Tree Automata and Tree Constraints

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Plan of this Talk

1. Tree Constraints

2. Solving Tree Constraints by Syntactic Methods

3. Solving Monadic Tree Constraints by Automata

4. Solving Tree Constraints by Tree Automata

(Tree) Constraints

A constraint system is given by

- a first-order signature $\langle \Sigma, P \rangle$
- a class of 1st-order formulas over this signature
- a fixed interpretation (with universe $= T(\Sigma)$)

Questions:

- Validity of constraints
- How to simplify constraints
- $\ldots$

Usage of Tree Constraints

- Constraint Logic Programming, Constraint Programming
- Computational Linguistics
- Constrained-Based Deduction
- Constraint-Based Program Analysis
- Constraint-Based Anything
- ...

Constraint-Based Theorem Proving

One example: ordered rewriting.

$$\left\{\begin{array}{rcl} (x \times y) \times z & = & x \times (y \times z) \\ x \times 1 & = & x \\ x \times x^{-1} & = & 1 \end{array}\right.$$

Is an equation like

$$(x \times y)^{-1} = y^{-1} \times x^{-1}$$

a consequence of these axioms?

Proof be rewriting

Idea: Turn the equations into a terminating rewrite system, like

$$x \times 1 \quad \rightarrow \quad x$$

Normalize both sides of an equation and test for syntactic equality.

Problem: In general, completeness is lost but can be regained by adding new rewrite rules (Knuth-Bendix Completion).

Non-orientable equations

$$x \times y = y \times x$$

The corresponding rewrite rule is looping:

$$x \times y \to y \times x$$

Constrained rewrite rule

$$x \times y \to y \times x \mid x > y$$

To be read: Rewrite ground term $s \times t$ into $t \times s$ only if $s > t$.

[Kirchner & Kirchner & Rusinowitch '90, ...]

In general, an equation $l = r$ yields two constrained rewrite rules:

$$\begin{aligned} l &\rightarrow r \mid l > r \\ r &\rightarrow l \mid r > l \end{aligned}$$

where $>$ is an ordering relation that is

1. total on ground terms

2. well-founded

3. monotonic

Examples of such orderings: recursive path ordering, lexicographic path ordering, Knuth-Bendix Ordering, ...

We may use a constraint system with

- Signature $\langle \Sigma, \{\leq\} \rangle$
- Interpretation of $\leq$: The chosen ordering (lexicographic path ordering, Knuth-Bendix Ordering, $\ldots$)
- Class of constraints: depends on the kind of problems one wants to solve (see next slides), includes at least atomic inequations $s < t$, closed under $\exists$ and $\wedge$

Consider constrained rewrite rules

$$l \to r \mid c(\bar{x})$$

where $\bar{x} = \mathit{free}(l) \cup \mathit{free}(r) \cup \mathit{free}(c)$.

$\exists^*$-Formulas as Constraints

To complete the rewrite system, we have to know whether two rules can overlap:

$$\begin{array}{ll} (R1) & s \to t \mid c(\bar{x}) \\ (R2) & u \to v \mid d(\bar{y}) \end{array}$$

overlap at the top when the following constraint is satisfiable:

$$s = u \wedge c(\bar{x}) \wedge d(\bar{y})$$

that is if

$$\exists \bar{x}, \bar{y}\; s = u \wedge c(\bar{x}) \wedge d(\bar{y})$$

is valid.

$\forall^*\exists^*$-Formulas as Constraints

$$\begin{array}{ll} (R1) & s \to t \mid c(\bar{x}) \\ (R2) & u \to v \mid d(\bar{y}) \end{array}$$

$(R2)$ subsumes $(R1)$ iff

$$\forall \bar{x}\Big(c(\bar{x}) \to \exists \bar{y}(s = u \wedge t = v \wedge d(\bar{y}))\Big)$$

Consequence: We need $\forall^*\exists^*$-formulas to decide subsumption.

