Iterated Chromatic Subdivisions are Collapsible*

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Abstract

The standard chromatic subdivision of the standard simplex is a combinatorial algebraic construction, which was introduced in theoretical distributed computing, motivated by the study of the view complex of layered immediate snapshot protocols. A most important property of this construction is the fact that the iterated subdivision of the standard simplex is contractible, implying impossibility results in fault-tolerant distributed computing. Here, we prove this result in a purely combinatorial way, by showing that it is collapsible, studying along the way fundamental combinatorial structures present in the category of colored simplicial complexes.

1 Introduction

Fault-tolerant distributed computing is concerned with determining algorithms, when possible, solving so-called *decision tasks* on a given distributed architecture, in the presence of faults. Distributed architectures can be message-passing distributed machines, where a set of processors compute in their local memory, synchronize and communicate by sending and receiving messages over a network. They can also be shared-memory concurrent machines, where processors compute and communicate through shared locations, where reads and writes are supposed to be "atomic", which means that they are run in mutual exclusion. Here, we are considering the latter model, which has an equivalent presentation where processors are executing the following steps: scanning the entire shared memory (and copying it in their local memory), computing in its local memory, and then updating its "own value", i.e. writing the outcome of its computation in a specific location in global memory, assigned to him only. There are two main types of fault models that are generally considered in the literature: crash failures, where any of the processors may die unexpectedly during computation while scanning or updating their value, and byzantine failures where any of the processors can carry on computation without dying, but instead update unpredictable values in global memory. In what follows, we study a mathematical model of the sets of reachable configurations of shared-memory protocols, in the presence of crash failures only.

The seminal result in this field was established by Fisher, Lynch and Patterson in 1985 [8]. They proved that there exists a simple task that cannot be

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solved in a message-passing (or equivalently a shared memory) system with at most one potential crash. In particular, there is no way in such a distributed system to solve the very fundamental consensus problem: each processor starts with an initial value in local memory (typically an integer), and should end up with a common value, which is one of the initial values.

This created a very active research area, see for instance [18, 12]. Later on, Biran, Moran and Zaks developed a characterization of the decision tasks that can be solved by a (simple) message-passing system in the presence of one failure [2]. The argument uses a "similarity chain", which could be seen as a connectedness result of a representation of the space of all reachable states, that we call here the *view complex* [16] or the (full information) *protocol complex* [14]. Of course, this argument turned out to be difficult to extend to models with more failures, as higher-connectedness properties of the view complex matter in these cases. This technical difficulty was first tackled, using homological calculations, by Herlihy and Shavit [13] (and independently [3, 21]): there are simple decision tasks, such as consensus once again, that cannot be solved in the wait-free asynchronous model, i.e. shared-memory distributed protocols on n processors, with up to $n-1$ crash failures. Then, the full characterization of wait-free asynchronous decision tasks with atomic read and writes on registers was described by Herlihy and Shavit [14]. This relies on the central notion of chromatic (or colored) simplicial complexes, and subdivisions of those. All results above are deduced from the contractibility of the so-called "standard" chromatic subdivision, which was completely formalized in [16] and corresponds to the *view complex* of distributed algorithms solving layered immediate snapshot protocols. Until recently, this so-called standard chromatic subdivision was not actually shown to be a subdivision. Here, we prove this result in a purely combinatorial way, by showing that it is collapsible, a fact that was also independently shown by Kozlov [17]. In doing so, we elaborate generic tools to make such proofs, and extend the result to the *iterated* subdivision, as arising in more general iterated protocols.

Contents of the paper and main contributions. In Section 2, we begin by defining a category of simplicial complexes (Section 2.1), from which we derive a category of colored simplicial complexes as a slice category (Section 2.2). We study combinatorial structures in those categories, most importantly by introducing a colored join operation, similar to the one defined in [4]. We also recall the classical notion of collapse from simple homotopy theory (Section 2.3). In Section 3, we abstractly define the standard chromatic subdivision of a colored simplicial complex by extending a folklore definition of the barycentric subdivision (Section 3.1), then study the collapsibility of the basic subdivision of the standard simplex, which can be seen as a "toy version" of the standard chromatic subdivision (Section 3.2), and finally extend these constructions in order to show that the standard chromatic subdivision is collapsible (Section 3.4), which requires to understand in details the interaction between joins and simple homotopy (Section 3.3). This is the first major contribution of this paper, that determines collapsibility sequences using different techniques than the ones used in [17]. We think that these techniques should lead us to extending our work to more general complexes, arising from more general distributed architectures. The second major contribution of this article is the extension of these results to the iterated subdivision (Section 4). In order to do so, we first embed simplicial complexes into a presheaf category (Section 4.1) which is much more well behaved with respect to collapses, we then briefly extend the tools in simple homotopy to this case (Section 4.2) and finally show in this setting that the iterated subdivision is collapsible (Section 4.3). We conclude with further directions for this work in Section 5.

2 Simplicial complexes

There are two main ways to represent the combinatorial objects that we are manipulating, which are closely related: simplicial complexes and presimplicial sets. While most constructions can be performed similarly in both settings, we chose to use mainly simplicial complexes, because they are easier to describe, and we recall the main definitions in this section, stated in a categorical setting (more details can be found in many places such as [15]). It will turn out however, that the second representation, as presheaves, has its own advantages and it will be explained and used in Section 4.

We write $[n]$ for the set $[n] = \{0, \dots, n\}$ with $n+1$ elements, and $\mathfrak{S}_n$ for the group of permutations on $[n]$. Given a set X, we write $\#X$ for the cardinal of X.

2.1 Simplicial complexes

We only consider *abstract* simplicial complexes in this article, and simply call them simplicial complexes in the following.

Definition 1. A **simplicial complex** $(\underline{K}, K)$ is a pair consisting of

- a set $\underline{K}$ of *vertices*,
- together with a set K of finite subsets of $\underline{K}$ called *simplices*,

such that

- K is non-empty,
- for every vertex $x \in \underline{K}$, we have $\{x\} \in K$,
- if $\sigma \in K$ and $\tau \subseteq \sigma$ then $\tau \in K$.

Given two simplices σ and τ, with $\tau \subseteq \sigma$, in a given simplicial complex, we say that τ is a *face* of σ and that σ is a *coface* of τ.

Remark 2. In the above definition, requiring K to be non-empty amounts to require $\emptyset \in K$ by the last condition of Definition 1. So, for instance the *empty* simplicial complex $(\emptyset, \{\emptyset\})$ is valid, but the *void* simplicial complex $(\emptyset, \emptyset)$ is not. This is not fundamental, and dropping this non-emptyness condition actually leads to a richer structure (the category is not anymore pointed if we allow the void simplicial complex, see Remark 16). However, the geometrical interpretation is less clear, and in particular there is no obvious geometric realization functor anymore. Notice that instead of requiring $\emptyset \in K$, we could as well have required that simplices are non-empty, but this makes some constructions, such as the join (Definition 9), less natural to express: the simplices of $K \star L$ are of the form $\sigma|\tau$ where σ or τ can be empty.

Remark 3. When we know that the set of vertices of a simplicial complex K is a subset of a given set X, i.e. $\underline{K} \subseteq X$, the set of vertices can be recovered by

$$\underline{K} \quad = \quad \{x \in X \mid \{x\} \in K\}$$

This explains why in the following we will often identify a simplicial complex $(\underline{K}, K)$ with the underlying set K of simplices, when a bigger set containing the vertices is clear from the context.

Definition 4. The **dimension** of a simplex σ is given by

$$\dim \sigma \quad = \quad \#\sigma - 1$$

More generally, given a set X, we write

$$\dim X \quad = \quad \#X - 1$$

in the following. The *dimension* of a complex K is

$$\dim K \quad = \quad \sup \{\dim \sigma \mid \sigma \in K\}$$

A complex K is *finite-dimensional* when $\dim K < \infty$.

Definition 5. A **morphism**

$$f \quad : \quad K \quad \to \quad K'$$

of simplicial complexes consists of a function

$$f \quad : \quad \underline{K} \quad \to \quad \underline{K'}$$

which

- preserves simplices: for every $\sigma \in K$, $f(\sigma) \in K'$,
- is *locally injective*: for every $\sigma \in K$, the restriction of f to σ is injective.

We write **SC** for the category of simplicial complexes.

Most of the morphisms we consider are going to be monomorphisms, which are those for which the function on vertices is injective (not only locally, in the above sense). This is in particular the case for *inclusions* $f : K \hookrightarrow K'$, where $K \subseteq K'$, and therefore $\underline{K} \subseteq \underline{K'}$, and f is this inclusion of $\underline{K}$ into $\underline{K'}$ (in this case, we also say that K is a *subcomplex* of K').

Remark 6. Requiring functions to be locally injective instead of considering all functions preserving simplices is roughly the same as using presimplicial sets instead of simplicial sets. This does not play a major role in the following constructions. As explained above, we could also have restricted to injective functions for most of the paper. The only place where it is really important to consider locally injective functions, which motivates our choice in the definition of the category, is when defining colored complexes using a slice construction in Section 2.2: it ensures that colors of two vertices in the same simplex are distinct.

Definition 7. Given a finite set I, the **standard I-simplicial complex** Δ^I is the complex with I as the set of vertices and all subsets of I as simplices. We write Δ^n instead of $\Delta^{[n]}$ for $n \in \{-1\} \cup \mathbb{N}$.

Remark 8. Notice that an endomorphism $f : \Delta^n \to \Delta^n$ in **SC** is simply a bijective function $[n] \to [n]$, i.e. we have a group isomorphism

$$\mathbf{SC}(\Delta^n, \Delta^n) \quad \cong \quad \mathfrak{S}_n$$

Therefore, given a simplicial complex K, $\mathbf{SC}(\Delta^n, K)$ is equipped with a structure of left $\mathfrak{S}_n$-module by precomposition with elements of $\mathbf{SC}(\Delta^n, \Delta^n)$. This can enable us to formulate an analogue to Yoneda lemma in this context as follows. Given a complex K, and $n \in \{-1\} \cup \mathbb{N}$, we write K_n for the set of n-dimensional simplices of K:

$$K_n \quad = \quad \{\sigma \in K \mid \dim \sigma = n\}$$

We have the following isomorphism of left $\mathfrak{S}_n$-modules:

$$\mathfrak{S}_n \times K_n \quad \cong \quad \mathbf{SC}(\Delta^n, K)$$

The join of two simplicial complexes can be defined as follows, see [6] for a study of this operation in the setting of simplicial sets.

Definition 9. Given two simplicial complexes K and L, their **join** $K \star L$ is the simplicial complex whose

- vertices are

$$\underline{K \star L} \quad = \quad \underline{K} \uplus \underline{L}$$

- simplices are

$$K \star L \quad = \quad \{\sigma \subseteq \underline{K} \uplus \underline{L} \mid \sigma \cap \underline{K} \in K \quad \text{and} \quad \sigma \cap \underline{L} \in L\}$$

Given $\sigma \in K$ and $\tau \in L$, we often write $\sigma|\tau$ for the simplex $\sigma \uplus \tau$ in $K \star L$ obtained by disjoint union:

$$K \star L \quad = \quad \{\sigma|\tau \mid \sigma \in K \quad \text{and} \quad \tau \in L\}$$

The neutral element for the join operation is the simplex $1 = \{\emptyset\}$. Given morphisms $f : K \to K'$ and $g : L \to L'$, we write

$$f \star g \quad : \quad K \star L \quad \to \quad K' \star L'$$

for the morphism defined as $f \uplus g : \underline{K} \uplus \underline{L} \to \underline{K'} \uplus \underline{L'}$ on vertices.

Remark 10. Notice that $\dim(\sigma|\tau) = \dim(\sigma) + \dim(\tau) + 1$.

Remark 11. Suppose given $\sigma|\tau \in K \star L$ and $\sigma'|\tau' \in K \star L$. We have $\sigma|\tau \subseteq \sigma'|\tau'$ if and only if $\sigma \subseteq \sigma'$ and $\tau \subseteq \tau'$. Moreover, we have $(\sigma|\tau) \cup (\sigma'|\tau') = (\sigma \cup \sigma')|(\tau \cup \tau')$.

Lemma 12. *The join operation together with* 1 *equips* **SC** *with a (symmetric) monoidal category structure.*

Remark 13. For this definition to work, we have to allow $\emptyset$ as an element of K as noticed in Remark 2.

Lemma 14. *Given two disjoint finite sets I and J, we have*

$$\Delta^I \star \Delta^J \quad = \quad \Delta^{I \cup J}$$

Lemma 15. *Given two simplicial complexes K and L their* ***coproduct*** *$K + L$ is the simplicial complex whose*

- *vertices are*

$$\underline{K + L} \quad = \quad \underline{K} \uplus \underline{L}$$

- *simplices are*

$$K + L \quad = \quad \{\sigma|\tau \in K \star L \mid \sigma = \emptyset \quad \textit{or} \quad \tau = \emptyset\}$$

The neutral element is the complex $0 = \{\emptyset\}$.

Remark 16. Notice that $0 = 1$. If we had allowed the void complex $\emptyset$ (not even containing $\emptyset$ as a simplex) then constructions would have been the same excepting that we would have defined $0 = \emptyset \neq \{\emptyset\} = 1$.

Remark 17. Since the terminal object is the same as the initial object in our category ($0 = 1$ is a zero object and the category is pointed), for every pair of objects K and L, there is a canonical arrow

$$K \quad \to \quad 1 = 0 \quad \to \quad L$$

Because of this, there is a canonical arrow

$$K + L \quad \to \quad K \star L$$

namely, writing $i_K : 0 \to K$ for the initial arrow, we have

$$K + L \quad \cong \quad K \star 1 + 1 \star L \quad = \quad K \star 0 + 0 \star L \quad \xrightarrow{\mathrm{id}_K \star i_L + i_K \star \mathrm{id}_L} \quad K \star L$$

More explicitly, this morphism sends $\sigma \in K$ to $\sigma|\emptyset$ and $\tau \in L$ to $\emptyset|\tau$.

Definition 18. Given a simplicial complex K and $\sigma \subseteq \underline{K}$ with $\sigma \neq \emptyset$, we define the simplicial complex $K \setminus \sigma$, called the **removal** of σ in K or a **restriction** of K, as the simplicial complex such that $\underline{K \setminus \sigma} \subseteq \underline{K}$ and

$$K \setminus \sigma \quad = \quad K \setminus \{\tau \in K \mid \sigma \subseteq \tau\}$$

Remark 19. Notice that $K \setminus \emptyset$ is *not* defined.

Definition 20. The **boundary** of Δ^I is $\partial\Delta^I = \Delta^I \setminus I$.

Lemma 21. *Given simplices $\sigma, \tau \in K$, we have*

$$(K \setminus \tau) \setminus \sigma = (K \setminus \sigma) \setminus \tau \qquad \textit{and} \qquad (K \setminus \sigma) \setminus \sigma = K \setminus \sigma$$

Given a set $\Sigma = \{\sigma_1, \ldots, \sigma_k\} \subseteq K$, we can therefore define

$$K \setminus \Sigma \quad = \quad ((K \setminus \sigma_1) \ldots) \setminus \sigma_k$$

Remark 22. Notice that there is a canonical inclusion morphism

$$K \setminus \Sigma \quad \hookrightarrow \quad K$$

since the set of vertices (resp. simplices) of the first is a subset of the vertices (resp. simplices) of the second.

Remark 23. There is an ambiguity in the notations, which hopefully will not lead to confusions: the expression $K \setminus \emptyset$ can either denote the removal in K of the empty set of vertices (in which case it is not defined as per Remark 19), or the removal in K of the empty set of simplices (in which case we have $K \setminus \emptyset = K$ following the definition given in Lemma 21).

