5^{th} ITYM — Problem 1: Pentominoes

Team France 2

Summary:
We solve completely the questions 1. to 3. Moreover, we also solve completely the question 3. for pentomino I, the questions 3.a for all pentominoes and 3.b for all pentominoes except Y, and answer partially the question 3c for all pentominoes.

1. We list of all the types of pentominoes, using a greedy algorithm ensures we have not forgotten any of them.
2. We prove, using a tool we called the *pretty line*, that every type of pentomino can tile the real plane.
3. We work on the case of pentomino P:
   a We prove that it can tile a rectangle of side $5 \times n$ if and only if n is even.
   b We find all the rectangles it can tile.
   c We find, for each size of rectangle, the maximum number of non-overlapping pentominoes P we can place in the rectangle.
4. We solve completely the question 3.a for every pentomino, the question 3.b for every pentomino except Y — for which we only have partial results. We also solve completely the question 3.c for pentomino I, and partially for all other types of pentominoes, obtaining a method that is sub-optimal but nevertheless efficient for rectangles with large side lengths.

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1 Classify all the pentominoes

Theorem 1:
There exists exactly 12 types of pentominoes, up to rotation and axial symmetry. ■

We show here how we came to this result: we built successively figures containing from 2 to 5 unit squares, making sure that we would forget none of them by looking at all possible expansions of such a figure when we had to add a unit square. We present our work with more detail in the following two pages.

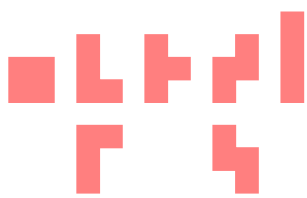
Definition 2:
Consider a flat figure composed unit squares, connected along their edges: we call it a

- *domino* if it contains 2 unit squares.
- *trimino* if it contains 3 unit squares.
- *tetromino* if it contains 4 unit squares.
- *pentomino* if it contains 5 unit squares.

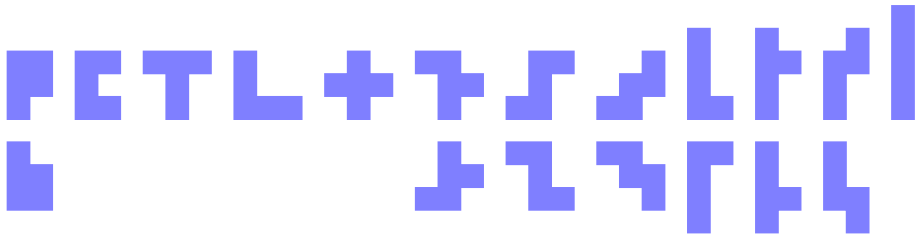
We consider these figures up to rotation and axial symmetry. ■

Proof of Theorem 1:
In order to make a complete list of the pentominoes, and to be sure we haven't forgotten any of them, we start with the dominoes. There is only one type of domino. If we add a unit square to this domino, we get a trimino. So we have tested all the possibilities and concluded that there are only 2 triminoes:

Then we have found all the tetrominoes we can get by adding a unit square to the two triminoes. There are 5 tetrominoes, 2 of which are not invariant (up to rotation) by axial symmetry:



Finally, we have found all the pentominoes using the same method. There are 12 pentominoes, 7 of which are not invariant (up to rotation) by axial symmetry:



P C T V X F Z W L Y N I

$\square$

We have associated a letter to each of them, in order to be able to name them for the rest of the article. The letters do not always have exactly the same shape than the pentominoes, but these are the standard pentominoes denominations, which is why we preferred them than more "personal" but less frequent letters the reader would not have been used to.

2 Determine the pentominoes that can tile the real plane

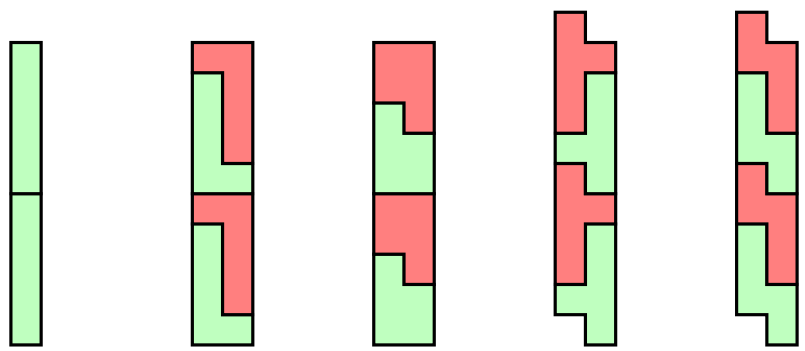
Theorem 3:
Every type of pentomino tiles the real plane. ■

In order to prove this result, we will consider each pentomino separately. In each case, we build parts of the real plane $\mathbb{R}^2$ that:

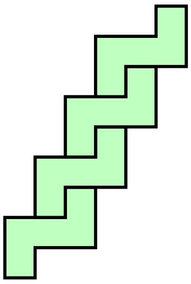
- are tiled by our pentomino using translations and, in some cases, rotations;
- tile the real plane by using translations only.

Definition 4:
A *pretty line* is a part P of $\mathbb{R}^2$ that tiles $\mathbb{R}^2$ using translations of vectors P tiles the plane by translations of $k\overrightarrow{v}$, with $k \in \mathbb{Z}$, where $\overrightarrow{v}$ is some vector with integer coordinates. ■

Proof of Theorem 3:
For each type of pentomino, we build blocks of one or two pentominoes, that tile some *pretty line* by translations of $k\overrightarrow{u}$, with $k \in \mathbb{Z}$, where $\overrightarrow{u}$ is some vector with integer coordinates. In each case, we show here from 2 to 6 such blocks (so that it is clear what our *pretty line* is), indicate the vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ mentioned above and draw the borders of the pentominoes of the block. □