Arbitrary Formulas as Constraints

$$\begin{array}{ll} (R1) & s \to t \mid c(\bar{x}) \\ (R2) & u \to v \mid d(\bar{y}) \end{array}$$

To make both rules disjoint replace $d(\bar{y})$ by

$$d(\bar{y}) \wedge \neg \exists \bar{x} \Big(c(\bar{x}) \wedge s = u \wedge t = v \Big)$$

Consequence: Constraints can be any FOL formula

Solving Tree Constraints by Syntactic Methods

Design a set of transformation rules that

- preserves the semantics of formulas
- is terminating
- has normal forms which allow to read off the answer

Example: Solving equational constraints

$$\frac{f(t_1, \ldots, t_n) = f(s_1, \ldots, s_n)}{s_1 = t_1 \wedge \ldots \wedge s_n = t_n}$$

Example: Quantifier elimination

$$\frac{\exists x \; x = t \wedge \phi}{\phi[x \mapsto t]}$$

Problem: Sometimes difficult to find.

Using Automata to Solve Tree Constraints

Running example: Non-Structural Subtyping

- Signature: $\langle \Sigma, \leq \rangle$
- Interpretation of $s \leq t$: For all $\pi \in O(s) \cap O(t)$ we have $s(\pi) \leq t(\pi)$ for some fixed order on Σ.
- Class of constraints: Let's see how far we get ...

Example Σ: $\bot/0 < f/2 < g/2 < \top/0$

Properties of Non-Structural Subtyping

- complete lattice
- minimum: $\perp$
- maximum: $\top$
- $f(s_1, s_2) \leq f(t_1, t_2) \Leftrightarrow s_1 \leq t_1 \wedge s_2 \leq t_2$

Old and New Results on NSS

- Satisfiability of constraints $\exists \bar{x}\ c_1 \wedge \ldots \wedge c_n$

  Decidable in n^3

  [Rehoff; Palsberg & Wand & O'Keefe]

- Entailment of constraints

  $\forall \bar{x}\ c_1 \wedge \ldots \wedge c_n \to c$

  (Still) open question

- Full first-order theory
  - Monadic signature: decidable
  - In general: undecidable

  [Aiken et al. POPL'02]

Simplification: Monadic Signatures

Monadic signature: only constants and unary function symbols.

A tree over a monadic signature Σ can be represented as a word over Σ.

Example: Σ: $\perp/0 < f/1 < g/1 < \top/0$

$$\begin{array}{c} f \\ | \\ g \\ | \\ \perp \end{array}$$

is represented by the word

$$f g \perp$$

An n-tuple of trees over a monadic signature Σ can be represented as a word over $(\Sigma \cup \{\#\})^n$

Example: Σ: $\bot/0 < f/1 < g/1 < \top/0$

$$\begin{array}{ccc} f & \top & g \\ | & & | \\ g & & \top \\ | & & \\ \bot & & \end{array}$$

is represented by the word

$$\begin{pmatrix} f \\ \top \\ g \end{pmatrix} \begin{pmatrix} g \\ \# \\ \top \end{pmatrix} \begin{pmatrix} \bot \\ \# \\ \# \end{pmatrix}$$

Coding function $\gamma : T(\Sigma) \mapsto (\Sigma \cup \{\#\})^n$

Basic Idea

A solution of a constraint $\phi(x_1, \ldots, x_n)$ is a n-tuple of trees over Σ

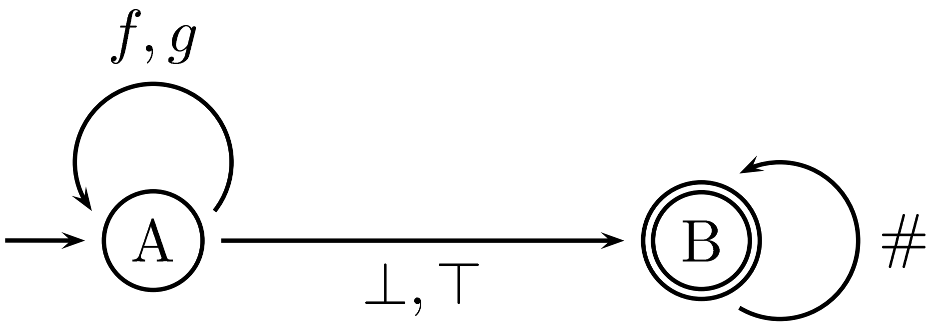
Try to build, for every formula $\phi(x_1, \ldots, x_n)$, an automaton A_ϕ with