Lemma 24. *Suppose given a complex K and two sets Σ and T of simplices of K. Then*

$$K \setminus \Sigma \quad = \quad K \setminus T$$

if and only if

$$\forall \tau \in T, \exists \sigma \in \Sigma, \sigma \subseteq \tau \qquad \textit{and} \qquad \forall \sigma \in \Sigma, \exists \tau \in T, \tau \subseteq \sigma$$

Proof. Suppose that the equality holds. In particular, $K \setminus \Sigma \subseteq K \setminus T$. Given $\tau \in T$, if for every $\sigma \in \Sigma$ we have $\sigma \not\subseteq \tau$ then $\tau \in K \setminus \Sigma$ and therefore $\tau \in K \setminus T$, which is absurd. Therefore there exists $\sigma \in \Sigma$ such that $\sigma \subseteq \tau$. The other direction is similar.

Conversely, suppose that the property on Σ and T holds. Let $\upsilon \in K \setminus \Sigma$, then for every $\sigma \in \Sigma$ we have $\sigma \not\subseteq \upsilon$. Therefore for every $\tau \in T$ we do not have $\tau \subseteq \upsilon$ because there exists $\sigma \in \Sigma$ such that $\sigma \subseteq \tau$ and $\upsilon \in K \setminus T$. We have shown $K \setminus \Sigma \subseteq K \setminus T$ and the other inclusion is similar. □

Remark 25. In the above lemma, the elements of Σ and T are supposed to be simplices of K and not simply subsets of $\underline{K}$. Otherwise the lemma is not true. Namely, consider $\sigma \subseteq \underline{K}$ which is not a simplex of K. Then $K \setminus \sigma = K \setminus \emptyset$ (where $\emptyset$ denotes the empty set of simplices, as opposed to the empty set of vertices whose removal would not be properly defined as per Remark 19) in contradiction with previous lemma.

Definition 26. Given a simplicial complex K, the **star** $\operatorname{st}(\sigma)$ of a simplex $\sigma \in K$ is the simplicial subcomplex of K whose simplices are

$$\operatorname{st}(\sigma) \quad = \quad \{\tau \in K \mid \sigma \cup \tau \in K\}$$

Definition 27. Given a simplicial complex K, the **open star** $\operatorname{ost}(\sigma)$ of a simplex $\sigma \in K$ is the set of *cofaces* of σ in K:

$$\operatorname{ost}(\sigma) \quad = \quad \{\tau \in K \mid \sigma \subseteq \tau \in K\}$$

We sometimes write $\operatorname{st}_K(\sigma)$ (resp. $\operatorname{ost}_K(\sigma)$) in order to make it clear that the (open) star is computed in the complex K. Given a set $\Sigma \subseteq K$ of simplices, we sometimes use the expected notation $\operatorname{st}(\Sigma) = \bigcup_{\sigma \in \Sigma} \operatorname{st}(\sigma)$ and similarly for $\operatorname{ost}(\Sigma)$.

Remark 28. The open star operation does not generally define a simplicial complex.

Remark 29. We always have $\mathrm{ost}(\sigma) \subseteq \mathrm{st}(\sigma)$.

Remark 30. Given a simplicial complex K, $\mathrm{st}(\emptyset) = \mathrm{ost}(\emptyset) = K$.

Remark 31. The condition of Lemma 24 can be reformulated as $\Sigma \subseteq \mathrm{ost}(T)$ and $T \subseteq \mathrm{ost}(\Sigma)$.

Lemma 32. *Given two simplicial complexes K and L and a simplex $\sigma|\tau \in K \star L$, we have*

$$\mathrm{st}_{K\star L}(\sigma|\tau) \quad = \quad \mathrm{st}_K(\sigma) \star \mathrm{st}_L(\tau)$$

and

$$\mathrm{ost}_{K\star L}(\sigma|\tau) \quad = \quad \mathrm{ost}_K(\sigma) \star \mathrm{ost}_L(\tau)$$

Proof. Follows immediately from Remark 11. □

2.2 Colored complexes

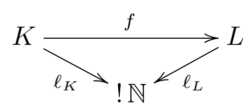
We now define the category of colored complexes, a variant of simplicial complexes where vertices have an attributed color, as a slice category. We write **Inj** for the category of sets and injective functions.

Definition 33. The **labeling functor** $! : \mathbf{Inj} \to \mathbf{SC}$ is the functor which to a set X associates the simplicial complex $!\,X$, with X as set of vertices and all finite subsets of X as simplices (i.e. for any finite set I, we have $!\,I = \Delta^I$), and to an injective function $f : X \to Y$ associates the function f itself.

Definition 34. The category of **colored complexes** is the slice category

$$\mathbf{CSC} \quad = \quad \mathbf{SC}/\,!\,\mathbb{N}$$

An object (K, ℓ_K) in this category consists of a simplicial complex K together with a morphism $\ell_K : K \to !\,\mathbb{N}$ called its **coloring**. A morphism $f : K \to L$ is a morphism of simplicial complexes such that $\ell_L \circ f = \ell_K$.



Remark 35. Because the morphisms of **SC** are locally injective functions, the labeling is given by a function $\ell : \underline{K} \to \mathbb{N}$ such that, for any two vertices $x, y \in \underline{K}$, if there exists $\sigma \in K$ such that $x \in \sigma$ and $y \in \sigma$ then $\ell(x) \neq \ell(y)$, i.e. two vertices in the same simplex have different colors.

Remark 36. It is the comma category $\mathbf{SC}/\,!$ that really deserves to be called the *category of colored complexes*. However, in the following, we will only consider the case where colors are integers and morphisms act trivially on colors, which is why we restricted to the category of Definition 34, making notations lighter. We refer the reader to [19] for the definition of comma categories.

Remark 37. Notice that, as with any slice category, projection on the first factor provides a forgetful functor $\mathbf{CSC} = \mathbf{SC}/\,!\,\mathbb{N} \to \mathbf{SC}$.

Definition 38. The **colored standard I-complex** is Δ^I colored with $\ell(i) = i$ for every vertex $i \in I$.

The following definition, which plays a fundamental role in this work, had a first occurrence in [4].

Definition 39. Given two colored complexes K and L, their **colored join** $K \star_{\mathrm{c}} L$ is the colored simplicial complex whose

- vertices are

$$\underline{K \star_{\mathrm{c}} L} \quad = \quad \underline{K} \uplus \underline{L}$$

  colored by $\ell_K \uplus \ell_L$

- simplices are

$$K \star_{\mathrm{c}} L \quad = \quad \{\sigma|\tau \mid \sigma \in K \quad \text{and} \quad \tau \in L \quad \text{and} \quad \ell_K(\sigma) \cap \ell_L(\tau) = \emptyset\}$$

We usually simply write $\star$ instead of $\star_{\mathrm{c}}$ in the following: given two colored complexes K and L, $K \star L$ will always denote their colored join. Intuitively, $K \star_{\mathrm{c}} L$ is the biggest subcomplex of $K \star L$ which is well-colored, in the sense that two distinct vertices of a simplex have distinct colors.

Remark 40. Suppose that K and L are complexes with disjoint sets of colors, i.e. that we have $\ell_K(\underline{K}) \cap \ell_L(\underline{L}) = \emptyset$. Then the join and the colored join coincide: more precisely, writing $U : \mathbf{CSC} \to \mathbf{SC}$ for the forgetful functor (see Remark 37), we have $U(K \star_{\mathrm{c}} L) = UK \star UL$.

The restriction operation can also be extended to the colored setting in a straightforward way.

2.3 Simple homotopy

The main tool we are going to use to show that a simplicial complex is contractible is the notion of simplicial collapse due to Whitehead [22], see also [5] for a modern introduction to the subject. Collapsibility to a point implies contractibility (see the classical Theorem 49), but the contrary is false, see for instance Bing's house with two rooms [1].

Definition 41. Suppose that K is a simplicial complex of finite dimension and $\sigma, \tau \in K$ are two simplices. The simplex τ is a **free face** of σ when the following two conditions are satisfied:

1. $\tau \subseteq \sigma$ and $\tau \neq \sigma$,

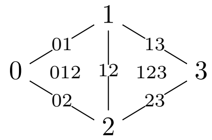
2. σ is a maximal (w.r.t. inclusion) simplex of K and no other maximal simplex of K contains τ.

For such a pair (τ, σ), we call $K \setminus \tau$ a **collapse step** of K. If additionally we have $\dim \tau = \dim \sigma - 1$, then this is called an **elementary** collapse step. A **collapse** is a finite sequence of collapse steps. A simplicial complex that has a finite sequence of collapses to a point is called **collapsible**, where a *point* is a complex isomorphic to Δ^0. The equivalence relation on simplicial complexes generated by collapses is called **simple homotopy**.

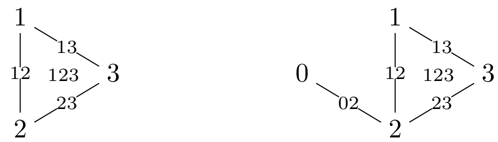
Remark 42. Notice that we only consider collapses of complexes which are of finite dimension, this will be implicitly checked in the following.

By remark 22, when K collapses to K' there is a canonical monomorphism $K' \to K$. We thus more generally say that a morphism $f : K' \to K$ is a **collapse** when there exists a collapse from K to K' such that the corresponding monomorphism is f. By extension, morphisms obtained from collapses by pre- and post-composing collapses by isomorphisms are also considered as collapses.

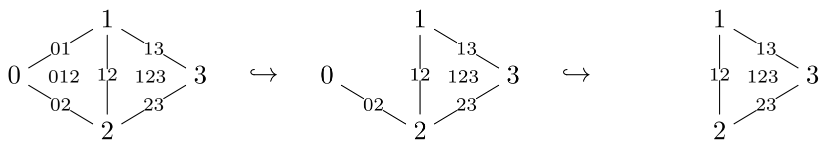
Example 43. Consider the following simplicial complex, whose set of vertices is $\{0, 1, 2, 3\}$ and whose maximal simplices are $\{0, 1, 2\}$ and $\{1, 2, 3\}$:



The simplices $\{0\}$ and $\{0, 1\}$ are free faces of $\{0, 1, 2\}$, respectively giving rise to the following collapses:



On the contrary, the simplices $\{1\}$ and $\{1, 2\}$ are not free faces. Note that the free face $\{0, 1\}$ induces an elementary collapse step. This is not the case for the free face $\{0\}$, but there is an equivalent sequence of elementary collapse steps from this free face:



Lemma 45 shows that this is in fact a general phenomenon: elementary collapse steps generate all collapses.

Remark 44. As noticeable in previous example, we use a lighter notation for sets in figures and simply write 012 instead of $\{0, 1, 2\}$, etc.

Lemma 45. *Any collapse step (and thus collapse) can be decomposed as a sequence of elementary collapse steps.*

Proof. Notice that in the following, we use the close notations $\sigma \setminus \tau$ (for the difference of sets of vertices) and $K \setminus \sigma$ (for the removal of a simplex σ in a simplicial complex K, see Definition 18).

Suppose that τ is a free face of σ in a simplicial complex K. We fix an enumeration $x_1, \ldots, x_k$ of the elements of $\sigma \setminus \tau$ and write $\tau_i = \tau \cup \{x_1, \ldots, x_i\}$ for $0 \leq i \leq k$: we have $\tau_0 = \tau$, $\tau_k = \sigma$ and $\tau_{i+1} = \tau_i \cup \{x_{i+1}\}$:

$$\tau = \tau_0 \overset{x_1}{\subseteq} \tau_1 \overset{x_2}{\subseteq} \tau_2 \overset{x_3}{\subseteq} \ldots \overset{x_k}{\subseteq} \tau_k = \sigma$$

First, notice that τ_{k-1} is a free face of σ in K, because τ is a free face of σ in K and $\tau \subseteq \tau_{k-1} \subsetneq \sigma$. Moreover, given i with $0 < i < k$, we show that τ_{i-1} is a free face of $\sigma \setminus \{x_i\}$ in $K \setminus \tau_i$:

- both cells are actually in $K \setminus \tau_i$ by an easy verification, and we have $\tau_{i-1} \subsetneq \sigma \setminus \{x_i\}$ because $\tau_i \subsetneq \sigma$ and $\tau_i = \tau_{i-1} \uplus \{x_i\}$,
- $\sigma \setminus \{x_i\}$ is maximal in $K \setminus \tau_i$: otherwise there is a simplex $\sigma' = (\sigma \setminus \{x_i\}) \cup \{x\}$ in $K \setminus \tau_i$, with $x \neq x_i$ (otherwise $\tau_i \subseteq \sigma' = \sigma$ and therefore $\sigma' \notin K \setminus \tau_i$), which contradicts the fact that τ is a free face of σ in K since $\tau \subseteq (\sigma \setminus \{x_i\}) \cup \{x\} \not\subseteq \sigma$,
- $\sigma \setminus \{x_i\}$ is the only maximal coface of τ_{i-1} in $K \setminus \tau_i$: given σ' in $K \setminus \tau_i$ such that $\tau_{i-1} \subseteq \sigma'$, we have $\tau \subseteq \sigma'$ in K and therefore $\sigma' \subseteq \sigma$ since τ is a free face of σ in K, and $x_i \notin \sigma'$ (otherwise $\tau_i \subseteq \sigma'$, and thus $\sigma' \notin K \setminus \tau_i$) therefore $\sigma' \subseteq \sigma \setminus \{x_i\}$.

We have thus constructed a sequence of elementary collapse steps

$$K \setminus \tau = K \setminus \tau_1 \setminus \tau \;\hookrightarrow\; K \setminus \tau_1 = K \setminus \tau_2 \setminus \tau_1 \;\hookrightarrow\; \ldots \;\hookrightarrow\; K \setminus \tau_{k-1} \;\hookrightarrow\; K \setminus \sigma$$

which allows us to conclude (above, we have $(K \setminus \tau_{i+1}) \setminus \tau_i = K \setminus \tau_i$ because $\tau_i \subseteq \tau_{i+1}$). □

Definition 46. Given a finite set $I \subseteq \mathbb{N}$ and $p \in I$, we write

$$\Lambda^I_p \quad = \quad \Delta^I \setminus \sigma$$

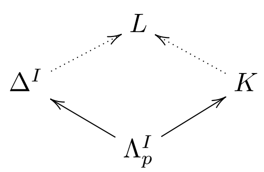
where σ is the simplex $\sigma = I \setminus \{p\}$.

Notice that the inclusion

$$\Lambda^I_p \quad \hookrightarrow \quad \Delta^I$$

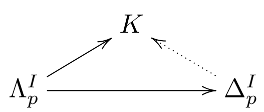
is an elementary collapse step, and conversely, every elementary collapse step of Δ^I can be described in this way, for some $p \in I$. More generally, every elementary collapse step can be generated by the above family of collapse steps as follows (this is simply a reformulation of the definition of an elementary collapse step).

Lemma 47. *An inclusion $K \hookrightarrow L$ is an elementary collapse step if and only if there exists an inclusion $\Lambda^I_p \hookrightarrow K$ for some finite set I and $p \in I$ such that $K \hookrightarrow L$ is part of a pushout cocone of the form*



(1)

and there is no inclusion $\Delta^I_p \hookrightarrow K$ making the following diagram commute:



(2)

Remark 48. Notice that with $K = L = \Delta^I$, the diagram (1) is a pushout, which shows why we need condition (2) : the identity on Δ^I is not an elementary collapse step. This situation is intuitively due to the fact that in a simplicial complex, two simplices having the same set of vertices are necessarily equal. We will see in Section 4 that we can embed the category **CSC** into a presheaf category in which we can represent "complexes" having multiple simplices with the same vertices, and is thus much better suited w.r.t. pushouts of collapses.

The following, which is a classical theorem, shows that collapsibility can be used to show the contractibility of a simplicial complex in an algebraic way [22].