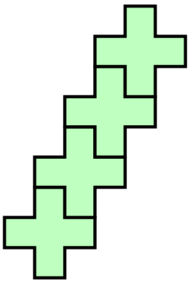
pentomino I	pentomino L	pentomino P	pentomino Y	pentomino N
$\overrightarrow{u} = \binom{0}{5}$	$\overrightarrow{u} = \binom{0}{5}$	$\overrightarrow{u} = \binom{0}{5}$	$\overrightarrow{u} = \binom{0}{5}$	$\overrightarrow{u} = \binom{0}{5}$
$\overrightarrow{v} = \binom{1}{0}$	$\overrightarrow{v} = \binom{2}{0}$	$\overrightarrow{v} = \binom{2}{0}$	$\overrightarrow{v} = \binom{2}{0}$	$\overrightarrow{v} = \binom{2}{0}$



pentomino Z

$\overrightarrow{u} = \binom{1}{2}$

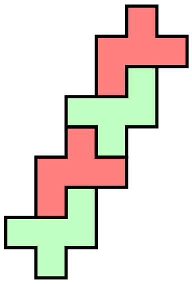
$\overrightarrow{v} = \binom{3}{1}$



pentomino X

$\overrightarrow{u} = \binom{1}{2}$

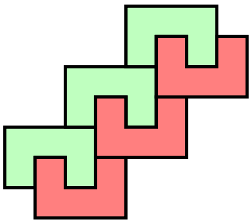
$\overrightarrow{v} = \binom{3}{1}$



pentomino F

$\overrightarrow{u} = \binom{2}{4}$

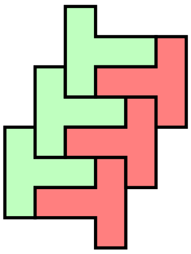
$\overrightarrow{v} = \binom{3}{1}$



pentomino C

$\overrightarrow{u} = \binom{2}{2}$

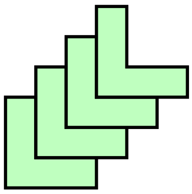
$\overrightarrow{v} = \binom{4}{-1}$



pentomino T

$\overrightarrow{u} = \binom{1}{2}$

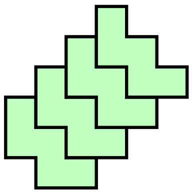
$\overrightarrow{v} = \binom{5}{0}$



pentomino V

$\overrightarrow{u} = \binom{1}{1}$

$\overrightarrow{v} = \binom{5}{0}$



pentomino W

$\overrightarrow{u} = \binom{1}{1}$

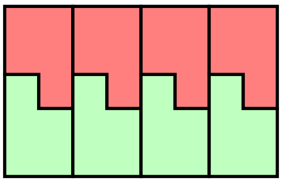
$\overrightarrow{v} = \binom{5}{0}$

3 Pentomino P

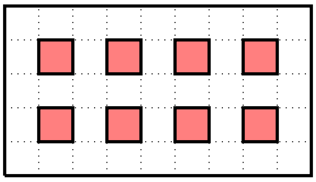
3.1 For which $n \in \mathbb{N}$ is it possible to tile completely a rectangle of size $5 \times n$?

Theorem 5:
The pentomino P tiles the rectangle $5 \times n$ if and only if n is even. ■

Proof of Theorem 5:
If n is even, we can tile a rectangle of size $5 \times n$ by using bricks of two pentominoes P as follows (here, in the case $n = 8$):



If n is odd, we color our rectangle as follows (here, in the case $n = 9$):



There are only $n - 1$ colored squares, and every pentomino P contains at least one of them. Therefore, we can put at most $n-1$ pentominoes P in the rectangle and we cannot tile the whole rectangle. □

3.2 For which $m, n \in \mathbb{N}$ is it possible to tile completely a rectangle of size $n \times m$?

Theorem 6:
A rectangle of size $m \times n$ can be tiled with pentominoes P if and only if the following conditions are verified:

1. at least one of m or n is divisible by 5,
2. neither m nor n is equal to 1 or 3, and
3. if either m or n is equal to 5, then the other side length is even.

■

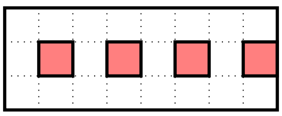
We will first prove that each of these conditions is necessary, then that their conjunction is sufficient.

Proposition 7:
If the side lengths m and n of a rectangle are not divisible by 5, this rectangle cannot be tiled by any type of pentomino. ■

Proof of Proposition 7:
Every pentomino is of area 5, so that it can tile only rectangles of area divisible by 5. If the dimensions side lengths m and n are not divisible by 5, neither is the area of the rectangle (by Gauss theorem), and the rectangle cannot be tiled by any type of pentomino. □

Proposition 8:
No rectangle of size $1 \times n$ nor $3 \times n$ can be tiled with pentominoes P. ■

Proof of Proposition 8:
First of all, note that no pentomino P fits into a single line, that is a $1 \times n$ rectangle. Then, we use a coloration similar to the one we used when proving Theorem 5:



There are only $\lfloor \frac{n}{2} \rfloor$ red squares, and every pentomino P contains at least one of them. Therefore, we can put at most $\lfloor \frac{n}{2} \rfloor \leq \frac{n}{2} < \frac{3n}{5}$ pentominoes P in the rectangle, and we cannot tile the whole rectangle. □

Proposition 9:
Consider some pentomino. If a, b, m, n are non-negative integers such that there exists

tilings of rectangles of sizes $m \times n$ and $m \times n'$, then there exists some tiling of the rectangle of size $m \times (an + bn')$. ■

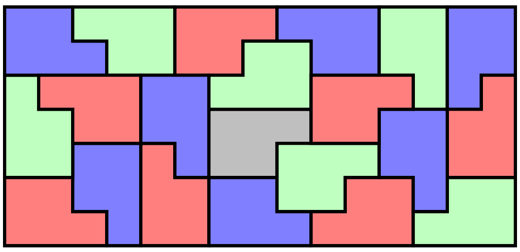
Proof of Proposition 9:
We just have to put side-to-side a tilings of the $m \times n$ rectangle and b tilings of the $m \times n'$ rectangle to get our tiling of the rectangle $m \times (an + bn')$. □

Proposition 10:
We can tile every rectangle of size $10 \times n$ such that $n \geq 7$. ■

Proof of Proposition 10:
Since we can tile rectangles of size 5×2, Proposition 9 implies that we can tile rectangles of size 10×2, and $10 \times n$ for every even integer n. Moreover, it also implies that we can tile rectangles of size 5×10 (or, equivalently, 10×5), and therefore that we can tile rectangles of size $10 \times (5 + 2n)$ for every integer $n \in \mathbb{N}$. In particular, since every integer $n \geq 7$ is either even or the sum of 5 and of some non-negative even number, Proposition 10 follows. □

Proposition 11:
We can tile every rectangle of size $15 \times n$ such that $n \geq 7$. ■

Proof of Proposition 11:
Since we can tile rectangles of size 5×2, Proposition 9 implies that we can tile rectangles of size 15×2, and $15 \times n$ for every even integer n. Moreover, the rectangle 15×7 can be tiled as follows:



Therefore, Proposition 9 implies that we can tile rectangles of size $10 \times (7 + 2n)$ for every integer $n \in \mathbb{N}$. In particular, since every integer $n \geq 7$ is either even or the sum of 7 and of some non-negative even number, Proposition 11 follows. □

Helped with all these intermediate results, we can now answer to the question asked in the first place.