$$L(A_\phi) = \{\gamma(t_1, \ldots, t_n) \mid\models \phi[t_1, \ldots, t_n]\}$$

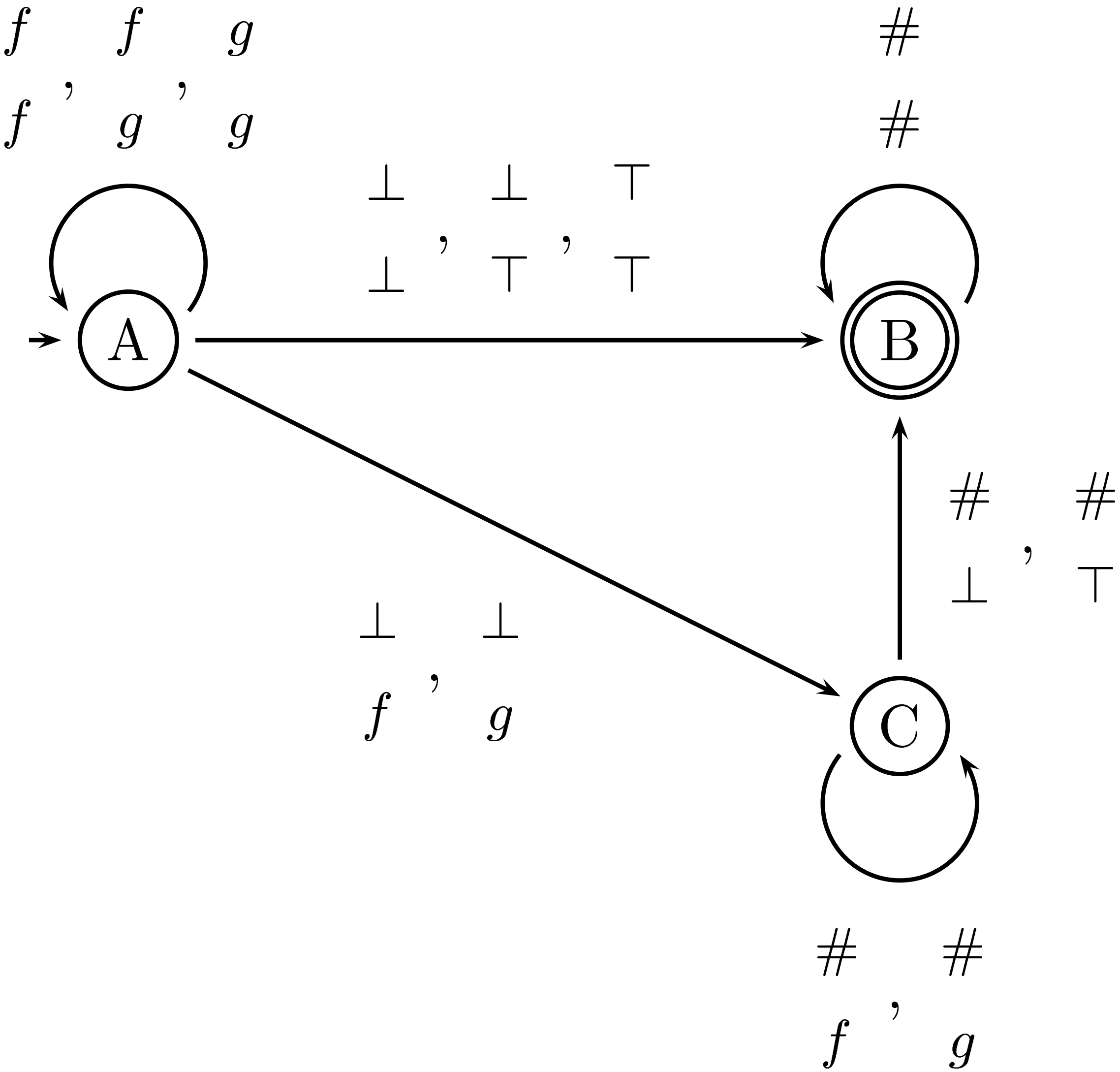
If ϕ is closed: ϕ is valid iff $L(A_\phi) \neq \emptyset$.