Theorem 49. *Every collapsible complex is contractible.*

Lemma 50. Δ^I *is collapsible for* $I \subseteq \mathbb{N}$ *finite and non-empty.*

Proof. By induction on $\dim I$. The result is immediate when $\dim I = 0$ because Δ^I is a point. Otherwise, pick $i \in I$. The simplex $\{i\}$ is a free face of I in Δ^I because I is the unique maximal simplex and Δ^I is finite (see Remark 42). Therefore the inclusion $\Delta^I \setminus \{i\} \hookrightarrow \Delta^I$ is a collapse. It can easily be checked that $\Delta^I \setminus \{i\} = \Delta^{I\setminus\{i\}}$ and we can conclude using the induction hypothesis. □

Often, one can perform many different collapse steps on a given complex. It can be useful to perform them all at once. The following proposition provides a useful such case in which this is possible.

Proposition 51. *In a simplicial complex* K*, suppose that we are given a finite family* $(\tau_i, \sigma_i)_{i\in I}$ *of pairs of simplices such that, for every* $i, j \in I$*,* τ_i *is a free face of* σ_i *in* K *and* $\tau_i \subseteq \sigma_j$ *implies* $i = j$*. Then the inclusion*

$$K \setminus \tau_I \quad \hookrightarrow \quad K$$

is a collapse, where $\tau_I = \{\tau_i \mid i \in I\}$*.*

Proof. By induction on the cardinal of I. The base case where the family is empty is trivial. Otherwise, pick $i \in I$ and write $K' = K \setminus \tau_{I\setminus\{i\}}$. By definition of K', $\sigma_i \in K'$ because otherwise there would exist $j \in I \setminus i$ such that $\tau_j \subseteq \sigma_i$ which contradicts the hypothesis. Similarly, $\tau_i \in K'$ because otherwise there would exist $j \in I \setminus i$ such that $\tau_j \subseteq \tau_i \subseteq \sigma_i$ which contradicts the hypothesis. Moreover, since K' is included in K, σ_i is still maximal in K' and is the unique maximal coface of τ_i in K'. Indeed, suppose for contradiction that $\tau_i \subseteq \sigma'_i$ for some maximal simplex $\sigma'_i \in K'$. Since K is supposed to be finite-dimensional (see Remark 42), σ'_i is a face of some maximal simplex σ''_i in K. Since τ_i is a free face of σ_i, necessarily $\sigma''_i = \sigma_i$ and therefore $\sigma'_i = \sigma_i$ because $\sigma_i \in K'$ and σ'_i is supposed to be maximal. Finally, τ_i is a free face of σ_i in K' and therefore the inclusion $K \setminus \tau_I = K' \setminus \tau_i \hookrightarrow K$ is a collapse. □

Remark 52. In the previous proposition, the hypothesis that $\tau_i \subseteq \sigma_j$ implies $i = j$ can be replaced by the equivalent one that the σ_i are pairwise distinct.

Finally, the following proposition will be used in order to extend a collapse of a subcomplex to a collapse of the bigger complex.

Lemma 53. *Suppose that we are given two complexes* K *and* L *and an inclusion* $K \hookrightarrow L$*. Suppose moreover that the inclusion*

$$K \setminus \upsilon \quad \hookrightarrow \quad K$$

is a collapse for a given simplex $\upsilon \in K$. If the cofaces of υ in L are also in K, then the inclusion

$$L \setminus \upsilon \quad \hookrightarrow \quad L$$

is also a collapse.

Proof. Since $K \setminus \upsilon \hookrightarrow K$ is a collapse, there exists a sequence of pairs of simplices $(\tau_1, \sigma_1), \ldots, (\tau_k, \sigma_k)$ such that τ_{i+1} is a free face of σ_{i+1} in $K_i = K \setminus \{\tau_1, \ldots, \tau_i\}$, forming a sequence of collapse steps

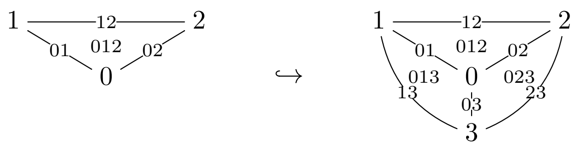
$$K \setminus \upsilon = K_k \quad \hookrightarrow \quad K_{k-1} \quad \hookrightarrow \quad \ldots \quad \hookrightarrow \quad K_1 \quad \hookrightarrow \quad K_0 = K$$

Since $K \setminus \upsilon = K_k = K \setminus \{\tau_1, \ldots, \tau_k\}$, we have $\upsilon \subseteq \tau_i$ for every index i by Lemma 24. Therefore the cofaces of τ_i in K and in L coincide: $\mathrm{ost}_K(\tau_i) = \mathrm{ost}_L(\tau_i)$ for every index i. Writing $L_i = L \setminus \{\tau_1, \ldots, \tau_i\}$, we deduce easily that $\mathrm{ost}_{K_i}(\tau_{i+1}) = \mathrm{ost}_{L_i}(\tau_{i+1})$, and therefore τ_{i+1} is a free face of σ_{i+1} in L_i. Finally, we have a sequence of collapse steps

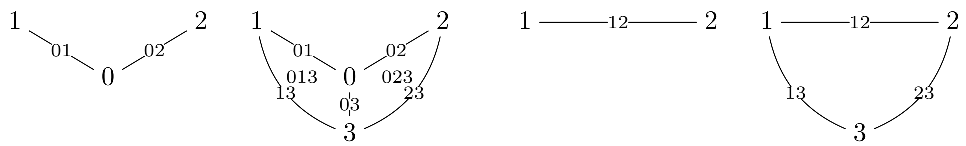
$$L \setminus \upsilon = L_k \quad \hookrightarrow \quad L_{k-1} \quad \hookrightarrow \quad \ldots \quad \hookrightarrow \quad L_1 \quad \hookrightarrow \quad L_0 = L$$

from which we can conclude, where the first equality is easy to justify using Lemma 24. ☐

Example 54. Consider the simplicial complexes K, whose only maximal simplex is $\{0, 1, 2\}$, and L, whose maximal simplices are $\{0, 1, 2\}$, $\{0, 1, 3\}$ and $\{0, 2, 3\}$, together with the obvious inclusion $K \hookrightarrow L$:



With $\upsilon = \{1, 2\}$, the inclusion $K \setminus \upsilon \hookrightarrow K$ is a collapse, the cofaces of υ in L are also in K, and therefore the inclusion $L \setminus \upsilon \hookrightarrow L$ is also a collapse. On the contrary, with $\upsilon' = \{0\}$, the inclusion $K \setminus \upsilon' \hookrightarrow K$ is a collapse, but the cofaces of υ' in L are not all in K (e.g. $\{0, 3\}$), and the inclusion $L \setminus \upsilon' \hookrightarrow L$ is not a collapse (for instance, it does not reflect contractibility):



$K \setminus \{1, 2\}$ $\qquad$ $L \setminus \{1, 2\}$ $\qquad$ $K \setminus \{0\}$ $\qquad$ $L \setminus \{0\}$

Proposition 55. *Given a complex K and a simplex $\sigma \in K$, if the inclusion*

$$\mathrm{st}_K(\sigma) \setminus \sigma \quad \hookrightarrow \quad \mathrm{st}_K(\sigma)$$

is a collapse then the inclusion

$$K \setminus \sigma \quad \hookrightarrow \quad K$$

is also a collapse.

Proof. As noticed in Remark 29, the cofaces of σ are all contained in $\mathrm{st}(\sigma)$. We can therefore apply Lemma 53. □

Remark 56. The previous proposition can easily be generalized to sets of simplices as follows: given a complex K and a set $\Sigma \subseteq K$ of simplices, if the inclusion map $\mathrm{st}_K(\Sigma) \setminus \Sigma \hookrightarrow \mathrm{st}_K(\Sigma)$ is a collapse then the inclusion $K \setminus \Sigma \hookrightarrow K$ is also a collapse.

3 The standard chromatic subdivision of the standard simplex is collapsible

3.1 The standard chromatic subdivision

In this section, we adapt the usual definition of the barycentric subdivision to the colored case. The abstract definition we based ours on is folklore, see for instance [9]. We first begin by introducing a category of graphs that will be used in the following.

Definition 57. We write **Graph** for the category of **graphs** $G = (V_G, E_G)$, with $E_G \subseteq V_G \times V_G$, which are *irreflexive*, i.e. $(x, y) \in E_G$ implies $x \neq y$. A morphism $f : G \to H$ consists of a function $f : V_G \to V_H$ such that for every $(x, y) \in E_G$, we have $(f(x), f(y)) \in E_H$.

A useful construction when studying a simplicial complex is its *poset of faces* which is the poset of non-empty simplices of the complex ordered by inclusion. In Definition 58, we introduce a variant of this definition, that we call the *graph of elements* of the complex (by analogy with the category of elements of a presheaf such as a presimplicial set). We introduce this variant because its generalization to colored simplicial complexes (see Definition 64) naturally gives rise to graphs with cycles (see Example 65) which do not correspond to posets anymore: this also explains the peculiar definition of graphs we consider in Definition 57, which can be seen as a generalization of the notion of poset allowing some cycles.

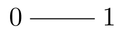
Definition 58. The **graph of elements** $\mathrm{El}(K)$ of a simplicial complex K is the graph whose elements are the non-empty simplices of K and there is an edge $\tau \to \sigma$ whenever $\tau \subsetneq \sigma$. This construction extends to a functor $\mathrm{El} : \mathbf{SC} \to \mathbf{Graph}$.

Definition 59. The **nerve of a graph** $G = (V_G, E_G)$ is the simplicial complex NG whose vertices are those of G and simplices are sets of the form $\{x_1, \ldots, x_n\}$ with an edge $x_i \to x_j$ for any $i < j$. This construction extends to a functor $N : \mathbf{Graph} \to \mathbf{SC}$.

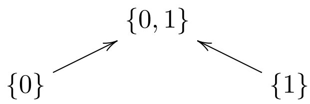
Remark 60. Given a morphism of graphs $f : G \to H$, the fact that f preserves edges ensures that $Nf : NG \to NH$ preserves simplices, and the fact that graphs are supposed to be irreflexive ensures that Nf is locally injective. The functor N is thus well-defined, and this explains why we have chosen to restrict to irreflexive graphs in Definition 57.

Definition 61. The **barycentric subdivision** functor is $\chi = N \circ \mathrm{El}$.

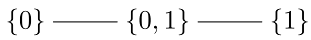
Example 62. Consider Δ^1:



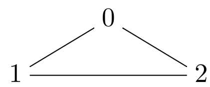
we have $\mathrm{El}(\Delta^1)$:



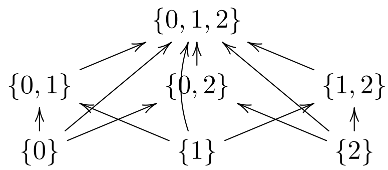
and $\chi(\Delta^1) = N(\mathrm{El}(\Delta^1))$:



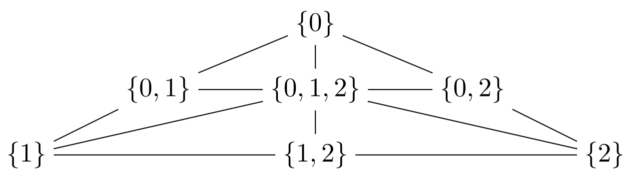
Similarly for Δ^2:



we have $\mathrm{El}(\Delta^2)$:



and $\chi(\Delta^2) = N(\mathrm{El}(\Delta^2))$:



The previous definitions can be adapted to the colored case as follows, which is an abstract reformulation of the usual definition of the standard chromatic subdivision [14]. We define a functor $! : \mathbf{Inj} \to \mathbf{Graph}$ which to a set X associates the graph with X as set of vertices and pairs $(x, y) \in X \times X$ with $x \neq y$ as edges.

Definition 63. The category of **colored graphs** is the slice category $\mathbf{Graph}/\,!\,\mathbb{N}$.

Notice that an object of this category can be seen as a pair (G, ℓ) where G is a graph and $\ell : V_G \to \mathbb{N}$ is a function such that for every edge (x, y) we have $\ell(x) \neq \ell(y)$.

Definition 64. We can define a functor

$$\mathrm{El} \quad : \quad \mathbf{SC}/\,!\,\mathbb{N} \quad \to \quad \mathbf{Graph}/\,!\,\mathbb{N}$$

which to every simplicial complex K associates the graph whose vertices are pairs (σ, i) where $\sigma \in K$ and $i \in \ell_K(\sigma)$, labeled by i, and there is an edge $(\tau, i) \to (\sigma, j)$ whenever

1. $i \neq j$,

2. $\tau \subseteq \sigma$,

3. $j \notin \ell_K(\tau)$ or $\sigma \subseteq \tau$.

In the other direction, we can define a functor

$$N \quad : \quad \mathbf{Graph}/!\,\mathbb{N} \quad \to \quad \mathbf{SC}/!\,\mathbb{N}$$

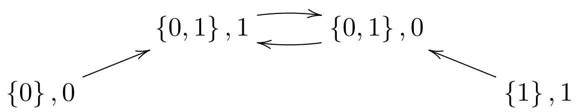
which to a colored graph (G, ℓ) associates the simplicial complex with V_G as set of vertices, labeled by ℓ, simplices being sets of the form $\{x_1, \ldots, x_n\}$ such that there is an edge $x_i \to x_j$ whenever $i < j$. The **standard chromatic subdivision** functor is

$$\chi \quad = \quad N \circ \mathrm{El} \quad : \quad \mathbf{CSC} \quad \to \quad \mathbf{CSC}$$

Example 65. Consider the labeled complex Δ^1:

$$0 \text{ --- } 1$$

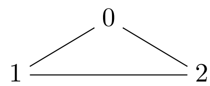
we have $\mathrm{El}(\Delta^1)$:



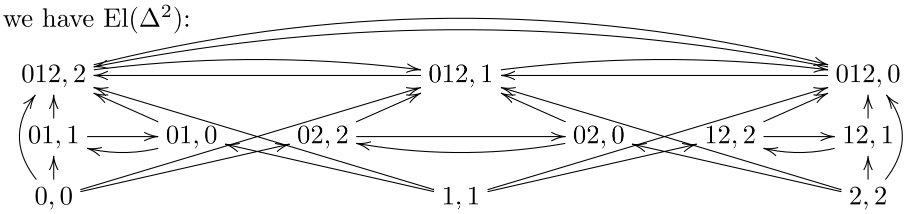
and $\chi(\Delta^1) = N(\mathrm{El}(\Delta^1))$:

$$\{0\}, 0 \text{ --- } \{0,1\}, 1 \text{ --- } \{0,1\}, 0 \text{ --- } \{1\}, 1$$

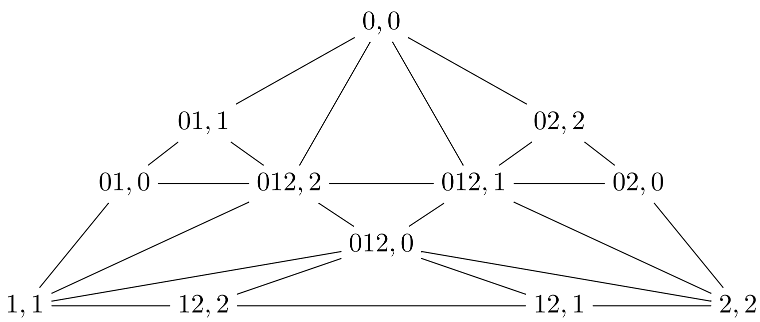
Consider the labeled complex Δ^2:



we have $\mathrm{El}(\Delta^2)$:



and $\chi(\Delta^2) = N(\mathrm{El}(\Delta^2))$ is



3.2 The basic chromatic subdivision of the standard simplex

Before showing the contractibility of the standard chromatic subdivision of the standard simplicial complex, we first investigate a simpler complex as a toy example: the colored join

$$K^I \quad = \quad \partial\Delta^I \star \Delta^I$$

where $I \subseteq \mathbb{N}$ is a finite set, which we call the *basic chromatic subdivision* following [4]. We write $n = \dim I$. The simplices of K^I are of the form $\sigma|\tau$ with $\sigma, \tau \subseteq I$ such that

1. $\sigma \neq I$ (by definition of $\partial\Delta^I$)
2. $\sigma \cap \tau = \emptyset$ (by definition of the colored join)

Given an integer n, we sometimes write K^n instead of $K^{[n]}$. The simplex $\emptyset|I$ is called the *central simplex* of K^I.

