Totality, towards completeness

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Contributions

1. Total finiteness spaces \((A, T(A))\), a semantics for classical linear logic.

2. Barycentric boolean calculus, a parallel syntax which is total:

   \[
   s ::= x \in V \mid \lambda x.s \mid (s)S \\
   \quad \mid T \mid F \mid \text{if } s \text{ then } R \text{ else } S
   \]

   \[
   R, S ::= \sum_{i=1}^{m} a_i s_i \quad \text{where } \sum_{i=1}^{m} a_i = 1.
   \]

3. Full completeness at the first order boolean type.

Theorem (Completeness)

*Every total function of \(T(B^n \Rightarrow B)\) is the interpretation of a term of the boolean barycentric calculus.*
## Context

### Linear Logic et differential lambda-calculus

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### Denotational semantics

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<td>The quest for sequentiality, through the full adequacy issue.</td>
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   - Algebraic $\lambda$-calculus

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   - Finiteness spaces

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What about parallel algorithm

Non-deterministic algorithm are made of different programs of the same type that are reduced in parallel.

We use algebraic $\lambda$-calculus which is equipped with sums and scalar coefficients which give account of the number of way to compute a result (cf. Boudol, Vaux, Ehrhard-Regnier).

We tackle the full completeness question from both traditional viewpoint and

- vary the model to fit a language,
- vary the language to fit the model.
Algebraic \( \lambda \)-Calculus

Simply typed \( \lambda \)-calculus:

\[
\begin{align*}
\Gamma, x : A \vdash x : A & \quad (\text{var}) \\
\Gamma, x : A \vdash s : B & \quad (\text{abs}) \\
\Gamma \vdash \lambda x. s : A \Rightarrow B & \quad (\text{app})
\end{align*}
\]

Algebraic extension:

\[
\begin{align*}
\Gamma \vdash 0 : A & \quad (0) \\
\Gamma \vdash s_1 : A & \quad (\text{sum}) \\
\Gamma \vdash s_2 : A & \\
\Gamma \vdash s_1 + s_2 : A & \quad (\text{scal}) \\
\Gamma \vdash s : A & \\
a \in \mathbb{K} & \\
\Gamma \vdash as : A
\end{align*}
\]

Zero proves any formula, it stands for non total proof like the Daimon \( \mathbf{\times} \) of Girard
The barycentric boolean calculus

Let $\mathbb{k}$ be a infinite field and $\mathcal{V}$ be a countable set of variables.

**Definition ($\lambda + +$)**

Atomic terms $s$ and barycentric terms $T$ are inductively defined

$$s ::= x \in \mathcal{V} \mid \lambda x.s \mid (s)S$$
The barycentric boolean calculus

Let $\mathbb{k}$ be a infinite field and $\mathcal{V}$ be a countable set of variables.

**Definition ($\lambda + \text{Barycentric} +$ )**

Atomic terms $s$ and barycentric terms $T$ are inductively defined

\[
s ::= x \in \mathcal{V} \mid \lambda x. s \mid (s)S
\]

\[
R, S ::= \sum_{i=1}^{m} a_i s_i \quad \text{where } \left\{ \begin{array}{l}
\forall i \leq m, a_i \in \mathbb{k}, \\
\sum_{i=1}^{m} a_i = 1.
\end{array} \right.
\]
The barycentric boolean calculus

Let $\mathbb{k}$ be a infinite field and $\mathcal{V}$ be a countable set of variables.