Idea goes back to [Buechi '60]

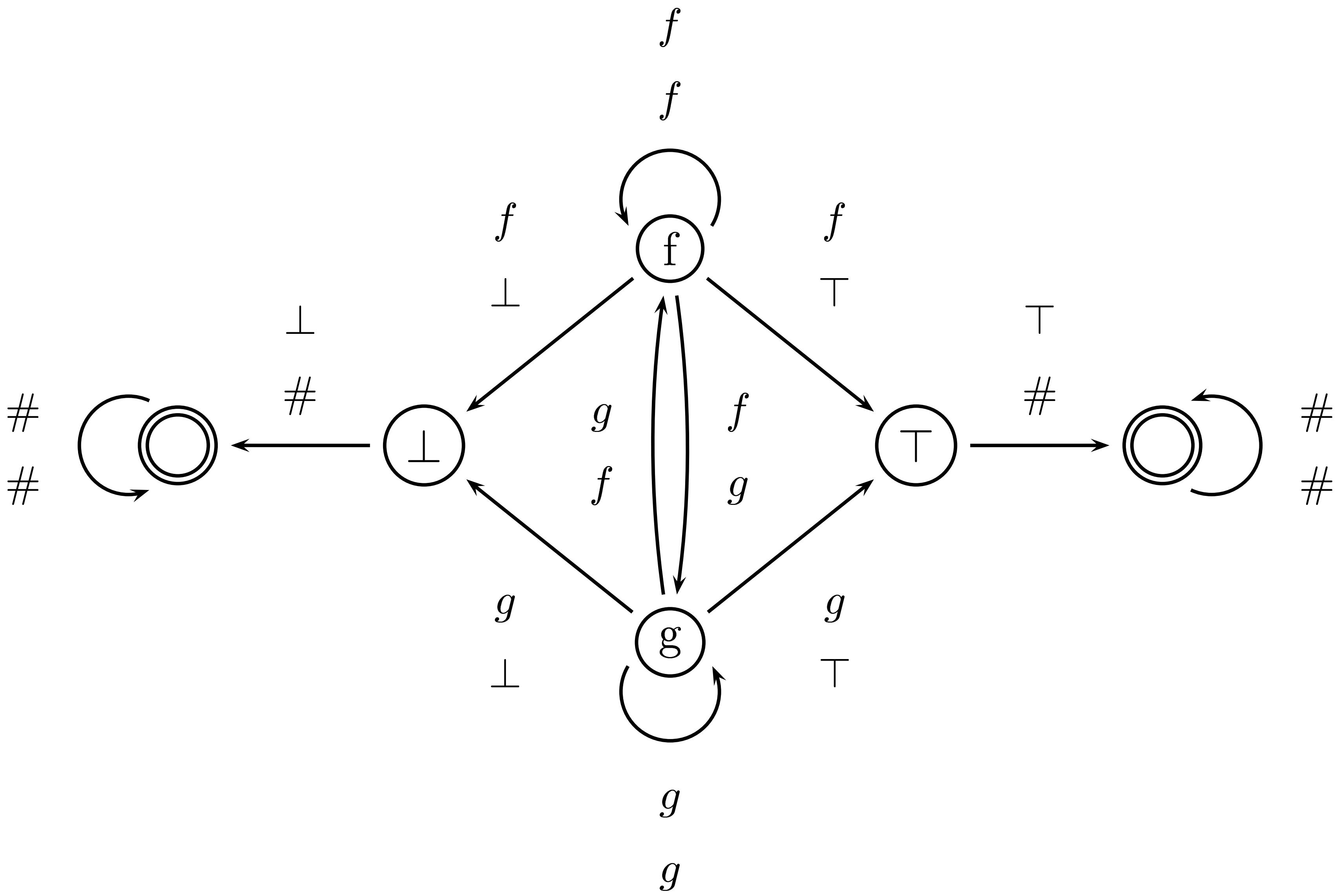
An automaton that recognizes $\gamma(T(\Sigma))$



An automaton for the solutions of $x_1 \leq x_2$



An automaton for the solutions of $x_1 = f(x_2)$



Building automata for composite formulas

Inductive construction using the closure properties of automata:

- conjunction $\Rightarrow$ intersection
- negation $\Rightarrow$ complement
  (using determinisation)
- existential quantification $\Rightarrow$ projection