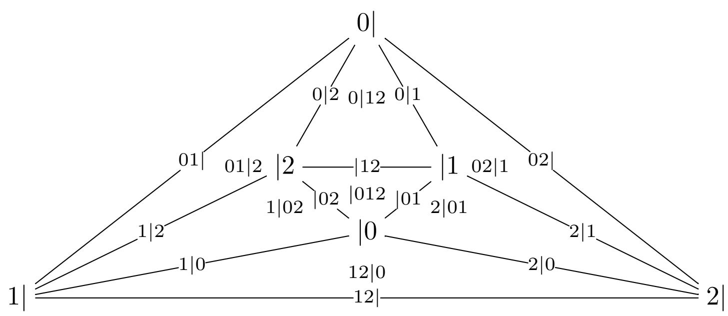
Example 66. K^0 is

$$|0$$

(we sometimes write $|\sigma$ instead of $\emptyset|\tau$), K^1 is

$$0| \xrightarrow{\;0|1\;} |1 \xrightarrow{\;|01\;} |0 \xrightarrow{\;1|0\;} 1|$$

and K^2 is



Lemma 67. *The canonical inclusion*

$$\begin{array}{ccc} \Delta^I & \hookrightarrow & \partial\Delta^I \star \Delta^I = K^I \\ \sigma & \mapsto & \emptyset|\sigma \end{array}$$

is a collapse.

Proof. Given $k \geq 0$, we write

$$\Sigma_k \quad = \quad \left\{\sigma|\emptyset \in K^I \;\middle|\; \dim\sigma = k\right\}$$

and

$$K^I_k \quad = \quad K^I \setminus \bigcup_{k' \geq k} \Sigma_{k'}$$

i.e.

$$K^I_k \quad = \quad \begin{cases} K^I & \text{when } k > n \\ K^I_{k+1} \setminus \Sigma_k & \text{when } 0 \leq k \leq n \end{cases}$$

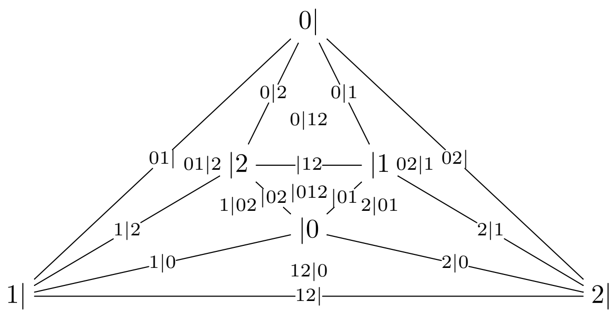
Notice that K^I_k is K^I restricted to simplices $\sigma|\tau$ satisfying $\dim \sigma < k$. A simplex $\sigma|\emptyset \in \Sigma_k$ is therefore a free face of $\sigma|\tau$ in K^I_{k+1} with $\tau = I \setminus \sigma$ and we thus have a collapse $K^I_k = K^I_{k+1} \setminus \Sigma_k \hookrightarrow K^I_{k+1}$ by Proposition 51. We constructed a sequence of collapses

$$K^I_0 \quad \hookrightarrow \quad K^I_1 \quad \hookrightarrow \quad \dots \quad \hookrightarrow \quad K^I_{n-1} \quad \hookrightarrow \quad K^I_n = K^I$$

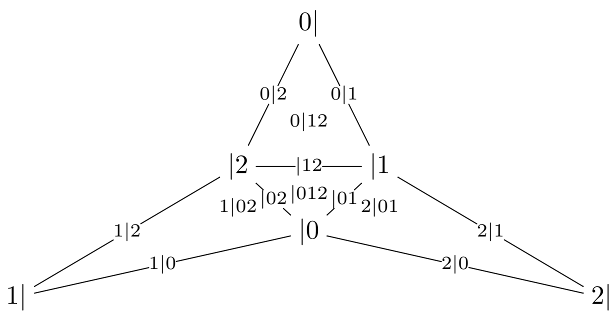
Moreover, a simplex of K^I_0 is a simplex $\sigma|\tau$ of K^I with $\dim \sigma < 0$. The simplices of K^I_0 are thus of the form $\emptyset|\tau$ with $\tau \in \Delta^I$. □

Example 68. The collapse of K^2 onto Δ^2 goes as follows:

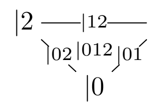
$K^2 = K^2_2 \quad =$



$K^2_1 \quad =$



$\Delta^2 = K^2_0 \quad =$



Corollary 69. *The simplex K^I is collapsible.*

Proof. We have shown in previous lemma that K^I collapses to Δ^I, and Δ^I is collapsible by Lemma 50. □

An interesting remark one can make using the same ideas on those simplicial complexes is that after removing the central simplex $\emptyset|I$, the complex contracts to its boundary:

Lemma 70. *The canonical inclusion*

$$\begin{array}{ccc} \partial\Delta^I & \hookrightarrow & \left(\partial\Delta^I \star \Delta^I\right) \setminus \emptyset|I = K^I \setminus \emptyset|I \\ \sigma & \mapsto & \sigma|\emptyset \end{array}$$

is a collapse.

Proof. Given $k \geq 0$, we write

$$T_k \quad = \quad \left\{ \emptyset|\tau \in K^I \;\middle|\; \dim \tau = k \right\}$$

and

$$K^I_k \quad = \quad K^I \setminus \bigcup_{k' \geq k} T_{k'}$$

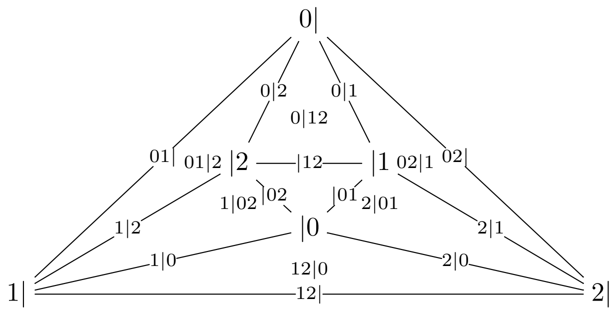
Notice that K^I_k is K^I restricted to simplices $\sigma|\tau$ satisfying $\dim \tau < k$. In particular, $K^I_n = K^I \setminus \emptyset|I$. Given $0 \leq k < n$, a simplex $\emptyset|\tau \in T_k$ is a free face of $\sigma|\tau$ in K^I_{k+1} with $\sigma = I \setminus \tau$ and we have a collapse $K^I_k = K^I_{k+1} \setminus T_k \hookrightarrow K^I_{k+1}$ by Proposition 51. We thus get a sequence of collapses

$$K^I_0 \quad \hookrightarrow \quad K^I_1 \quad \hookrightarrow \quad \ldots \quad \hookrightarrow \quad K^I_{n-1} \quad \hookrightarrow \quad K^I_n = K^I \setminus \emptyset|I$$

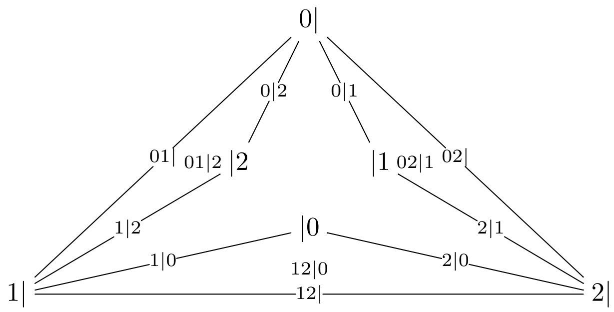
Finally, K^I_0 is the restriction of K^I to simplices of the form $\sigma|\emptyset$ with $\sigma \in \partial\Delta^I$. $\square$

Example 71. The collapse of $K^2 \setminus \emptyset|012$ onto $\partial\Delta^2$ goes as follows:

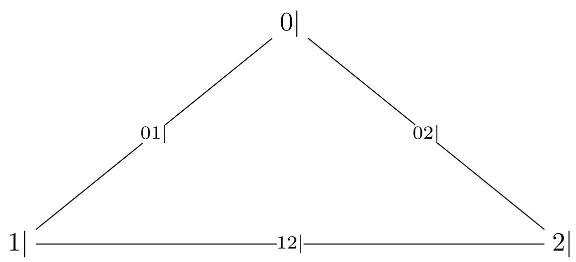
$K^2_2 \quad =$



$K^2_1 \quad =$



$\partial\Delta^2 = K^2_0 \quad =$



Finally, we can remark that we can mimic elementary collapse steps on this construction.

Lemma 72. *Given $p \in I$, the canonical inclusion*

$$\begin{array}{ccc} \Lambda^I_p & \hookrightarrow & \partial\Delta^I \star \Delta^I = K^I \\ \sigma & \mapsto & \sigma|\emptyset \end{array}$$

is a collapse.

Proof. We first remark that, writing $I' = I \setminus \{p\}$, the simplex $I'|\emptyset$ is a free face of $I'|\{p\}$. We thus have a collapse step

$$K' \quad \hookrightarrow \quad K^I$$

with $K' = K^I \setminus (I'|\emptyset)$. We write

$$\Sigma_k \quad = \quad \{\sigma|p \in K' \mid \dim\sigma = k\}$$

and

$$K'_k \quad = \quad K' \setminus \bigcup_{k' \geq k} \Sigma_{k'}$$

i.e. K'_k is K' restricted to simplices $\sigma|\tau$ such that $p \in \tau$ implies $\dim\sigma < k$. Notice that $K'_k = K'$ for $k \geq n-1$. Moreover, given $-1 \leq k < n-1$ and $\sigma|p \in \Sigma_k$, the simplex $\sigma|p$ is a free face of $\sigma|(I \setminus \sigma)$ in K'_{k+1} and we have a collapse $K'_k = K'_{k+1} \setminus \Sigma_k \hookrightarrow K'_{k+1}$ by Proposition 51. We have thus constructed a sequence of collapses

$$K'_{-1} \quad \hookrightarrow \quad K'_0 \quad \hookrightarrow \quad \ldots \quad \hookrightarrow \quad K'_{n-1} = K'$$

Notice that $K'_{-1} = K \setminus \{\emptyset|p, I'|\emptyset\}$. Finally, we write

$$T_k \quad = \quad \left\{\emptyset|\tau \in K'_{-1} \;\middle|\; \dim\tau = k\right\}$$

and

$$K''_k \quad = \quad K'_{-1} \setminus \bigcup_{k' \geq k} T_{k'}$$

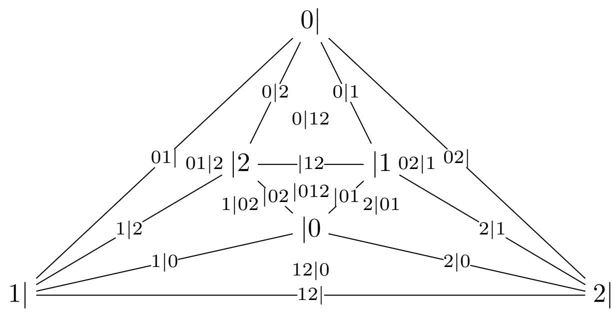
Notice that $K''_k = K'_{-1}$ for $k \geq n$ and more generally, K''_k consists of simplices $\sigma|\tau \in K'$ such that $p \notin \tau$ and $\dim\tau < k$. Moreover, given $0 \leq k < n$ and $\emptyset|\tau \in T_k$, the simplex $\emptyset|\tau$ is a free face of $(I \setminus \tau)|\tau$ in K''_{k+1}, and we have a collapse $K''_k = K''_{k+1} \setminus T_k \hookrightarrow K''_{k+1}$ by Proposition 51. We thus have constructed a sequence of collapses

$$\Lambda^I_p = K''_0 \quad \hookrightarrow \quad K''_1 \quad \hookrightarrow \quad \ldots \quad \hookrightarrow \quad K''_n = K'_{-1}$$

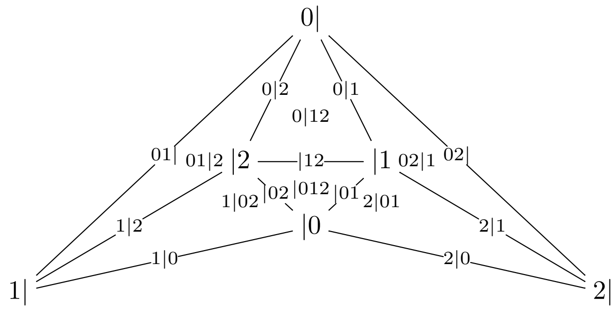
By the preceding remark, the simplices $\sigma|\tau \in K''_0$ are those in K^I such that $\tau = \emptyset$ and $\sigma \neq I'$, thus justifying the equality $\Lambda^I_p = K''_0$. ∎

Example 73. The collapse of K^2 onto $\Lambda_0^{[2]}$ goes as follows.

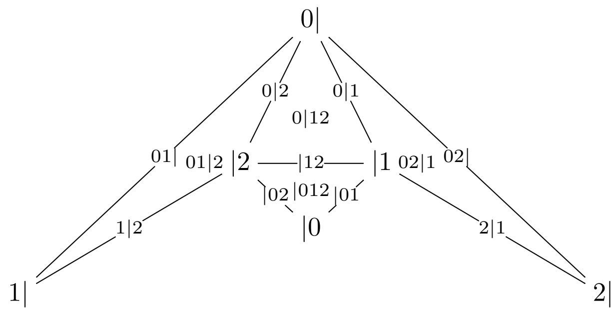
$K^2 \quad =$



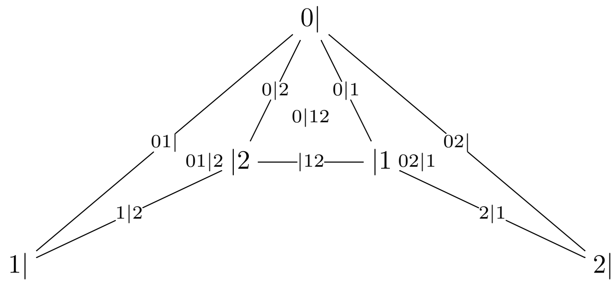
$K' = K'_1 \quad =$



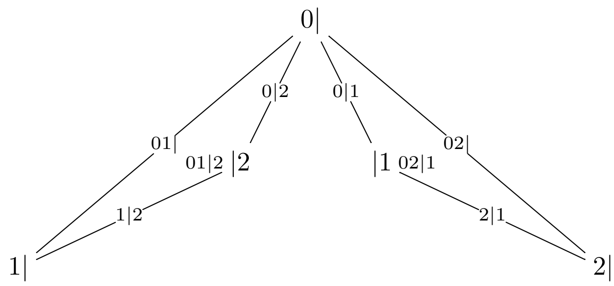
$K'_0 \quad =$



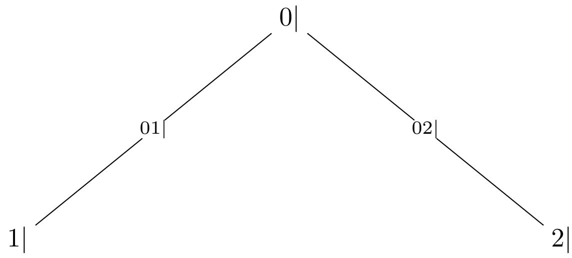
$K'_{-1} = K''_2 \quad =$



$K'_1 \quad =$



$\Lambda_0^{[2]} = K'_0 \quad =$



3.3 Joins and simple homotopy

In this section, we show a bunch of useful lemmas relating joins and free faces. For simplicity, we state them in the non-colored case, but those extend straightforwardly to the colored case, using Remark 40, when we suppose that the colored complexes over which the join is taken have disjoint sets of colors, which will be the case in our applications. We suppose that K is a simplicial complex and $I \subseteq \mathbb{N}$ a finite set. We omit proofs when they are immediate.