Definition ($\lambda +$ Barycentric + Boolean)

Atomic terms $s$ and barycentric terms $T$ are inductively defined

\[
\begin{align*}
  s &::= x \in \mathcal{V} \mid \lambda x.s \mid (s)S \\
  &\mid T : B \mid F : B \mid \text{if } s \text{ then } R \text{ else } S : B \Rightarrow A \Rightarrow A \Rightarrow A
\end{align*}
\]

$R, S ::= \sum_{i=1}^{m} a_i s_i$ where

\[
\begin{align*}
  \forall i \leq m, a_i \in \mathbb{k}, \\
  \sum_{i=1}^{m} a_i = 1.
\end{align*}
\]

The booleans are affine combinations of true ($T$) and false ($F$) and $B = 1 \oplus 1$ from linear logic.
Semantics of $\Lambda_B$

- Webbed model enriched with coefficients and sums, hence vector spaces.
- Boolean type leads to finite dimensional vector spaces:
  \[
  \begin{align*}
  [T] &= (1, 0), \quad [F] = (0, 1), \\
  [\text{if } (aT + bF) \text{ then } Q \text{ else } R] &= a [Q] + b [R], \\
  [\sum a_i s_i] &= \sum a_i [s_i].
  \end{align*}
  \]
- Functional type leads to infinite dimensional vector spaces:
  \[
  A \Rightarrow B = !A \rightarrow B
  \]

The web of exponential isn’t finite, hence we need topology!
Non-determinism

For every proof $\pi$ and $\pi'$ of linear logic formulæ,

$$\frac{\pi \vdash A}{[\pi ; \pi'] \vdash B} \quad \text{Cut} \quad [\pi ; \pi'] = \left( \sum_{a \in |A|} [\pi]_a [\pi']_{a,b} \right)_{b \in |B|}$$

The sum

- allows non-determinism, since result of different computations are added;
- is controlled since, in the simple typed case, it is finite.

Finiteness spaces use orthogonality between $\pi \vdash A$ and $\pi' \vdash A^\perp$

$$|[\pi]| \cap |[\pi']| \text{ finite.}$$

to make explicit the controlled non-determinism.
Relational Finiteness Spaces

Let $\mathcal{I}$ be countable, for each $\mathcal{F} \subseteq \mathcal{P}(\mathcal{I})$, let us denote

$$\mathcal{F}^\perp = \{ u' \subseteq \mathcal{I} ; \forall u \in \mathcal{F}, u \cap u' \text{ finite} \}. $$

**Definition**

A *relational finiteness space* is a pair $A = (|A|, \mathcal{F}(A))$ where the web $|A|$ is countable and the collection $\mathcal{F}(A)$ of finitary subsets satisfies $(\mathcal{F}(A))^\perp \perp = \mathcal{F}(A)$.

**Example**

*Booleans.*

$$\mathcal{B} = (\mathbb{B}, \mathcal{P}(\mathbb{B})) \quad \text{with} \quad \mathbb{B} = \{ T, F \}, \quad \mathcal{P}(\mathbb{B}) = \{ \emptyset, \{ T \}, \{ F \}, \{ T, F \} \}. $$

*Integers.*

$$\mathcal{N} = (\mathbb{N}, \mathcal{P}_{\text{fin}}(\mathbb{N})) \quad \text{and} \quad \mathcal{N}^\perp = (\mathbb{N}, \mathcal{P}(\mathbb{N})). $$
Linear Finiteness Spaces

Let $k$ be an infinite discrete field. For every sequence $x \in k^{|A|}$, the support of $x$ is $|x| = \{a \in |A| ; x_a \neq 0\}$.

**Definition**

The *linear finiteness space* associated to $A = (|A|, \mathcal{F}(A))$ is

$$k\langle A \rangle = \{x \in k^{|A|} ; |x| \in \mathcal{F}(A)\}.$$

The *linearized topology* is generated by the neighborhoods of $0$

$$V_J = \{x \in k\langle A \rangle ; |x| \cap J = \emptyset\}, \text{ with } J \in \mathcal{F}(A)\perp.$$

**Example**

*Booleans.* $\mathbb{B} = \{T, F\}$ \hspace{1cm} $k\langle \mathbb{B} \rangle = k^2$.

*Integers.* $|\mathbb{N}| = \mathbb{N}$ \hspace{1cm} $k\langle \mathbb{N} \rangle = k^{(\omega)}$ and $k\langle \mathbb{N}^{\perp} \rangle = k^{\omega}$.
Finiteness Spaces, Functions.