Plus cylindrification:

To build the conjunction of $\Phi(X_1, X_2)$ and $\Psi(X_2, X_3)$

- Build A_Φ and A_Ψ
- Cylindrify to the variables (X_1, X_2, X_3)
- Intersection automaton

Tree Automata: Definition

[Doner, Thatcher, Wright: 1965-1970]

A tree automaton (TA) is given by

- a tree signature Σ
- a finite set Q of states
- a set $F \subseteq Q$ of final states
- a set Δ of transition rules of the form

$$f(q_1, q_2) \rightarrow q$$

Tree Automata: Example

- $\Sigma = \{0/0, s/1, nil/0, cons/2\}$
- $Q = \{N, L\}$
- $F = \{L\}$
- $\Delta = \left\{ \begin{array}{llll} 0 & \to N & s(N) & \to N \\ nil & \to L & cons(L, N) & \to L \end{array} \right\}$

The move relation $\to_A$ of an automaton A is the smallest relation with

- $a \to_A q(a)$ if $q \to a \in \Delta$
- if $t_1 \to_A q_1(t_1)$

  $t_2 \to_A q_2(t_2)$

  $f(q_1, q_2) \to q \in \Delta$

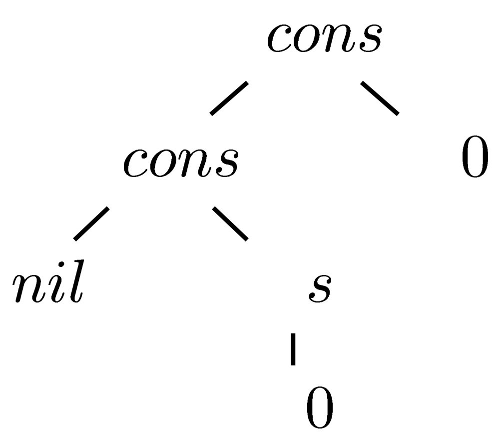
  then $f(t_1, t_2) \to_A q(f(t_1, t_2))$

The automaton A accepts the tree t if $t \to_A q(t)$ for some $q \in F$.

The language recognized by A is

$$L(A) = \{t \mid A \text{ accepts } t\}$$

Run of a Tree Automaton: Example



Transition rules:

$$\begin{array}{llll} 0 & \to N & s(N) & \to N \\ nil & \to L & cons(L,N) & \to L \end{array}$$

$$L(\quad\quad)$$

$$L(\quad\quad) \qquad N(\ )$$

$$L(\quad) \qquad N(\ )$$

$$N(\ )$$

Properties of Tree Automata

Tree automata enjoy all nice properties of word automata:

- closure under $\cup$, $\cap$, complement, linear homomorphisms, inverse homomorphisms, cylindrification, projection.
- decidability of emptiness
- determinisation
- minimisation

A Tree Automaton for $\gamma(T(\Sigma))$

$\Sigma = \{\perp/0, \top/0, f/2\}$

$$\begin{array}{rcl}
\# & \rightarrow & B \\
\#(B, B) & \rightarrow & B \\
\perp & \rightarrow & A \\
\perp(B, B) & \rightarrow & A \\
\top & \rightarrow & A \\
\top(B, B) & \rightarrow & A \\
f(A, A) & \rightarrow & A
\end{array}$$