Lemma 74. *Given $\sigma \in K$, we have*

$$\left(K \star \Delta^I\right) \setminus (\sigma|\emptyset) \quad = \quad (K \setminus \sigma) \star \Delta^I$$

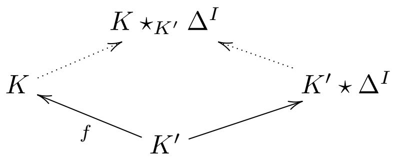
and the canonical inclusions into $K \star \Delta^I$ coincide.

Proof. We have $K \star \Delta^I = \left\{\sigma|\tau \;\middle|\; \sigma \in K \text{ and } \tau \in \Delta^I\right\}$. Thus,

$$\begin{aligned}
\left(K \star \Delta^I\right) \setminus (\sigma|\emptyset) &= \left\{\sigma'|\tau \;\middle|\; \sigma' \in K \text{ and } \tau \in \Delta^I \text{ and } \sigma|\emptyset \not\subseteq \sigma'|\tau\right\} \\
&= \left\{\sigma'|\tau \;\middle|\; \sigma' \in K \text{ and } \tau \in \Delta^I \text{ and } \sigma \not\subseteq \sigma'\right\} \\
&= \left\{\sigma'|\tau \;\middle|\; \sigma' \in K \setminus \sigma \text{ and } \tau \in \Delta^I\right\} \\
&= (K \setminus \sigma) \star \Delta^I
\end{aligned}$$

□

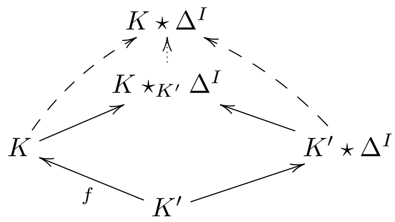
Definition 75. Given a morphism $f : K' \to K$, we write $K \star_{K'} \Delta^I$ for the complex defined by the pushout



where the map $K' \hookrightarrow K' \star \Delta^I$ is the canonical inclusion map $\sigma \mapsto \sigma|\emptyset$ given by Remark 17. The morphism f is often clear from the context, which is why we did not include it in the notation. The universal morphism

$$K \star_{K'} \Delta^I \quad \to \quad K \star \Delta^I$$

given by the inclusion $K \hookrightarrow K \star \Delta^I$ (see Remark 17) and $f \star \Delta^I : K' \star \Delta^I \to K \star \Delta^I$ is called the *canonical inclusion*:



Lemma 76. *In the previous definition, when $f : K' \hookrightarrow K$ is an inclusion, the simplex $K \star_{K'} \Delta^I$ is the subcomplex of $K \star L$ whose simplices are*

$$K \star_{K'} \Delta^I \quad = \quad \{\sigma|\tau \in K \star L \mid \sigma \in K' \text{ or } \tau = \emptyset\}$$

Lemma 77. *Given $\sigma \in K$, we have*

$$\left(K \star \Delta^I\right) \setminus \{\sigma|\,\{i\} \mid i \in I\} \quad = \quad K \star_{K \setminus \sigma} \Delta^I$$

(where $f : K \setminus \sigma \hookrightarrow K$ is the canonical inclusion, see Remark 22) and the canonical inclusions into $K \star \Delta^I$ coincide.

Proof. We have

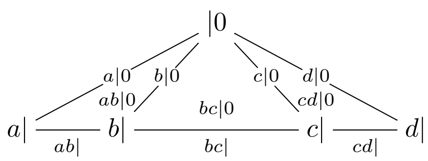
$$
\begin{aligned}
&(K \star \Delta^I) \setminus \{\sigma|\,\{i\} \mid i \in I\} \\
&\qquad = \left\{\sigma'|\tau \;\middle|\; \sigma' \in K \text{ and } \tau \in \Delta^I \text{ and } \forall i \in I, \sigma|\,\{i\} \not\subseteq \sigma'|\tau\right\} \\
&\qquad = \left\{\sigma'|\tau \;\middle|\; \sigma' \in K \text{ and } \tau \in \Delta^I \text{ and } \forall i \in I, (\sigma \not\subseteq \sigma' \text{ or } \{i\} \not\subseteq \tau)\right\} \\
&\qquad = \left\{\sigma'|\tau \;\middle|\; \sigma' \in K \text{ and } \tau \in \Delta^I \text{ and } (\sigma \not\subseteq \sigma' \text{ or } \forall i \in I, \{i\} \not\subseteq \tau)\right\} \\
&\qquad = \left\{\sigma'|\tau \;\middle|\; \sigma' \in K \text{ and } \tau \in \Delta^I \text{ and } (\sigma \not\subseteq \sigma' \text{ or } \tau = \emptyset)\right\} \\
&\qquad = K \star_{K \setminus \sigma} \Delta^I
\end{aligned}
$$

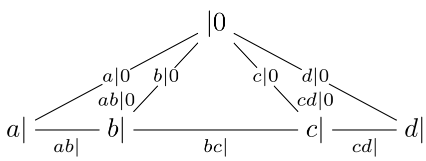
□

Example 78. Consider the complex K^1 of Example 66:

$$a \overset{ab}{\text{———}} b \overset{bc}{\text{———}} c \overset{cd}{\text{———}} d$$

and $\sigma = bc$. Then $K^1 \star \Delta^0$ is pictured on the left and $(K^1 \star \Delta^0) \setminus (\sigma|I) = K^1 \star_{K \setminus \sigma} \Delta^0$ is pictured on the right:





The following lemma looks like a particular case of the previous one, which it is not because $K \setminus \emptyset$ is not defined (see Remark 19).

Lemma 79. *We have*

$$\left(K \star \Delta^I\right) \setminus (\emptyset|I) \quad = \quad K \star (\Delta^I \setminus I) \quad = \quad K \star \partial\Delta^I$$

Proof. The second equality is simply by definition of $\partial\Delta^I$. The first one is shown as follows:

$$\begin{aligned}
\left(K \star \Delta^I\right) \setminus (\emptyset|I) &= \left\{\sigma|\tau \in K \star \Delta^I \;\middle|\; \sigma \in K \text{ and } \tau \in \Delta^I \text{ and } \emptyset|I \not\subseteq \sigma|\tau\right\} \\
&= \left\{\sigma|\tau \in K \star \Delta^I \;\middle|\; \sigma \in K \text{ and } \tau \in \Delta^I \text{ and } I \not\subseteq \tau\right\} \\
&= \left\{\sigma|\tau \in K \star \Delta^I \;\middle|\; \sigma \in K \text{ and } \tau \in \Delta^I \setminus I\right\} \\
&= K \star \left(\Delta^I \setminus I\right)
\end{aligned}$$

□

From the following simple proposition will follow lemmas which will be useful when combining free faces and join operation.

Proposition 80. *Given two simplicial complexes K and L, the following properties hold.*

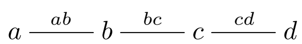
1. *If σ' is a free face of σ in K and τ' is a free face of τ in L then $\sigma'|\tau'$ is a free face of $\sigma|\tau$ in $K \star L$.*
2. *If σ is a maximal face in K and τ' is a free face of τ in L then $\sigma|\tau'$ is a free face of $\sigma|\tau$ in $K \star L$.*
3. *If σ' is a free face of σ in K and τ is a maximal face of L then $\sigma'|\tau$ is a free face of $\sigma|\tau$ in $K \star L$.*

Proof. Immediate from Remark 11. □

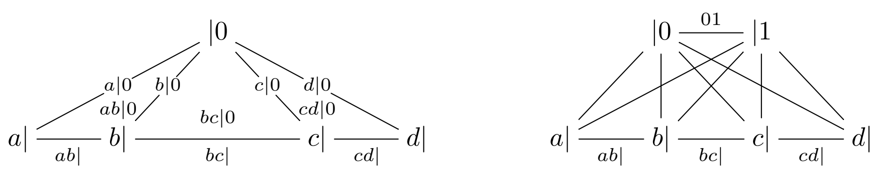
Lemma 81. *Suppose that σ is a maximal face in K. Then $\sigma|\emptyset$ is a free face of $\sigma|I$ in $K \star \Delta^I$.*

Proof. Corollary of Proposition 80 (property 2). □

Example 82. Consider the complex K^1 of Example 66:



where bc is a maximal face. Then $bc|$ is free face of $bc|0$ in $K^1 \star \Delta^0$ and of $bc|01$ in $K^1 \star \Delta^1$:

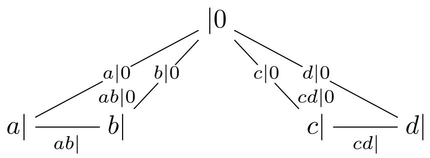


Lemma 83. *Suppose that τ is a free face of σ in K. Then, given $J \subseteq I$, $\tau|J$ is a free face of $\sigma|I$ in $K \star \Delta^I$.*

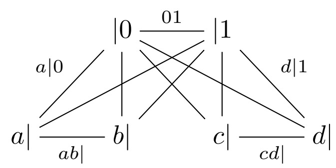
Proof. Corollary of Proposition 80 (property 1). □

Example 84. Consider the following complex $K = K^1 \setminus bc$:

where b and c are free faces of ab and cd respectively. Then $b|$ and $c|$ are free faces of $ab|0$ and $bc|0$ respectively in $K \star \Delta^1$:



and $b|$ and $c|$ are free faces of $ab|01$ and $bc|01$ respectively in $K \star \Delta^2$:



Lemma 85. *Suppose that K collapses to K'. Then $K \star \Delta^I$ collapses to $K' \star \Delta^I$. More precisely, given a collapse*

$$f \quad : \quad K' \quad \hookrightarrow \quad K$$

the morphism

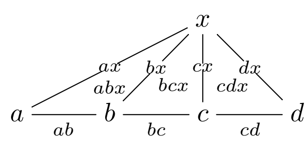
$$f \star \Delta^I \quad : \quad K' \star \Delta^I \quad \hookrightarrow \quad K \star \Delta^I$$

is also a collapse.

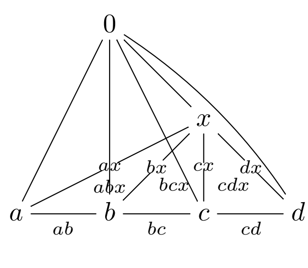
Proof. The collapsing f can be decomposed in a sequence of collapsing steps, it is therefore enough to handle the case where $K' = K \setminus \tau$ where τ is a free face of σ in K. By Lemma 83, $\tau|\emptyset$ is a free face of $\sigma|I$ in $K \star \Delta^I$. Therefore the inclusion $\left(K \star \Delta^I\right) \setminus (\tau|\emptyset) = (K \setminus \tau) \star \Delta^I \hookrightarrow K \star \Delta^I$ is a collapse by Lemma 74. ☐

Remark 86. As mentioned in the introduction of the section, in the colored case, we have to suppose that $\ell_K(\underline{K}) \cap I = \emptyset$.

Example 87. Consider the following complex K:



where bc is a free face of bcx. Then $bc|0$ is a free face of $bcx|0$ in $K \star \Delta^0$:



Lemma 88. *Suppose that $K' \hookrightarrow K$ is a collapse. Then the inclusion*

$$K \star_{K'} \Delta^I \quad \hookrightarrow \quad K \star \Delta^I$$

is also a collapse.

Proof. We can suppose that $K' = K \setminus \tau$ where τ is a free face of σ in K. We fix an enumeration of $I = \{i_1, \ldots, i_n\}$ and write $I_k = \{i_1, \ldots, i_k\}$. Suppose given k such that $1 \leq k < n$. Similarly to Proposition 80, we have that $\tau| \{i_k\}$ is a free face of $\sigma|I_k$ in $K \star \Delta^I \setminus \{\tau| \{i_{k+1}\}, \ldots, \tau| \{i_n\}\}$. Writing $T_k = \{\tau| \{i_{k+1}\}, \ldots, \tau| \{i_n\}\}$, we thus have constructed a sequence of collapse steps

$$K \star \Delta^I \setminus T_0 \quad \hookrightarrow \quad K \star \Delta^I \setminus T_1 \quad \hookrightarrow \quad \ldots \quad \hookrightarrow \quad K \star \Delta^I \setminus T_n = K \star \Delta^I$$

and we have $K \star_{K'} \Delta^I = K \star \Delta^I \setminus T_0$ by Lemma 77. □

3.4 Collapsibility of $\chi(\Delta^n)$

We now consider the complex $\chi(\Delta^I)$ with $I \subseteq \mathbb{N}$ finite and write $n = \dim I$. We recall that the standard chromatic subdivision $\chi(\Delta^I)$ of Δ^I is defined as follows (we are just paraphrasing the definition given in Section 3.1 applied to the complex Δ^I).

Definition 89. The **standard chromatic subdivision** $\chi(\Delta^I)$ of the standard I-simplicial complex Δ^I is the simplicial complex whose vertices are pairs (V, i) with $V \subseteq I$ and $i \in V$ and simplices are sets

$$\sigma \quad = \quad \{(V_0, i_0), \ldots, (V_d, i_d)\}$$

with $d \geq -1$ ($\sigma = \emptyset$ when $d = -1$) which are

1. *well-colored*: for every $k, l \in [d]$,

$$i_k = i_l \qquad \text{implies} \qquad k = l$$

2. *ordered*: for every $k, l \in [d]$,

$$V_k \subseteq V_l \qquad \text{or} \qquad V_l \subseteq V_k$$

3. *transitive*: for every $k, l \in [d]$,

$$i_l \in V_k \qquad \text{implies} \qquad V_l \subseteq V_k$$

This complex is colored via the second projection: $\ell(V, i) = i$.