Proof of Theorem 6:

First of all, Theorem 5 and Propositions 7 and 8 prove that the 3 conditions mentioned in Theorem 6 are necessary.
Conversely, let m and n be integers satisfying these 3 conditions. Without loss of generality, we assume that m is divisible by 5, and that $m \leq n$ if both m and n are divisible by 5.

- If n is even, Proposition 9 proves that we can tile some rectangle of size $5 \times n$, and therefore the rectangle of size $m \times n$ as well.
- If n is odd, then $m \neq 5$, so that $m \geq 10$. In this case, we cannot have $n = 5$ (because then we would also have $m = 5$), so that $n \geq 7$. Propositions 10 and 11 prove that we can tile the rectangles of sizes $10 \times n$ and $15 \times n$. Moreover,
  - if m is even, then it is divisible by 10, and
  - if m is odd, then m is the sum of 15 and of some non-negative multiple of 10.

  Therefore, in both cases, Proposition 9 proves that we can tile the rectangle of size $m \times n$, which closes the proof.

$\square$

3.3 How many pentominoes can a rectangle of size $n \times m$ contain?

Theorem 12:
Let m and n be positive integers. The maximal number of non-overlapping pentominoes P that can be placed in a rectangle of size $m \times n$ is:

- $\lfloor \frac{m}{2} \rfloor \lfloor \frac{n}{2} \rfloor$ if either m or n belongs to the set $\{1, 3, 5\}$, and
- $\lfloor \frac{mn}{5} \rfloor$ otherwise.

■

We will treat the first case, then the second one. In each case, we will prove simultaneously that the above values are upper bounds and lower bounds on the number of pentominoes P we can place without overlap.

Definition 13:
An *a-acceptable tiling* is a partial tiling of the rectangle $m \times n$ that lets only at most $4 + 5a$ unit squares empty or, equivalently, that contains at least $\lfloor \frac{mn}{5} \rfloor - a$ pentominoes. A *good tiling* is a 0-acceptable tiling. A *perfect tiling* is a tiling of the rectangle $m \times n$ which lets no unit square empty. ■

Proposition 14:
Consider some pentomino. If a, b, m, n are non-negative integers such that there exists a perfect tiling of the rectangle of size $m \times n$ and a b-acceptable tiling of the rectangle of size $m \times n'$, then there exists a b-acceptable tiling of the rectangle of size $m \times (an + n')$. ■

Proof of Proposition 14:
We just have to put side-to-side a perfect tilings of the $m \times n$ rectangle and the b-acceptable tiling of the $m \times n'$ rectangle to get our b-acceptable tiling of the rectangle $m \times (an + n')$. □

Since it is impossible to do better than having a good tiling, what we have to do is just to get as many good tilings as possible for rectangles with at least one small side length, then to use the rectangles pinpointed in Theorem 6 (those that have a perfect tiling) to get good tilings for rectangles with big side lengths.

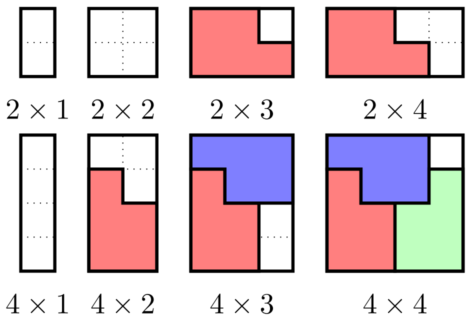
Proposition 15:
Consider some rectangle of size $m \times n$, with $m \in \{1, 3, 5\}$. Then, we can put a maximum of $\lfloor \frac{m}{2} \rfloor \lfloor \frac{n}{2} \rfloor$ non-overlapping pentominoes P in the rectangle. ■

Proof of Proposition 15:
The colorings used for proving Theorem 5 and Proposition 8 show that, indeed, we cannot hope for putting more pentominoes P than indicated in Proposition 15.

Conversely, for $m = 3$, as we can put 1 pentomino P in each rectangle of size 3×2; for $m = 5$, we can put 2 pentominoes P in each rectangle of size 5×2. Thus, for every choice of m, we can put $\lfloor \frac{m}{2} \rfloor$ pentominoes P in each rectangle of size $m \times 2$. It directly follows that we can put $\lfloor \frac{m}{2} \rfloor \lfloor \frac{n}{2} \rfloor$ pentominoes P in the rectangle of size $m \times n$. $\square$

Proposition 16:
Consider some rectangle of size $m \times n$, with $m \in \{2, 4\}$. Then there exists a good tiling of the rectangle. $\blacksquare$

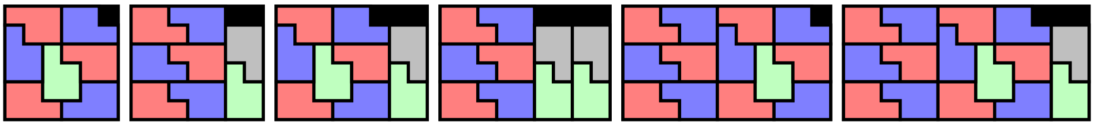
Proof of Proposition 16:
There exists a good tiling of each rectangle of size $2 \times n$ or $4 \times n$ with $1 \leq n \leq 4$. We give such tilings when $1 \leq n \leq 4$ below:



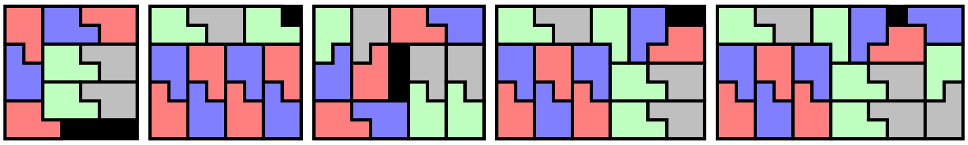
Moreover, there exists a perfect tiling of both rectangles of sizes 2×5 and 4×5. Therefore, Proposition 14 states that each rectangle of size $2 \times n$ or $4 \times n$ has a good tiling. $\square$

Proposition 17:
Consider some rectangle of size $m \times n$, $6 \leq m, n \leq 15$. Then there exists a good tiling of the rectangle. $\blacksquare$

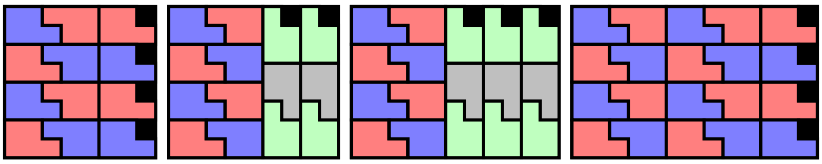
Proof of Proposition 17:
First of all, if $m \in \{10, 15\}$ and $n \geq 6$, then Propositions 10 and 11 state that the rectangle of size $m \times n$ has a perfect tiling, thus a good tiling.
Then, if $m \in \{12, 14\}$ and $n \geq 6$, Proposition 10 states that the rectangle of size $10 \times n$ has a perfect tiling. Moreover, Proposition 16 states that the rectangle of size $(m-10) \times n$ has a good tiling. Therefore, Proposition 14 implies that the rectangle of size $m \times n$ has a good tiling.
Finally, we look at the case where $m \in \{6, 7, 8, 9, 11, 13\}$. Without loss of generality, we assume that $m \leq n$. Then, we just said that the rectangles of sizes $m \times n$ such that $n \in \{10, 12, 14, 15\}$ have a good tiling. We complete the proof by noting that, if $n \in \{6, 7, 8, 9, 11, 13\}$, then the rectangle of size $m \times n$ also has a good tiling, as shown below (the unit squares not tiled are colored in black):



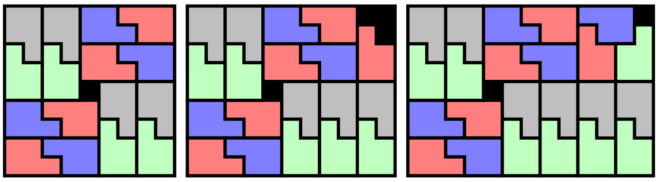
6×6 6×7 6×8 6×9 6×11 6×13



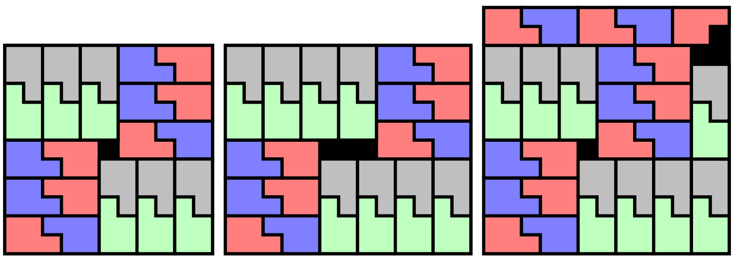
7×7 7×8 7×9 7×11 7×13



8×8 8×9 8×11 8×13



9×9 9×11 9×13



11×11 11×13 13×13

□

Having worked on all those particular cases, we can now tackle the initial question.

Proof of Theorem 12:
Without loss of generality, let us assume that $m \leq n$. First of all, if either m or n belongs to the set $\{1, 3, 5\}$, we have already stated the result in Proposition 15. Then, if m or n belongs to the set $\{2, 4\}$, Proposition 16 also states the result. Finally, if $6 \leq m \leq n$, consider the positive integers k and ℓ such that $6 \leq m - 10k \leq 15$ and $6 \leq n - 10\ell \leq 15$. It follows from Theorem 6 that the rectangles of size $(10k) \times n$ and $(m - 10k) \times (10\ell)$ have a perfect tiling. Moreover, Proposition 17 proves that the rectangle of size $(m - 10k) \times (n - 10\ell)$ has a good tiling. Therefore, by applying twice Proposition 14, the rectangle of size $m \times n$ also has a good tiling. □

4 Look for the other types of pentominoes

4.1 Pentomino I

We first look at which rectangles can be entirely tiled with pentominoes I. **Theorem 18:**
A rectangle of size $m \times n$ can be tiled with pentominoes I if and only if at least one of m or n is divisible by 5. ■

Proof of Theorem 18:
First, Proposition 8 already tells us that, if 5 divides neither m nor n, the rectangle of size $m \times n$ cannot be tiled with pentominoes I.
Conversely, let us assume that 5 divides m. Since the pentomino I is itself a rectangle of size 5×1, it tiles the rectangle of size $5 \times n$, and therefore the rectangle of size $m \times n$ as well. □

Then, we compute how many pentominoes I we can place, at most, in any rectangle.

Theorem 19:
A rectangle of size $m \times n$, with $m \leq n$, can be partially tiled with at most:

1. $m\lfloor \frac{n}{5} \rfloor$ pentominoes I if $m \leq 4$,
2. $\lfloor \frac{mn}{5} \rfloor - 1$ pentominoes I if $m \geq 5$ and $\{m, n\} \equiv \{2, 3\} \pmod 5$,
3. $\lfloor \frac{mn}{5} \rfloor$ pentominoes I in every other case.

■

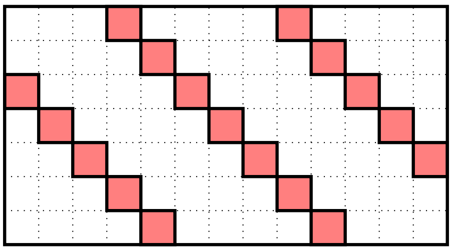
We treat first the case $4 \leq m$, then show that $\lfloor \frac{mn}{5} \rfloor - 1$ is indeed an upper bound when $\{m, n\} \equiv \{2, 3\} \pmod 5$, and finally show that the two values given in the cases 2 and 3 are actually maximal values.