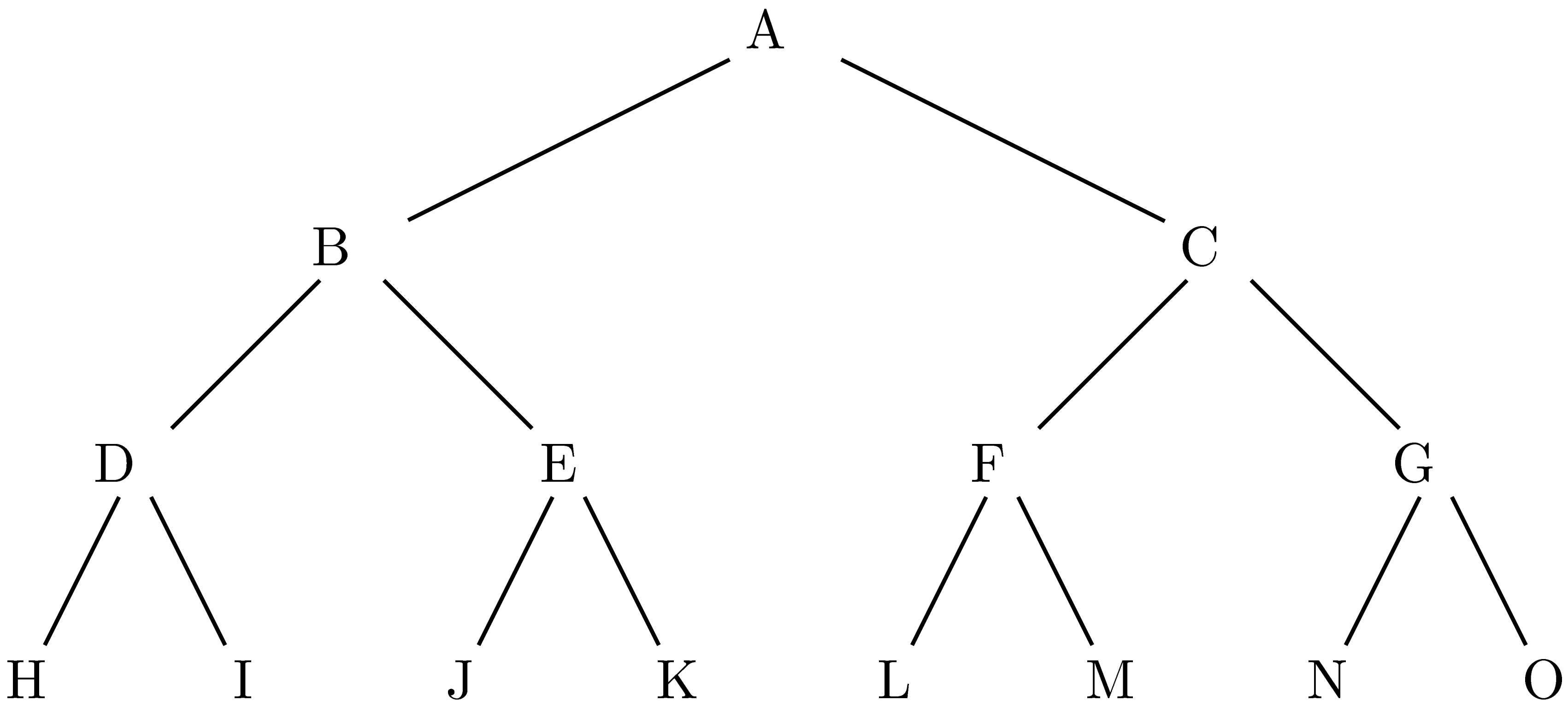
Accepting states: $\{A\}$.

A Tree Automaton for the Solutions of $x_1 \leq x_2$

Intersection of three automata:

1. check whether first component in $\gamma(T(\Sigma))$
2. check whether second component in $\gamma(T(\Sigma))$
3. check that no "bad" symbol occurs: $(f, \perp)$, $(\top, f)$, $(\top, \perp)$

A Tree Automaton for $x_1 = f(x_2, \_)$?



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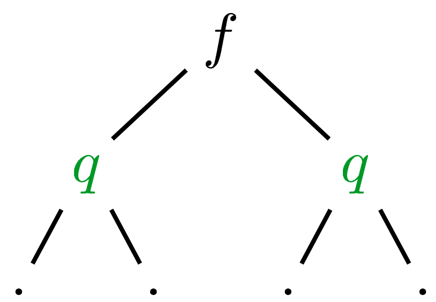
Tree Automata with Tests (TAT)

[Mongy and Dauchet, early 80's]: Have transitions with tests.