Remark 90. In [16], Kozlov shows that the cells of the complex are in bijection with pairs of sequences

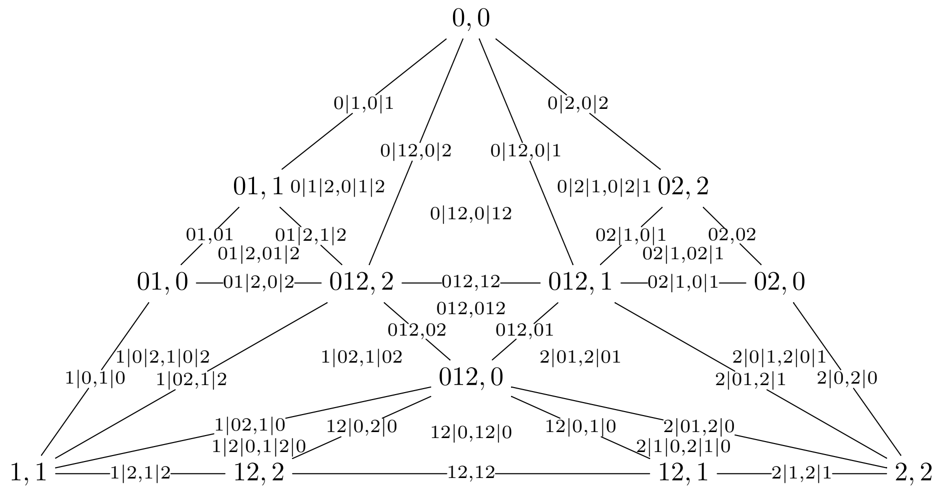
$$\sigma \quad = \quad B_0| \ldots |B_p, C_0| \ldots |C_p$$

with $B_i, C_i \subseteq I$ such that

- $\forall i, j \in [p], B_i \cap B_j = \emptyset$

- $\forall i \in [p], C_i \neq \emptyset$
- $\forall i \in [p], C_i \subseteq B_i$

The dimension of such a cell is $\#C_0 + \ldots + \#C_p - 1$. In the following, we sometimes use this notation since it makes it easy to draw planar simplicial complexes of dimension 2, and the notation for vertices coincides with the one of Definition 89. However, none of the following proofs rely on it. For instance $\chi(\Delta^2)$ is



By direct inspection of Definition 89, we have:

Lemma 91. *Given two finite sets I and J such that $J \subseteq I$, we have a canonical inclusion*

$$\chi(\Delta^J) \quad \hookrightarrow \quad \chi(\Delta^I)$$

Maximal simplices are those of dimension n. The maximal simplex

$$\{I\} \times I \quad = \quad \{(I,0),(I,1),\ldots,(I,n)\}$$

is called the **central simplex**. Notice that more generally, there is a canonical inclusion

$$\begin{array}{ccc} \Delta^I & \hookrightarrow & \chi(\Delta^I) \\ \sigma & \mapsto & \{I\} \times \sigma \end{array} \tag{3}$$

and we define the sets of **central k-simplices** by

$$\Sigma^I_k \quad = \quad \big\{\{I\} \times \sigma \;\big|\; \sigma \in \Delta^I \quad \text{and} \quad \dim \sigma = k\big\}$$

indexed by $k \geq 0$, and write

$$\Sigma^I \quad = \quad \bigcup_{k \geq 0} \Sigma^I_k$$

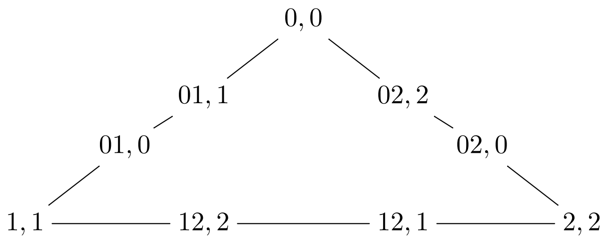
i.e. given a simplex $\sigma \in \Sigma^I$, $(V,i) \in \sigma$ implies $V = I$.

Remark 92. Notice that $\emptyset \notin \Sigma^I$ when $I \neq \emptyset$.

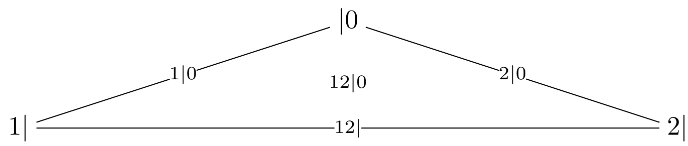
Definition 93. The **boundary** $\partial\chi(\Delta^I)$ of the complex Δ^I is the complex

$$\partial\chi(\Delta^I) \quad = \quad \chi(\Delta^I) \setminus \Sigma^I$$

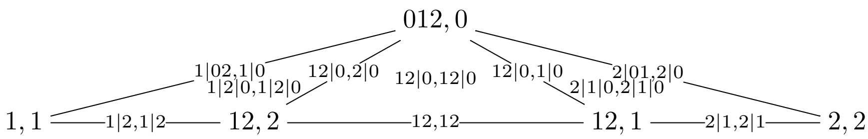
Example 94. $\partial\chi(\Delta^2)$ is



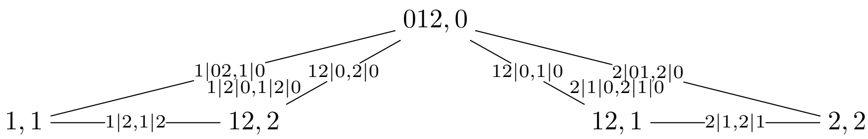
Now, we would like to reason as in the previous section, but things are more complicated because the complex is more subdivided. For instance, the following subcomplex of K^2 (see Example 66)



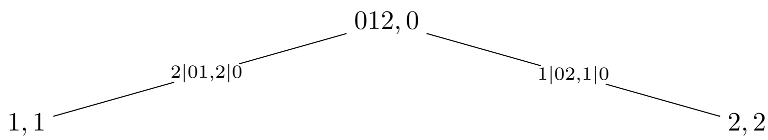
has been replaced by



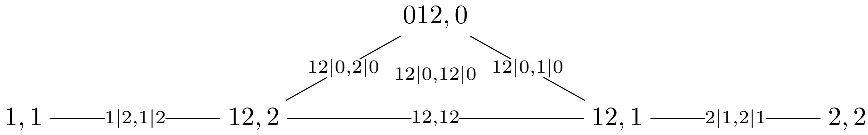
which is more subdivided. In the first complex, 12| is a free face of 12|0, and we have a collapse by removing the simplex 12|. In the second one, the face 12| has been subdivided. However, we can still simulate the previous collapse in multiple steps as follows:



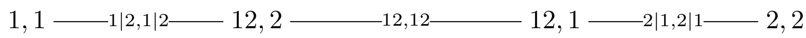
and then



(notice that we first remove the face corresponding to the central simplex, then the faces corresponding to the faces of the central simplex). In the first one, we can also remove the free face $|0$, and again this can be done in multiple steps in the second one:



and then



(notice that this time, we first remove the faces corresponding to the exterior of the lower subsimplex). In order to show that $\chi(\Delta^I)$ is collapsible, we are thus going to proceed essentially as in Section 3.2 (the proofs of Lemma 67 and Corollary 69 in particular), simulating each collapse step by a sequence of multiple collapse steps.

The following lemma shows that $\chi(\Delta^I)$ is a subcomplex of the join $\partial\chi(\Delta^I)\star\Delta^I$. It will enable us to use theorems about joins to show results concerning the subdivision.

Lemma 95. *The map defined by*

$$\phi \quad : \quad \chi(\Delta^I) \quad \hookrightarrow \quad \partial\chi(\Delta^I) \star \Delta^I$$

which to every simplex $\upsilon = \{(V_1, i_1), \ldots, (V_k, i_k)\}$ of $\chi(\Delta^I)$ associates the simplex $\sigma|\tau$ of $\partial\chi(\Delta^I) \star \Delta^I$, with

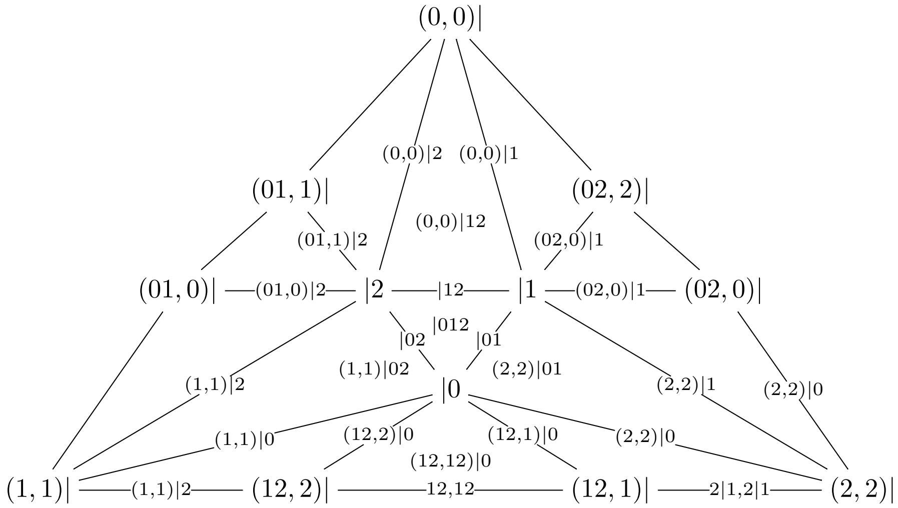
$$\sigma = \{(V, i) \in \upsilon \mid V \neq I\} \qquad \textit{and} \qquad \tau = \{i \in I \mid (I, i) \in \upsilon\}$$

is well-defined and injective. The simplices in its image are those of the form $\sigma|\tau$ satisfying

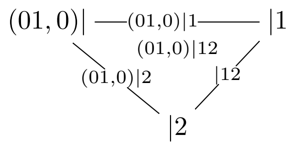
$$\forall (V, i) \in \sigma, \qquad V \subseteq I \setminus \tau \tag{4}$$

Proof. Consider the image $\sigma|\tau = \phi(\upsilon)$. The complex $\partial\chi(\Delta^I) = \chi(\Delta^I) \setminus \Sigma^I$ is obtained by restricting $\chi(\Delta^I)$ to simplices of the form $\{(V_1, i_1), \ldots, (V_k, i_k)\}$ with $V_p \neq I$ for every index $1 \leq p \leq k$. Since $\sigma \subseteq \upsilon$, we know that σ is a simplex of $\chi(\Delta^I)$, and thus of $\partial\chi(\Delta^I)$ by the preceding remark. By the well-coloring property of Definition 89, the sets of colors of σ and τ are disjoint. The map ϕ is thus well defined, and it is clearly injective. Moreover, its images satisfy condition (4) because of transitivity property of Definition 89. Conversely, consider a simplex $\sigma|\tau \in \partial\chi(\Delta^I) \star \Delta^I$ satisfying (4), and write $\upsilon = \sigma \cup (\{I\} \times \tau)$. It can be checked that υ satisfies the conditions of Definition 89: it is well-colored because the colors of σ and τ are disjoint (by definition of the colored join), it is ordered because σ is and for every $(V, i) \in \sigma$ we have $V \subseteq I$, and finally it is transitive because of (4). Finally, we have $\sigma|\tau = \phi(\upsilon)$. □

Example 96. The simplicial complex $\chi(\Delta^2)$ can be seen as the following subcomplex of $\partial\chi(\Delta^2) \star \Delta^2$:



(in order for the figure to be readable, we did not write the name for all simplices). Notice that this embedding is not full. For instance, the simplex



is present in $\partial\chi(\Delta^2) \star \Delta^2$ but not in the embedding shown above.

Remark 97. The fact that the map given by Lemma 95 is not full, as illustrated in previous example, is due to the fact that we restricted to *transitive* simplices in the standard chromatic subdivision (in Definition 89, which corresponds to condition 3 of Definition 64): without this condition, the map would be an isomorphism. The subdivision without the transitivity condition is the one we would get if we considered layered protocols which are not immediate snapshot [14]. This case is briefly discussed in Section 5.

Lemma 98. *The star of* Σ^I *in* Δ^I *satisfies*

$$\mathrm{st}(\Sigma^I) \quad = \quad \chi(\Delta^I)$$

Proof. By Lemma 95, $\chi(\Delta^I)$ can be seen as a subcomplex of $\partial\chi(\Delta^I) \star \Delta^I$, and its simplices can be written in the form $\sigma|\tau$ with $\sigma \in \partial\chi(\Delta^I)$ and $\tau \in \Delta^I$. Suppose given such a simplex $\sigma|\tau$. If $\tau \neq \emptyset$ then $\sigma|\tau = (\sigma|\tau) \cup (\emptyset|\tau)$ with $\emptyset|\tau \in \Sigma^I$ and therefore $\sigma|\tau \in \mathrm{st}(\Sigma^I)$. Now, suppose that $\tau = \emptyset$: Remark 92 shows that we cannot conclude as before. By Definition 89, the simplex σ is of the form $\sigma = \{(V_0, i_0), \ldots, (V_d, i_d)\}$ with $V_i \subseteq V_{i+1}$ for $i \in [d-1]$, and $V_d \neq I$ by Definition 93. Therefore, writing $\tau' = I \setminus V_d \neq \emptyset$, the simplex $\sigma|\tau'$ is in $\chi(\Delta^I)$ by the characterization (4) of Lemma 95. Since $\tau' \neq \emptyset$, we have seen that $\sigma|\tau' \in \mathrm{st}(\Sigma^I)$, and therefore its face $\sigma|\emptyset$ is also in $\chi(\Sigma^I)$. $\square$

Theorem 99. *The complex* $\chi(\Delta^I)$ *satisfies the following properties. We write* J *for a finite subset of* $\mathbb{N}$ *satisfying* $I \cap J = \emptyset$.

1. *The inclusion*

$$\partial\chi(\Delta^I) = \chi(\Delta^I) \setminus \Sigma^I \quad \hookrightarrow \quad \chi(\Delta^I) \setminus \Sigma^I_n$$

is a collapse.

2. *The inclusion*

$$\partial\chi(\Delta^I) \star \Delta^J \quad \hookrightarrow \quad \chi(\Delta^I) \star \Delta^J$$

is a collapse.

3. *The inclusion*

$$\Delta^I \quad \hookrightarrow \quad \chi(\Delta^I)$$

is a collapse.

4. *The inclusion*

$$\chi(\Delta^I) \star \partial\Delta^J \quad \hookrightarrow \quad \chi(\Delta^I) \star \Delta^J$$

is a collapse.

5. $\chi(\Delta^I)$ *is collapsible.*

Proof. By induction over the integer $n = \dim I$. The base case with $n = 1$ is left to the reader. Consider $\chi(\Delta^I)$ with $n = \dim I$ and suppose that all the five properties hold up to dimension $n - 1$.

1. We write

$$K^I_k \quad = \quad \chi(\Delta^I) \setminus \bigcup_{k' \geq k} \Sigma^I_{k'}$$

for $0 \leq k \leq n$. Now, fix k such that $0 \leq k < n$. By Lemma 95, the simplices of Σ^I_k and $\chi(\Delta^I)$, and thus of K^I_k, can be seen as elements of $\partial\chi(\Delta^I) \star \Delta^I$, and are thus of the form $\sigma|\tau$ with $\sigma \in \partial\chi(\Delta^I)$ and $\tau \in \Delta^I$. In particular, the simplices in Σ^I_k are of the form $\emptyset|\tau$ with $\tau \subseteq I$ such that $\dim\tau = k$, and the complex K^I_k is obtained from $\chi(\Delta^I)$ by restricting to simplices of the form $\sigma|\tau$ with $\dim\tau < k$. Now, fix a cell $\emptyset|\tau \in \Sigma^I_k$, and consider its associated star $\mathrm{st}(\emptyset|\tau)$ in the simplicial complex K^I_{k+1}. The simplices of $\mathrm{st}(\emptyset|\tau)$ are of the form $\sigma|\tau'$ with $\tau' \subseteq \tau$ and σ such that $\sigma|\tau \in K^I_{k+1}$, i.e. $\sigma \in \chi\left(\Delta^{I\setminus\tau}\right)$. We have therefore shown $\mathrm{st}(\emptyset|\tau) = \chi\left(\Delta^{I\setminus\tau}\right) \star \Delta^\tau$. By induction property 4, the inclusion

$$\mathrm{st}(\emptyset|\tau) \setminus (\emptyset|\tau) = \chi\left(\Delta^{I\setminus\tau}\right) \star \partial\Delta^\tau \quad \hookrightarrow \quad \chi\left(\Delta^{I\setminus\tau}\right) \star \Delta^\tau = \mathrm{st}(\emptyset|\tau)$$

is a collapse, the first equality being shown in Lemma 79. Therefore, the inclusion

$$K^I_{k+1} \setminus (\emptyset|\tau) \quad \hookrightarrow \quad K^I_{k+1}$$

is also a collapse by Proposition 55 and Definition 26 of the star of a cell. Since the above reasoning holds for any simplex $\emptyset|\tau \in \Sigma^I_k$, we thus have a collapse

$$K^I_k = K^I_{k+1} \setminus \Sigma^I_k \quad \hookrightarrow \quad K^I_{k+1}$$

since by Proposition 51, all those collapses can be performed at once. We have constructed a sequence of collapses

$$\partial\chi(\Delta^I) = \chi(\Delta^I)\setminus\Sigma^I = K^I_0 \quad \hookrightarrow \quad K^I_1 \quad \hookrightarrow \quad \ldots \quad \hookrightarrow \quad K^I_n = \chi(\Delta^I)\setminus\Sigma^I_n$$

which shows the required property.

