \rightarrow B \rangle = \{(P, Q) \in k[X_t, X_f]^2 \mid x_t + y_t = 1 \Rightarrow P(x_t, y_t) + Q(x_t, y_t) = 1\}.$$
Back to barycentric boolean lambda-calculus

**Definition**

We define inductively the terms of $\Lambda_{\mathcal{B}}$ by

$$R, S ::= \sum_{i=1}^{m} a_i \ s_i \quad \text{with} \quad \sum_{i=1}^{m} a_i = 1,$$

$$s, s_i ::= x \in \mathcal{V} \mid \lambda x. s \mid (s)S \mid T \mid F \mid \text{if } s \text{ then } S \text{ else } R.$$ 

**Types**

We consider only simply typed lambda-term with

$$\sum a_i s_i^A : A,$$

$$T, F : \mathcal{B},$$

$$\text{if } (-) \text{ then } (-) \text{ else } (-) : (\mathcal{B}^n \Rightarrow \mathcal{B})^3 \Rightarrow (\mathcal{B}^n \Rightarrow \mathcal{B}).$$

Notice that term of $\Lambda_{\mathcal{B}^n}$ is a term of $\Lambda_{\mathcal{B}}$ with type $\mathcal{B}^n \Rightarrow \mathcal{B}$. 
Semantics

We use the translation of the *intuitionist implication* into linear logic

\[ A \Rightarrow B \simeq !A \multimap B. \]

To each typed barycentric boolean term is associated a proof of affine linear logic.

\[ [S] \] is the semantics of the proof associated to \( S \).

**Theorem**

Let \( S \in \Lambda_B \). If \( S \) of type \( A \), then \( [S] \in T\langle A \rangle \).
Soundness and partial completeness

Corollary

For every term $S : B \Rightarrow B \simeq \!B \rightarrow B$

$[S] \in T(\!B \rightarrow B)$ which is equal to

$\{(P, Q) \in k[X_t, X_f]^2 | x_t + y_t = 1 \Rightarrow P(x_t, y_t) + Q(x_t, y_t) = 1\}$

Reciprocally, we have already seen

Theorem

For every pair of polynomials $(P, Q) \in T(\!B \rightarrow B)$, there is $S \in \Lambda_{\!B}$ such that $[S] = (P, Q)$.

This is a completeness theorem for first order boolean terms which has even been proved for $\otimes^n \!B \rightarrow B$. 
Conclusion

Completeness

- Total completeness for LL?
  - no, it is not even complete for MALL: \((\mathcal{B} \multimap \mathcal{B}) \multimap \mathcal{B}\)
- Total completeness for higher order hierarchy \(\Lambda_{\mathcal{B}}\)?
- How to complete \(\Lambda_{\mathcal{B}}\) to get completeness?

Totality

Totality spaces constitute an elegant affine model of linear logic where linear logic construction are algebraically defined and completeness also seem to have an algebraic characterization.