Example:

$$f(q,q) \rightarrow^{12=21} q$$



Emptiness undecidable even when only comparisons between cousin positions [Seynhaeve et. al. 97]

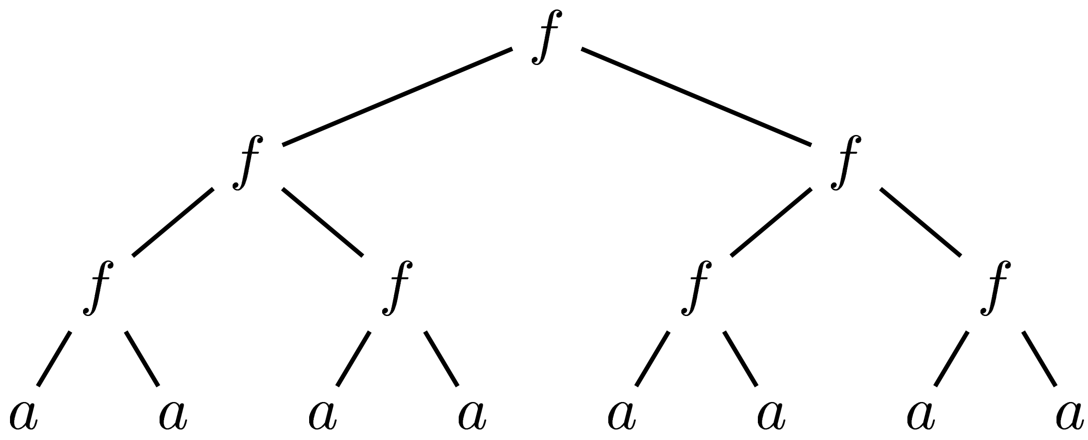
Restriction to tree automata with tests between brothers [Bogaert & Tison 92]:

- Closure under Boolean operations.
- Emptiness decidable.

Example:

$$a \to q \quad f(q,q) \to^{1=2} q$$

defines the complete binary trees.



Restriction to tree automata with a bounded number of tests on any branch (Reduction Automata) [Caron & Comon & Coquidé & Dauchet & Jacquemard '94]

- Closure under $\cup$, $\cap$ (but not under complement).
- Emptiness decidable.

Example: Reduction automaton for the set of terms that are reducible by some (non-linear) rewrite system.

Tree Automata with Component-wise Tests (TACT)

If we want to translate any logical formula of a logic into automata then we need closure under cylindrification.

TACT: Tests on Components:

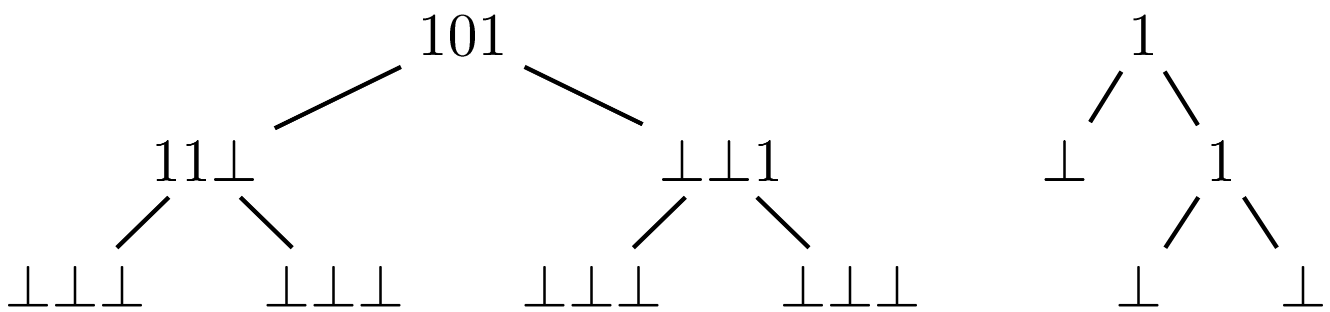
$$f(q_1, q_2) \rightarrow^{1.i=2.i} q$$

In general: any boolean combination of tests:

$$f(q_1, q_2) \rightarrow^{1.i=2.i \vee (1.j \neq 2.j \wedge 1.k=2.k)} q$$

Component of a Tree

Example 3rd component:



Results on Tree Automata with Component-Wise Tests

- Emptiness undecidable in case of
  - Tests between brothers [RT, FOSSACS'00],
  - One test, and this test is at the root.
- Not closed under arbitrary projection.

Conclusion

- Monadic case: Automata work well.
- General case: Problem for automata to represent equations.

Possible solution?

- Find a tree automata model which
  - is strong enough to model constraints
  - emptiness decidable
- Translate problems on constraints into Word automata [Müller et. al. LICS'98]