2. By property 1, we have a collapse

$$\partial\chi(\Delta^I) \quad \hookrightarrow \quad \chi(\Delta^I) \setminus \Sigma_n^I$$

Therefore, the map

$$\partial\chi(\Delta^I) \star \Delta^J \quad \hookrightarrow \quad \big(\chi(\Delta^I) \setminus \Sigma_n^I\big) \star \Delta^J$$

obtained by applying the functor $- \star \Delta^J$ is also a collapse by Lemma 85 and the fact that $I \cap J = \emptyset$. Finally, we have that $\Sigma_n^I = \{\sigma\}$ where σ is the central face of $\chi(\Delta^I)$. By Lemma 81, since σ is a maximal face in $\chi(\Delta^I)$, $\sigma|\emptyset$ is a free face of $\sigma|J$ in $\chi(\Delta^I) \star \Delta^J$. Therefore the inclusion

$$\big(\chi(\Delta^I) \setminus \sigma\big) \star \Delta^J = \big(\chi(\Delta^I) \star \Delta^J\big) \setminus (\sigma|\emptyset) \quad \hookrightarrow \quad \chi(\Delta^I) \star \Delta^J$$

is a collapse, the first equality being shown in Lemma 74. Composing the two previous collapses, we have shown that the inclusion

$$\partial\chi(\Delta^I) \star \Delta^J \quad \hookrightarrow \quad \chi(\Delta^I) \star \Delta^J$$

is a collapse, as desired.

3. Given $0 \leq k \leq n$, we define

$$\Sigma^{I-k} \quad = \quad \bigcup_{\substack{I' \subseteq I \\ \dim I - \dim I' = k}} \Sigma^{I'}$$

We also write, given $1 \leq k \leq n$,

$$K_k^I \quad = \quad \chi(\Delta^I) \setminus \bigcup_{0 \leq k' \leq k} \Sigma^{I-k'}$$

and by convention $K_0^I = \chi(\Delta^I)$. If we see $\chi(\Delta^I)$ as a subcomplex of $\partial\chi(\Delta^I) \star \Delta^I$ by Lemma 95, K_k^I is the subcomplex constituted of the cells $\sigma|\tau$ such that given $(V, i) \in \sigma$, $\dim V < n - k$. Now suppose fixed k such that $0 < k \leq n$ and pick a cell $\emptyset|\tau \in K_{k-1}^I$ with $\dim \tau = k - 1$. Using Lemma 91, we can construct an inclusion $\chi(\Delta^{I\setminus\tau}) \hookrightarrow \chi(\Delta^I)$, which corestricts to an inclusion $\chi(\Delta^{I\setminus\tau}) \hookrightarrow \partial\chi(\Delta^I)$, and similarly we have an inclusion $\Delta^\tau \hookrightarrow \Delta^I$. We therefore have an inclusion $\chi(\Delta^{I\setminus\tau}) \star \Delta^\tau \hookrightarrow \partial\chi(\Delta^I) \star \Delta^I$ since $(I \setminus \tau) \cap \tau = \emptyset$, which corestricts into an inclusion

$$\chi(\Delta^{I\setminus\tau}) \star \Delta^\tau \quad \hookrightarrow \quad \chi(\Delta^I)$$

via the map defined in Lemma 95, the corestriction being shown using the characterization given in this lemma and the definition of the colored join. By induction property 2, the inclusion

$$\Big(\chi(\Delta^{I\setminus\tau}) \star \Delta^\tau\Big) \setminus \Big(\Sigma^{I\setminus\tau}|\emptyset\Big) = \partial\chi(\Delta^{I\setminus\tau}) \star \Delta^\tau \quad \hookrightarrow \quad \chi(\Delta^{I\setminus\tau}) \star \Delta^\tau$$

is a collapse, the first equality being justified by Lemma 74. Moreover, by Lemma 98, we have $\mathrm{st}(\Sigma^{I\setminus\tau}) = \chi(\Delta^{I\setminus\tau})$, the star being computed in K_{k-1}^I, and therefore

$$\mathrm{st}\left(\left\{\sigma|\emptyset \;\middle|\; \sigma \in \Sigma^{I\setminus\tau}\right\}\right) \quad = \quad \chi(\Delta^{I\setminus\tau}) \star \Delta^\tau$$

by Remark 30 and Lemma 32. Therefore the inclusion

$$K^I_{k-1} \setminus \left(\Sigma^{I\setminus\tau}|\emptyset\right) \quad \hookrightarrow \quad K^I_{k-1}$$

is a collapse by Proposition 55 (in the variant mentioned in Remark 56). Since the above reasoning holds for any simplex $\tau \subseteq I$ with $\dim\tau = k-1$, i.e. $\#\tau = k$, we get a collapse

$$K^I_k = K^I_{k-1} \setminus \Sigma^{I-k} \quad \hookrightarrow \quad K^I_{k-1}$$

since by Proposition 51, all those collapses can be performed at once. We therefore have a sequence of collapses

$$\Delta^I \cong K^I_n \quad \hookrightarrow \quad K^I_{n-1} \quad \hookrightarrow \quad \dots \quad \hookrightarrow \quad K^I_0 = \chi(\Delta^I)$$

4. By property 3, the inclusion

$$\Delta^I \quad \hookrightarrow \quad \chi(\Delta^I)$$

is a collapse. By Lemma 88, the inclusion

$$\chi(\Delta^I) \star_{\Delta^I} \Delta^J \quad \hookrightarrow \quad \chi(\Delta^I) \star \Delta^J$$

is therefore also a collapse. Finally, the inclusion

$$\chi(\Delta^I) \quad \hookrightarrow \quad \chi(\Delta^I) \star_{\Delta^I} \Delta^J$$

is also a collapse: the inclusion

$$(\Delta^I \star \Delta^J) \setminus (\emptyset|\Delta^J) = \Delta^I \quad \hookrightarrow \quad \Delta^I \star \Delta^J$$

is easily shown to be a collapse, and by Proposition 55 and Remark 56 we have

$$\chi(\Delta^I) = (\chi(\Delta^I) \star_{\Delta^I} \Delta^J) \setminus (\emptyset|\Delta^J) \quad \hookrightarrow \quad \chi(\Delta^I) \star_{\Delta^I} \Delta^J$$

since $\mathrm{st}(\emptyset|\Delta^J) = \Delta^I \star \Delta^J$, where the star is computed in $\chi(\Delta^I) \star_{\Delta^I} \Delta^J$.

5. By property 3, the inclusion

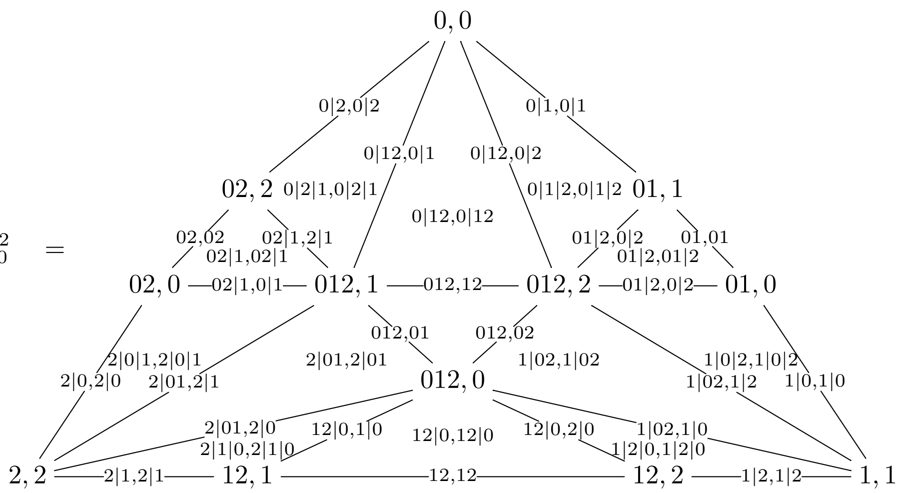
$$\Delta^I \hookrightarrow \chi(\Delta^I)$$

is a collapse, and Δ^I is collapsible by Lemma 50. □

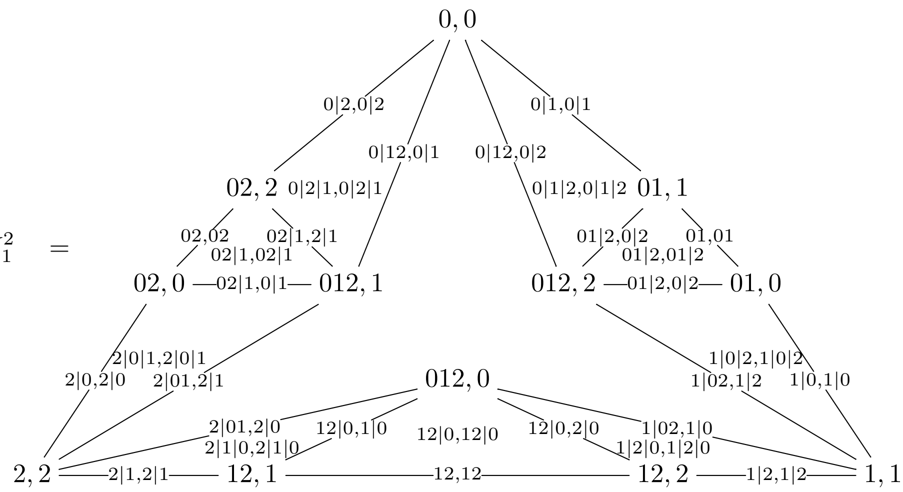
Example 100. The first point of the above theorem can be illustrated on $\chi(\Delta^2)\backslash\Sigma^2$

as follows:

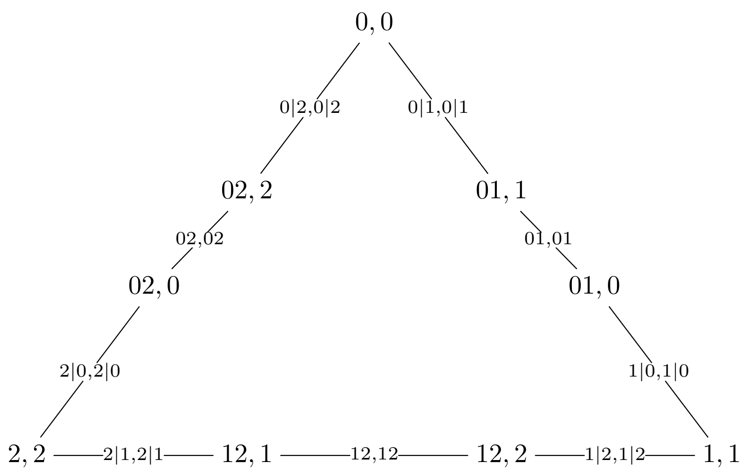
$\chi(\Delta^2) \setminus \Sigma^2 = K_0^2 \quad =$



$K_1^2 \quad =$

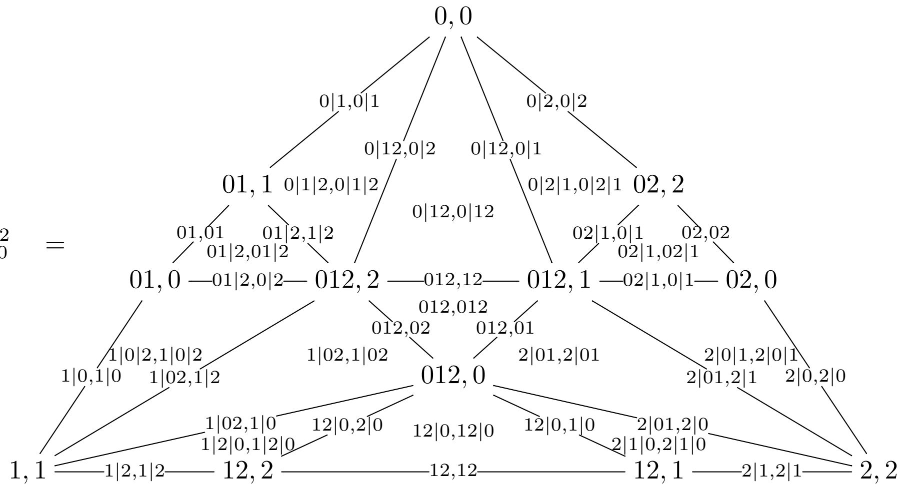


$\chi(\Delta^2) \setminus \Sigma_2^2 = K_2^2 \quad =$

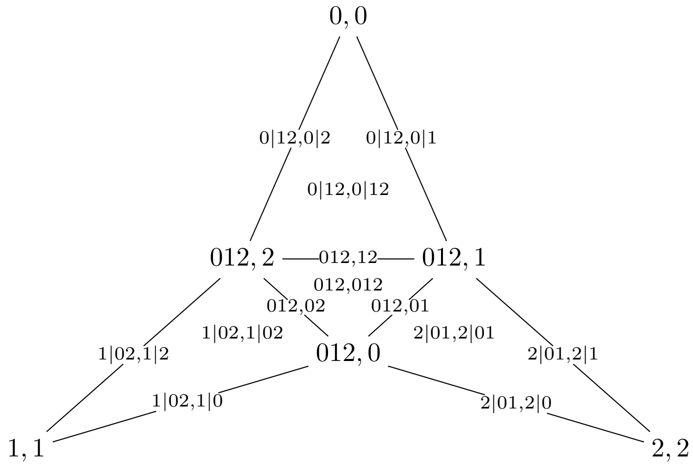


The second point is simply the suspension of the previous example. The third point is:

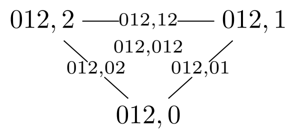
$\chi(\Delta^2) = K_0^2 \quad =$



$K_1^2 \quad =$



$\Delta^2 \cong K_2^2 \quad =$



This result was obtained independently by Kozlov [17], arriving by different means to a sequence collapse similar to ours. His paper moreover notices that the collapses can be performed in batches which are equivariant w.r.t. to the action of the symmetric group on Δ^I, which can also be directly checked in our case by examining the proof.

3.5 Simulation of elementary collapses

Consider an elementary collapse step for Δ^I. It is of the form

$$\Lambda_p^I \quad \hookrightarrow \quad \Delta^I$$

We show here that after subdivision, it is still possible to simulate this collapse step.

Proposition 101. *For every finite set I and $p \in I$, the inclusion*

$$\chi(\Lambda_p^I) \quad \hookrightarrow \quad \chi(\Delta^I)$$

obtained as the image under the functor χ of the canonical inclusion $\Lambda_p^I \hookrightarrow \Delta^I$ is a collapse.

Proof. The steps described in Lemma 72 in the case of the basic subdivision can be extended to the subdivision as in the proof of Theorem 99. □

This will be used in the next section as a crucial step in order to show that the iterated standard chromatic subdivision is collapsible.

4 The iterated subdivision is collapsible

In this section, we finally show that the iterated standard chromatic subdivision of the standard chromatic complex is collapsible: for every $n \in \mathbb{N}$, $\chi^n(\Delta^I)$ is collapsible. The general proof strategy is clear: we proceed by induction, and supposing that $\chi^n(\Delta^I)$ is collapsible, we show that $\chi^{n+1}(\Delta^I)$ is collapsible because, after the subdivision, we can still "simulate" the collapsing sequence in $\chi^n(\Delta^I)$ using Proposition 101. While a proof could certainly be performed "by hand", we give a fairly abstract proof of this result: we embed the category of simplicial complexes into a presheaf category whose elements are called chromatic presimplicial sets, which is much better suited w.r.t. colimits (in particular, the pushout of a collapse along another map is still a collapse), and then use the pair of adjoint functors given by nerve and realization in order to conclude.