Proposition 20:
A rectangle of size $m \times n$, with $m \leq 4$, can be partially tiled with at most $m\lfloor \frac{n}{5} \rfloor$ pentominoes I. ■

Proof of Proposition 20:
Let us assume that our rectangle has m rows, each of length n. Then, all our pentominoes I must be placed horizontally, which immediately proves Proposition 20. □

Proposition 21:
A rectangle of size $m \times n$, with $\{m, n\} \equiv \{2, 3\} \pmod 5$, has no good tiling by the pentomino I. ■

Proof of Proposition 21:
Without loss of generality, let us assume that $m \equiv 3 \pmod{5}$ and $n \equiv 2 \pmod{5}$. We color our rectangle as follows (here, $m = 13$ and $n = 7$):



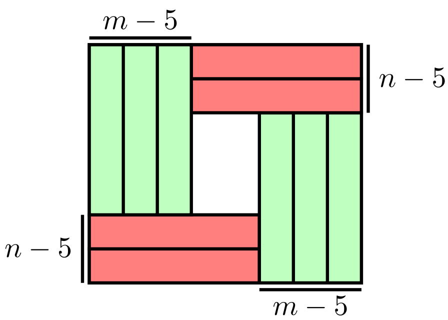
More precisely, we color the cell at row i and column j (with $0 \leq i < m$ and $0 \leq j < n$) if and only if $i + j \equiv 4 \pmod{5}$. Then, every pentomino I will contain at least one such colored cell. Moreover, the $n-2$ first rows contain exactly $\frac{m(n-2)}{5}$ colored cells overall, while each of the top 2 rows contains exactly $\frac{m-3}{5}$ colored cells. Therefore, the rectangle of size $m \times n$ contains excatly $\frac{m(n-2)+2(m-3)}{5} = \lfloor \frac{mn}{5} \rfloor - 1$ colored cells. It follows that the rectangle has no good tiling by the pentomino I. □

Proposition 22:
A rectangle of size $m \times n$, with $5 \leq m, n \leq 9$, has a

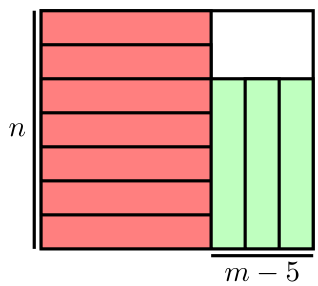
- 1-acceptable tiling if $\{m, n\} \equiv \{2, 3\} \pmod{5}$;
- good tiling otherwise.

■

Proof of Proposition 22:
There exists partial tilings that respectively contain $2(m+n-10)$ and $m+n-5$ pentominoes I, as follows (here, in the case $m = 8$ and $n = 7$):



Tiling with
$2(m+n-10)$ pentominoes



Tiling with
$m+n-5$ pentominoes

The first partial tiling lets $(10-m)(10-n)$ empty unit squares; the second one lets $(m-5)(n-5)$ empty unit squares. Therefore,

- if $m \leq 6$ or $n \leq 6$, the second tiling is a good tiling;
- if $\{m,n\} \in \{\{7,9\},\{8,8\},\{8,9\},\{9,9\}\}$, the first tiling is a good tiling;
- if $\{m,n\} = \{7,7\}$, the second tiling is a good tiling;
- if $\{m,n\} = \{7,8\}$, both tilings are 1-acceptable tilings.

$\square$

Proof of Theorem 19:
First of all, Propositions 20 and 21 indicate that the values given in Theorem 19 are upper bounds on the number of pentominoes I we can place in any rectangle of size $m \times n$. Moreover, if $m \leq 4$, Proposition 20 shows that this value can be attained. In addition, if $5 \leq m,n \leq 9$, Proposition 21 gives us 1-acceptable tilings, and even good tilings if $\{m,n\} \not\equiv \{2,3\}$ (mod. 5). Finally, let us consider non-negative integers a and b such that $5 \leq m-5a \leq 9$ and $5 \leq n-5b \leq 9$:

- if $\{m,n\} \equiv \{2,3\}$ (mod. 5), then the rectangle of size $(m-5a) \times (n-5b)$ has a 1-acceptable tiling and the rectangles of sizes $(m-5a) \times (5b)$ and $(5a) \times n$ have perfect tilings, so that the rectangle of size $m \times n$ has a 1-acceptable tiling;
- if $\{m,n\} \not\equiv \{2,3\}$ (mod. 5), then the rectangle of size $(m-5a) \times (n-5b)$ has a good tiling and the rectangles of sizes $(m-5a) \times (5b)$ and $(5a) \times n$ have perfect tilings, so that the rectangle of size $m \times n$ has a good tiling.

This closes the proof of Theorem 19. $\square$

4.2 Pentomino L

Theorem 23:
A rectangle of size $m \times n$ can be tiled with pentominoes L if and only if the following conditions are verified:

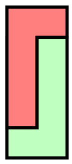
1. at least one of m or n is divisible by 5,
2. neither m nor n is equal to 1 or 3, and
3. if either m or n is equal to 5, then the other side length is even.

■

Note that these are exactly the rectangles that can be tiled with pentominoes P, as stated by Theorem 6. We therefore prove this result by using the same process as for pentominoes P.