Theorem (Taylor-Ehrhard expansion)

*Every program* $P : A \Rightarrow B$ *is interpreted by an analytic function* $\llbracket P \rrbracket : \mathbb{k}\langle A \rangle \to \mathbb{k}\langle B \rangle$

$$P = \sum_{n \in \mathbb{N}} P^{(n)}(0) \times \otimes n.$$  

Example

$$\mathbb{k}\langle !B \multimap 1 \rangle = \mathbb{k} [X_t, X_f],$$
$$\mathbb{k}\langle !B \multimap B \rangle = \mathbb{k}\langle !B \multimap 1 \oplus 1 \rangle = \mathbb{k}\langle !B \multimap 1 \rangle^2$$
$$= \mathbb{k} [X_t, X_f] \times \mathbb{k} [X_t, X_f].$$
Towards completeness

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What is totality?

A way to refine the semantics and step to full completeness.

For every proof $\pi$ and $\pi'$ of linear logic formulæ,

$$
\frac{\pi \vdash A \quad \pi' \vdash A^\perp}{[\pi ; \pi'] \vdash \bot \quad \text{Cut} \quad [\pi ; \pi'] = \langle [\pi] , [\pi'] \rangle = 1}
$$

Let $A$ be a finiteness space $A = (|A|, \mathcal{F}(A))$. The associate linear space is $\mathbb{k} \langle A \rangle = \{ x \in \mathbb{k}^{|A|} ; |x| \in \mathcal{F}(A) \}$.

**Definition**

A totality candidate is an affine subspace $\mathcal{T}$ of $\mathbb{k} \langle A \rangle$ such that $\mathcal{T}^\bullet \bullet = \mathcal{T}$ with

$$
\mathcal{T}^\bullet = \{ x' \in \mathbb{k} \langle A \rangle ; \forall x \in \mathcal{T}, \langle x', x \rangle = 1 \}.
$$

A totality space is a pair $(A, \mathcal{T}(A))$ with $\mathcal{T}(A)^\bullet \bullet = \mathcal{T}(A)$. 

An algebraic description

Every construction of linear logic has an algebraic description as a closed affine subspace.

\[ \mathcal{B} = 1 \oplus 1 \]

Affine combinations:

\[ f \in \mathcal{T}(A \rightarrow B) \text{ whenever } \forall x \in \mathcal{T}(A), f(x) \in \mathcal{T}(B); \]

\[ F \in \mathcal{T}(A \Rightarrow B) \text{ whenever } \forall x \in \mathcal{T}(A), F(x) \in \mathcal{T}(B). \]

Example \((\mathcal{B} \Rightarrow \mathcal{B} = \mathcal{B} \rightarrow \mathcal{B})\)

\[ \llangle \mathcal{B} \Rightarrow \mathcal{B} \rrangle = \llangle X_T, X_F \rrangle \times \llangle X_T, X_F \rrangle, \]

\[ (P_T, P_F) \in \mathcal{T}(\mathcal{B} \Rightarrow \mathcal{B}) \iff \forall (x_T, x_F) \in \mathcal{T}(\mathcal{B}), P_T(x_T, x_F) + P_F(x_T, x_F) = 1. \]
Full Completeness at the first order boolean type.

Syntax of $\Lambda_B$:

$$s ::= x \in \mathcal{V} \mid \lambda x.s \mid (s)S$$
$$\mid T \mid F \mid \text{if } s \text{ then } R \text{ else } S$$

$$R, S ::= \sum_{i=1}^{m} a_i s_i \quad \text{where } \sum_{i=1}^{m} a_i = 1.$$
Corollary

Parallel functions such as
- Parallel-Or
- Berry function
can be encoded in $\Lambda_B$.

Generalisations

- Possible for finitary types built over 1 and $\oplus$.
- Impossible for infinite types such as $\mathcal{N}$.
- Generalisation to higher order boolean type?