4.1 Chromatic presimplicial sets

We first introduce the notion of "geometric" realization of a colored simplicial complex. In the case where colors are elements of $\mathbb{N}$ (and thus totally ordered), we can use the classical nice abstract machinery developed for (pre)simplicial sets, as we explain below. We refer the reader to [20] for details and proofs in the general case.

Definition 102. The **presimplicial category** Δ is the category whose objects are integers and morphisms $f : m \to n$ are injective increasing functions $f : [m] \to [n]$. The category of **presimplicial sets** is the presheaf category $\hat{\Delta}$.

Definition 103. We write Λ for the **chromatic presimplicial category** whose objects are finite non-empty subsets I of $\mathbb{N}$ and morphisms are inclusions. Presheaves on this category are called **chromatic presimplicial sets**.

Proposition 104. *We can define a functor U : **CSC** $\to \hat{\Lambda}$ which to every colored simplicial complex K associates the chromatic presimplicial set UK such that given $I \subseteq \mathbb{N}$,*

$$UK(I) \quad = \quad \{\sigma \in K \mid \ell(\sigma) = I\}$$

and given $J \subseteq I$ the associated function is

$$\begin{array}{rcl} UK(I) & \to & UK(J) \\ \sigma & \mapsto & \{x \in \sigma \mid \ell(x) \in J\} \end{array}$$

Conversely, we can define a functor $F : \hat{\Lambda} \to \mathbf{CSC}$ *which to a chromatic presimplicial set* $P \in \hat{\Lambda}$ *associates the simplicial complex whose set of vertices is*

$$\underline{FP} \quad = \quad \biguplus_{i \in \mathbb{N}} P(\{i\})$$

a vertex $x \in P(\{i\})$ *being labeled by* $\ell(x) = i$*, and set of simplices is*

$$FP \quad = \quad \left\{ P(\iota_i^I)(\sigma) \;\middle|\; \sigma \in P(I) \text{ and } i \in I \right\}$$

where $\iota_i^I : \{i\} \hookrightarrow I$ *is the inclusion. The functor* F *is left adjoint to* U *and the induced comonad* $F \circ U$ *is the identity comonad: in particular, the functor* U *is an embedding.*

This proposition allows us to consider any simplicial complex as a chromatic presimplicial set in $\hat{\Lambda}$ (in particular, we will simply write Δ^I, Λ^I_p, etc. for the presheaves corresponding to the simplicial complexes). By extension w.r.t. simplicial complexes, given $P \in \hat{\Lambda}$ and $I \in \Lambda$, an element of $P(I)$ is called an I*-simplex*.

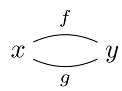
Remark 105. The adjunction is not an equivalence of categories. For instance, consider the chromatic presimplicial set $P \in \hat{\Lambda}$ defined by

$$P(\{0\}) = \{x\} \qquad P(\{1\}) = \{y\} \qquad P(\{0,1\}) = \{f, g\}$$

and other sets $P(I)$ are empty, with

$$P(\iota_0^{\{0,1\}})(f) = P(\iota_0^{\{0,1\}})(g) = x \quad \text{and} \quad P(\iota_1^{\{0,1\}})(f) = P(\iota_1^{\{0,1\}})(g) = y$$

Graphically,



Then $UF(P)$ is such that

$$UFP(\{0\}) = \{x\} \qquad UFP(\{1\}) = \{y\} \qquad UFP(\{0,1\}) = \{\{x, y\}\}$$

with $UFP(\iota_0^{\{0,1\}})(\{x, y\}) = x$ and $UFP(\iota_1^{\{0,1\}})(\{x, y\}) = y$:

$$x \overset{\{x,y\}}{\text{———}} y$$

and the two above presheaves are not isomorphic.

Remark 106. In the non-colored case the situation is of course similar: there is a forgetful functor from the category $\mathbf{SC}$ of simplicial complexes to the category $\hat{\Delta}$ of presimplicial sets, which admits a left adjoint.

Definition 107. To a functor

$$F \quad : \quad \Lambda \quad \to \quad \mathcal{C}$$

with $\mathcal{C}$ cocomplete, we associate a **nerve** functor

$$N_F \quad : \quad \mathcal{C} \quad \to \quad \hat{\Lambda}$$

defined on $A \in \mathcal{C}$ by

$$N_F A \quad = \quad \mathcal{C}(F-, A)$$

and a **realization** functor

$$R_F \quad : \quad \hat{\Lambda} \quad \to \quad \mathcal{C}$$

defined on $P \in \hat{\Lambda}$ by

$$R_F P \quad = \quad \operatorname{colim}\left(\mathrm{El}(P) \xrightarrow{\pi} \Lambda \xrightarrow{F} \mathcal{C}\right)$$

Remark 108. Above, $\mathrm{El}(P)$ denotes the category of elements of the presheaf P: its objects are pairs (I, x) with $I \in \Lambda$ and $x \in P(I)$ and a morphism $f : (I, x) \to (J, y)$ is a morphism $f : I \to J$ in Λ such that $P(f)(y) = x$. The functor $\pi : \mathrm{El}(P) \to \Lambda$ is the first projection. The category of elements can also equivalently be defined as the slice category y/P where $y : \Lambda \to \hat{\Lambda}$ is the Yoneda embedding.

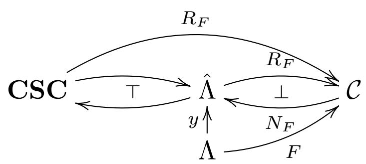
It can then be shown [20]:

Proposition 109. *For any functor F, the functor R_F is left adjoint to N_F.*

Definition 110. Given a functor $F : \Lambda \to \mathcal{C}$, by precomposing the realization functor $\hat{\Lambda} \to \mathcal{C}$ with the forgetful functor $\mathbf{CSC} \to \hat{\Lambda}$, we obtain a functor

$$R_F \quad : \quad \mathbf{CSC} \quad \to \quad \mathcal{C}$$

that we still write R_F and call it the realization (the typing makes clear the distinction between the two functors). The situation can thus be pictured as



(this is not a commutative diagram) where $y : \Lambda \to \hat{\Lambda}$ is the Yoneda embedding and $R_F = \mathrm{Lan}_y F$ is the left Kan extension of F along y.

While we will not need this in the following, we mention here that this abstract setting can be used to easily define the usual notion of geometric realization of a (colored) simplicial complex:

Definition 111. Given a set $I \subseteq \mathbb{N}$, there is a unique increasing bijection

$$\iota_I \quad : \quad I \quad \to \quad [\dim I]$$

We can thus define a functor

$$\Lambda \quad \to \quad \mathbf{\Delta}$$

which to a set I associates its cardinal $\#I = \dim I + 1$ and to a morphism $f : I \to J$ associates the function $\iota_J \circ f \circ \iota_I^{-1} : [\dim I] \to [\dim J]$. By post-composing with the usual functor

$$\Delta \quad \to \quad \mathbf{Top}$$

which to n associate the standard geometric n-simplex, we obtain a functor

$$\Lambda \quad \to \quad \mathbf{Top}$$

The associated realization functor

$$R \quad : \quad \mathbf{CSC} \quad \to \quad \mathbf{Top}$$

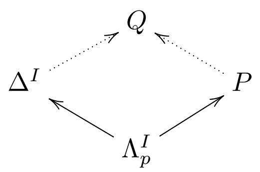
is called the **geometric realization** of colored simplicial complexes.

4.2 Simple homotopy in chromatic presimplicial sets

By Proposition 104, the category **CSC** embeds into $\hat{\Lambda}$. It is thus natural to wonder whether the constructions performed on simplicial complexes extend to chromatic presimplicial sets, and they actually do: we could have written the whole article using $\hat{\Lambda}$ instead of **CSC**, at the expense of heavier notations.

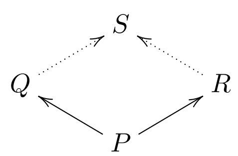
We only explain here how to extend the definition of simplicial collapses. Given $P \in \hat{\Lambda}$, an object $I \in \Lambda$ and an element $p \in I$, the inclusion $I' \hookrightarrow I$, with $I' = I \setminus \{p\}$, induces by functoriality of P a function $\partial^I_p : P(I) \to P(I')$ which to an I-simplex $x \in P(I)$ associates its *p-th face* $\partial^I_p(x)$. Given $J \subseteq I$ such that $I = J \uplus \{p_1, \ldots, p_n\}$, we say that $y \in P(J)$ is a *face* of $x \in P(I)$ when $y = \partial_{p_n} \ldots \partial_{p_1}(x)$. In this case y is a *free face* of x when x is maximal (it is the face of only itself), distinct from y, and is the only maximal simplex admitting y as a face. Based on this, it is easy to define the notion of *(elementary) collapse step* similarly as in Section 2.3. In this setting, Lemma 45 still holds (a collapse can be decomposed into a sequence of elementary collapse steps), and more interestingly an analogous of Lemma 47 holds, but is simpler to express since we do not need the extra condition, as discussed in Remark 48:

Lemma 112. *Given $P, Q \in \hat{\Lambda}$, an inclusion $P \hookrightarrow Q$ is an elementary collapse step if and only if there exists an inclusion $\Lambda^I_p \hookrightarrow P$ for some $I \in \Lambda$ and $p \in I$ such that $P \hookrightarrow Q$ is obtained by a pushout of the form*



As a direct corollary we have that

Lemma 113. *Given $P, Q, R \in \hat{\Lambda}$, together with inclusions $P \hookrightarrow Q$ and $P \hookrightarrow R$, such that $P \hookrightarrow Q$ is a collapse, then the map $R \hookrightarrow S$ obtained by the pushout is also a collapse*



More succinctly: collapses are stable under pushout.

Proof. The collapse $P \hookrightarrow Q$ can be decomposed as a sequence of elementary collapse steps by Lemma 45 and we conclude using Lemma 112 and the fact that pushouts are stable under composition. $\square$

4.3 The iterated subdivision is collapsible

In the non-colored case, the barycentric subdivision of a presimplicial set P can be defined as $\chi(P) = N \circ \mathrm{El}(P)$ where $\mathrm{El} : \hat{\Delta} \to \mathbf{Cat}$ is the functor of elements (see Remark 108) and $N : \mathbf{Cat} \to \hat{\Delta}$ is the usual nerve functor. This functor is left adjoint (to a functor named $\mathrm{Ex} : \hat{\Delta} \to \hat{\Delta}$), it is thus cocontinuous [9]. Moreover, every presheaf is a colimit of representables [20]: $P \cong R_y P$. The functor χ is thus characterized by its image on representables: we have

$$P \quad \cong \quad R_{\chi \circ y} P \quad = \quad \operatorname{colim}\left(\mathrm{El}(P) \xrightarrow{\pi} \Delta \xrightarrow{y} \hat{\Delta} \xrightarrow{\chi} \hat{\Delta}\right)$$

In the colored case, this situation generalizes as follows:

Proposition 114. *We can define a functor*

$$\chi^{\Delta} \quad : \quad \Lambda \quad \to \quad \mathbf{CSC}$$

which to every object $I \in \Lambda$ associates $\chi(\Delta^I)$ and to every morphism $J \to I$ witnessing an inclusion $J \subseteq I$ associates the corresponding inclusion $\chi(\Delta^J) \hookrightarrow \chi(\Delta^I)$ given by Lemma 91. The associated realization functor

$$R_{\chi^{\Delta}} \quad : \quad \mathbf{CSC} \quad \to \quad \mathbf{CSC}$$

is the standard chromatic subdivision:

$$\chi \quad = \quad R_{\chi^{\Delta}}$$

Similarly, the functor

$$\chi^{\Delta} \quad : \quad \Lambda \quad \to \quad \hat{\Lambda}$$

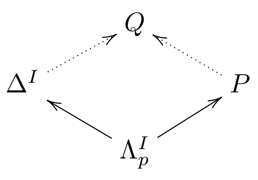
induces a functor

$$R_{\chi^{\Delta}} \quad : \quad \hat{\Lambda} \quad \to \quad \hat{\Lambda}$$

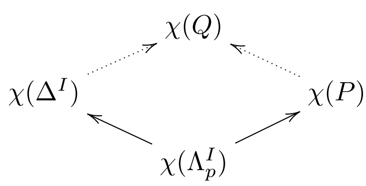
which coincides with the previous functor up to the embedding $\mathbf{CSC} \to \hat{\Lambda}$. This justifies the use of the same notation for both.

Lemma 115. *For every collapse $P \hookrightarrow Q$ with $P, Q \in \hat{\Lambda}$, the map $\chi(P) \hookrightarrow \chi(Q)$ obtained as the image of the functor χ is also a collapse.*

Proof. By functoriality, it is enough to consider the case where the collapse $P \hookrightarrow Q$ is a collapse step, and by Lemma 45 we can even suppose the collapse step to be elementary. By Lemma 112, Q can be obtained as the pushout of a diagram of the form



for some $I \in \Lambda$ and $p \in I$. By Proposition 114, the functor χ is a left adjoint and therefore preserves colimits. Thus, the image of the above pushout diagram is a pushout diagram



We have shown in proposition 101 that the map $\chi(\Lambda^I_p) \hookrightarrow \chi(\Delta^I)$ is still a collapse and therefore the map $\chi(P) \hookrightarrow \chi(Q)$ is also a collapse by Lemma 113. □

Theorem 116. *Suppose that K is a collapsible simplicial complex. Then $\chi(K)$ is also collapsible. In particular, for every $I \in \Lambda$ and $n \in \mathbb{N}$, $\chi^n(\Delta^I)$ is collapsible.*

Proof. The simplicial complex K can be seen as a chromatic presimplicial set via the embedding of Proposition 104. By assumption, K is collapsible to a point, i.e. there exists a collapse $\Delta^0 \hookrightarrow K$. By Lemma 115, its image $\chi(\Delta^0) \hookrightarrow \chi(K)$ under χ is also a collapse and we conclude that $\chi(K)$ is collapsible since $\chi(\Delta^0) \cong \Delta^0$. Given $I \in \Lambda$, Δ^I is collapsible by Lemma 50, and we conclude that $\chi^n(\Delta^I)$ is collapsible for every $n \in \mathbb{N}$ by induction. □

5 Conclusion

We have shown that the iterated standard chromatic subdivision of the standard simplicial complex is contractible and explored some of the combinatorial and categorical structures present in the category of colored simplicial complexes.

In the future, we plan to extend this work to other variants of view complexes. In particular, the proof that the standard chromatic subdivision of the standard simplex is contractible can be extended to the subdivision corresponding to non-layered protocols [10]. This has been recently been independently investigated [17], so that we did not think it was necessary to include it here. Our methods should be well suited for studying the more intricate iterated non-layered snapshot model, which is the real depiction of the reachable states in an atomic read write distributed model. We plan on extending this work for the study of full information protocols for other, more modern synchronization primitives, such as test&set, fetch&add, compare&swap etc. Finally, a companion paper will explicit the relationships between the topology of the view complex (the full information protocol complex) and the (directed) topology [11] of the semantics [7] of the protocols involved. We wish also to develop further the strong links between this work and tools used in order to elaborate model structures for our chromatic simplicial complexes (in fact, for their chromatic presimplicial sets counterparts). The chromatic join we defined extends naturally, as in [6] to joins on a category of (augmented) chromatic presimplicial sets, with a right adjoint, similar to the well-known *Ex* functor which is a central tool in simplicial approximation theorems, and model structures on simplicial sets. We believe that the results of [14] (which are nothing but some form of chromatic simplicial approximation theorem) can be reformulated in that more general abstract setting, and will be published elsewhere.

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