Proposition 24:
The pentomino L tiles the rectangle ot size $5 \times n$ if and only if n is even. ■

Proof of Proposition 24:
If n is even, we can tile the rectangle of size 5×2 as follows, and therefore we can also tile the rectangle of size $5 \times n$:

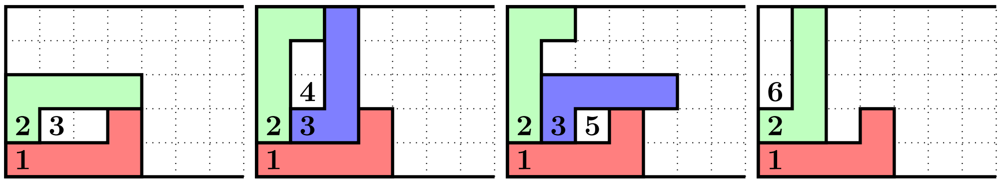


If some rectangle of size $5 \times n$ with n odd can be tiled with the pentomino L, consider a minimal such odd n, and look at the leftmost cells of the rectangle. Here starts a long, but conceptually straightforward, disjunction of cases, which allows us to conclude that we cannot tile our rectangle, thereby completing the proof.
There are 5 ways to tile the cell 1:

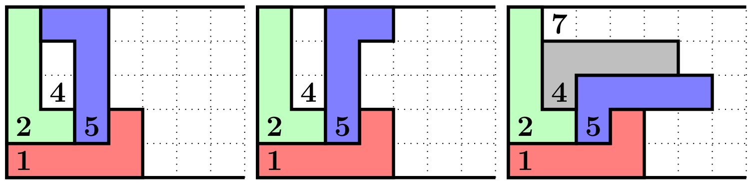
- In the way #1, there are 4 ways to tile the cell 2:
  - In the way #1.1, it becomes impossible to tile the cell 3.
  - In the way #1.2, there are 2 ways to tile the cell 3:
    * In the way #1.2.1, it becomes impossible to tile the cell 4.
    * In the way #1.2.2, it becomes impossible to tile the cell 5.
  - In the way #1.3, it becomes impossible to tile the cell 6.
  - In the way #1.4, there are 3 ways to tile the cell 5:

        * In the way #1.4.1, it becomes impossible to tile the cell 4.
        * In the way #1.4.2, it becomes impossible to tile the cell 4.
        * In the way #1.4.3, there remains only 1 possibility for tiling the cell 4, and it becomes impossible to tile the cell 7.

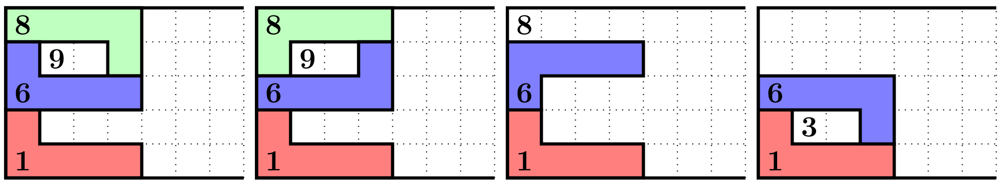
- In the way #2, there are 4 ways to tile the cell 6:
    - In the way #2.1, there remains only 1 possibility for tiling the cell 8, and it becomes impossible to tile the cell 9.
    - In the way #2.2, there remains only 1 possibility for tiling the cell 8, and it becomes impossible to tile the cell 9.
    - In the way #2.3, it becomes impossible to tile the cell 8.
    - In the way #2.4, it becomes impossible to tile the cell 3.
- In the way #3 there are 3 ways to tile the cell 6:
    - In the way #3.1, there remains only 1 possibility for tiling the cell 8, and it becomes impossible to tile the cell 9.
    - In the way #3.2, there remains only 1 possibility for tiling the cell 8, and it becomes impossible to tile the cell 9.
    - In the way #3.3, it becomes impossible to tile the cell 8.
- In the way #4, there are 3 ways to tile the cell 8:
    - In the way #4.1, there remains only 1 possibility for tiling the cell 9, and it becomes impossible to tile the cell 10.
    - In the way #4.2, there remains only 1 possibility for tiling the cell 9, and it becomes impossible to tile the cell 4.
    - In the way #4.3, there remains to tile a rectangle of size $5 \times (n-2)$, which is impossible by minimality of n.
- In the way #5, there is 1 way to tile the cell 8, and there remains 3 ways to tile the cell 10:
    - In the way #5.1, it becomes impossible to tile the cell 4.
    - In the way #5.2, it becomes impossible to tile the cell 4.
    - In the way #5.3, there remains only 1 possibility for tiling the cell 4, and it becomes impossible to tile the cell 11.
- In the way #6, there is 1 way to tile the cell 2, and then there remains to tile a rectangle of size $5 \times (n-2)$, which is impossible by minimality of n.



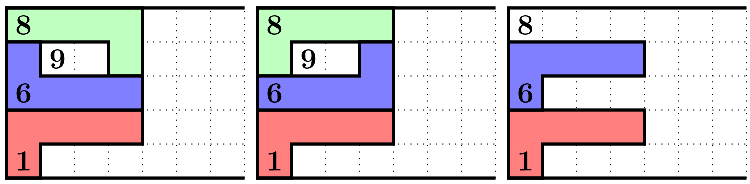
way #1.1 way #1.2.1 way #1.2.2 way #1.3



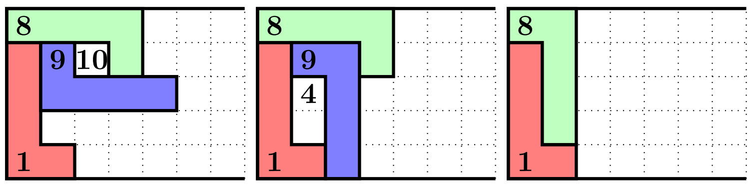
way #1.4.1 way #1.4.2 way #1.4.3



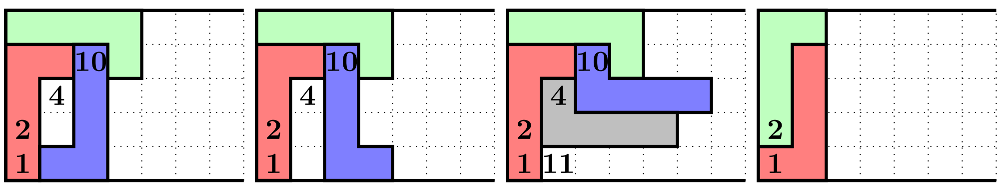
way #2.1 way #2.2 way #2.3 way #2.4



way #3.1 way #3.2 way #3.3



way #4.1 way #4.2 way #4.3



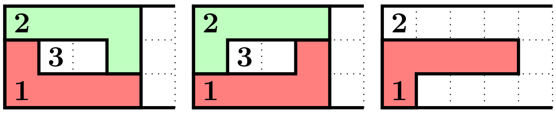
way #5.1 way #5.2 way #5.3 way #6

□

Proposition 25:
No rectangle of size $1 \times n$ nor $3 \times n$ can be tiled with pentominoes L. ■

Proof of Proposition 25:
For the rectangle of size $1 \times n$, we can not place any pentominoes L, so we can't tile the rectangle.
For the rectangle of size $3 \times n$, we try to tile the leftmost cells of the rectangle, then reach a contradiction.
There are 3 ways to tile the cell 1:

- In the case #1, there remains only 1 possibility for tiling the cell 2, and it becomes impossible to tile the cell 3.
- In the case #2, there remains only 1 possibility for tiling the cell 2, and it becomes impossible to tile the cell 3.
- In the case #3, it becomes impossible to tile the cell 2.

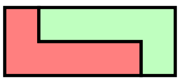


way #1 way #2 way #3

Our systematic failures to get a correct tiling prove that we could in fact not tile any rectangle of size $3 \times n$. □

Proposition 26:
We can tile every rectangle of size $10 \times n$ such that $n \geq 7$. ■

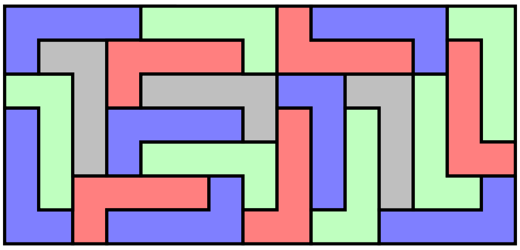
Proof of Proposition 26:
Since we can tile a rectangle of size 5×2 with pentominoes L, the proof of Proposition 10 also works here — we just have to replace "pentomino P" by "pentomino L" everywhere.



□

Proposition 27:
We can tile every rectangle of size $15 \times n$ such that $n \geq 7$. ■

Proof of Proposition 27:
We show below a tiling of the rectangle of size 7×15 by pentominoes L. Therefore, the proof of Proposition 11 also works here — we just have to replace "pentomino P" by "pentomino L" everywhere.



□

Proof of Theorem 23:
Theorem 23 and Propositions 24, 25, 26 and 27 are analogs to Theorems 6 and 5 and Propositions 8, 10 and 11 — we just have replaced “pentomino P” by “pentomino L” everywhere. Therefore, the proof of Theorem 6 also works here. □

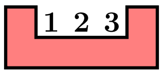
4.3 Pentomino Y

We do not find a complete characterization of those rectangles that can be tiled with the pentomino Y. However, we still get the partial result that we were first asked about the pentomino P: which rectangles of size $5 \times n$ can be tiled with the pentomino Y?

Theorem 28:
The pentomino Y tiles the rectangle of size $5 \times n$ if and only if 10 divides n. ■

We might proceed directly like for the pentominoes P or L, by a brutal disjunction of cases. However, it was already quite long in the case of pentomino L, and would be practically unreadable in the case of the pentomino Y because it would be really huge. Therefore, we first prove some untilability results, based on symmetries and on "untilable patterns", then look for all possible tilings of a rectangle (but we will go faster than in the previous parts, since it has become quite clear now how we want to proceed).

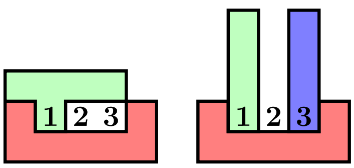
Proposition 29:
If the colored cells are either already tiled or out of the rectangle, you cannot tile the cells 1, 2 and 3 simultaneously.



■

Proof of Proposition 29:
Up to symmetries, there are (at most) 5 ways to tile the cell 1:

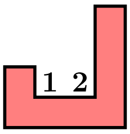
- In the way #1, it becomes impossible to tile the cell 2.
- In the ways #2 to #5, all the green cells — in the 4×1 rectangle that contains the cell 1 — at least are tiled, but neither the cell 2 nor 3 is; then, whichever way of tiling the cell 3 we use, all the blue cells — in the 4×1 rectangle that contains the cell 3 — at least will be tiled, but not the cell 2; then, it becomes impossible to tile the cell 2.



way #1 ways #2 to #5

□

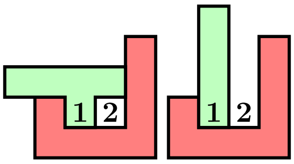
Proposition 30:
If the colored cells are either already tiled or out of the rectangle, you cannot tile the cells 1 and 2.



■

Proof of Proposition 30:
There are (at most) 5 ways to tile the cell 1:

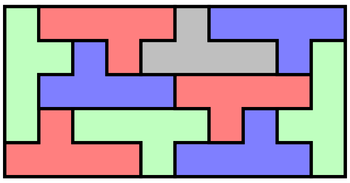
- In the way #1, it becomes impossible to tile the cell 2.
- In the ways #2 to #5, all the green cells — in the 4×1 rectangle that contains the cell 1 — at least are tiled, but not the cell 2; then, it becomes impossible to tile the cell 2.



way #1 way #2

□

Proof of Theorem 28:
First of all, note that it is indeed possible to tile a rectangle of size 5×10 with the pentomino Y, as follows:



Now, we prove that every tiling of a rectangle of size $5 \times n$ by the pentomino Y starts with a tiling of a rectangle of size 5×10, which will complete the proof.
There are 4 ways to tile the cell 1:

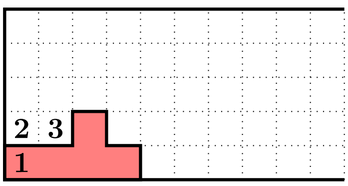
- In the way #1, it becomes impossible to tile the cells 2 and 3, according to the Proposition 29.
- In the way #2, it becomes impossible to tile the cells 3 and 4, according to the Proposition 29.
- In the way #3, there are 2 ways to tile the cells 2:

- In the way #3.1, it becomes impossible to tile the cells 5 and 6, according to the Proposition 29.
- In the way #3.2,there are 2 possibility for tiling the cell 6:
  * In the way #3.2.1, there remains 1 way to tile the cells 7 and 8. Then, there are 4 ways to tile the cell 9, including 3 ways — that we chose not to represent — that make at least one of the cells 10 and 11 untilable. Continuing, there is 1 way to tile the cell 11 and 2 ways to tile the cells 12 and 13. However, both these ways will tile the same cells, so we treat them both simultaneously. Finally, there are 3 ways to tile the cell 14:
    · In the way #3.2.1.1, we tile a rectangle of size 5×10.
    · In the way #3.2.1.2, it becomes impossible to tile the cell 15.
    · In the way #3.2.1.3, it becomes impossible to tile both the cells 16 and 17, according to the Proposition 29.
  * In the way #3.2.2, there remains 3 ways to tile the cell 18, including 1 way — not represented — that makes the cell 8 untilable:
    · In the way #3.2.2.1, we arrive at the situation we met in the way #3.2.1 after tiling the cells 6 and 7; therefore, we do not continue further in this way.
    · In the way #3.2.2.2, there are 3 ways to tile the cell 19, including 2 ways — not represented — that make the cell 20 untilable. Then, there is 1 way to tile the cell 13, and 2 ways to tile the cell 21:

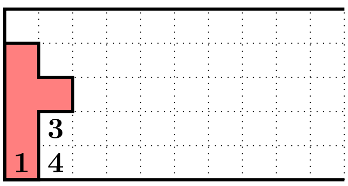
      In the way #3.2.2.2.1, there remains 1 way to tile the cell 10, then 2 ways — identical up to a vertical symmetry — to tile the cell 22. Choosing the one we show below, it becomes impossible to tile both the cells 17 and 23, according to the Proposition 29.

      In the way #3.2.2.2.2, there remains 1 way to tile the cell 12, then the cell 11, and we arrive at the situation we met in the way #3.2.1 after tiling the cells 12 and 13; therefore, we do not continue further in this way.

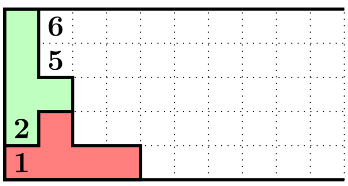
- In the way #4, there remains 2 ways to tile the cell 24, including 1 way — not represented — that makes the cell 5 untilable. Then, we arrive at the situation we met in the way #3.2, up to a vertical symmetry; therefore, we do not continue further in this way, and our long disjunction of cases is completed.



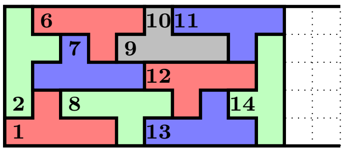
way #1



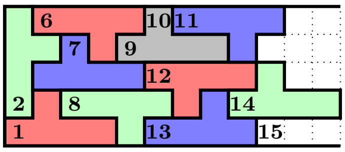
way #2



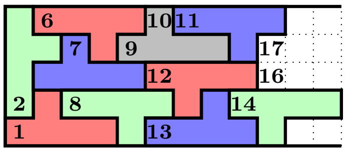
way #3.1



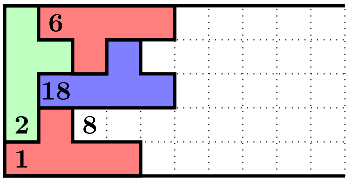
way #3.2.1.1



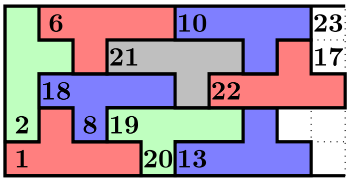
way #3.2.1.2



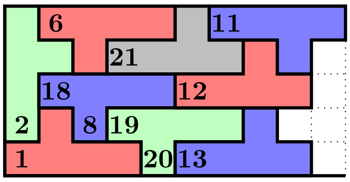
way #3.2.1.3



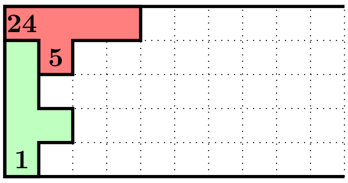
way #3.2.2.1



way #3.2.2.2.1



way #3.2.2.2.2



way #4

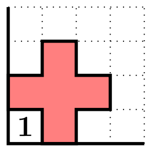
□

4.4 The remaining pentominoes cannot tile any rectangle

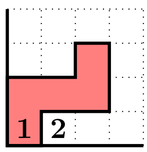
Theorem 31:
None of the pentominoes X, Z, F, T, C, V, W and N tiles any rectangle. ■

Proof of Theorem 31:
We treat separately the case of each type of pentomino.

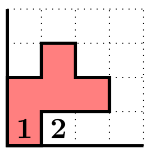
Pentomino X: It is impossible to tile the cell 1, hence to tile the rectangle itself:



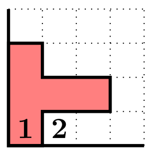
Pentominoes Z, F and T: Up to symmetry of the rectangle, there is only one way to tile the cell 1, and it becomes impossible to tile the cell 2:



pentomino Z

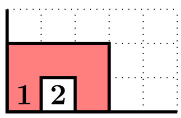


pentomino F

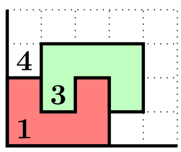


pentomino T

Pentomino C: We have to study more cases. Up to symmetry of the rectangle, there are 2 ways to tile the cell 1. However, in the way #1, it is impossible to tile the cell 2. In the way #2, there remains only 1 possibility for tiling the cell 3, and it becomes impossible to tile the cell 4:



way #1

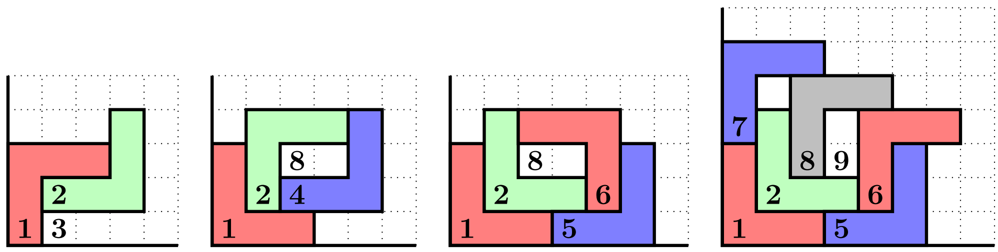


way #2

Pentomino V: We have to study even more cases. Up to symmetry, there are 2 ways to tile the cell 1:

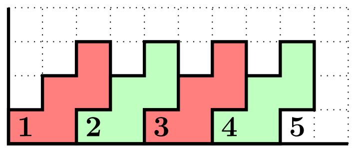
- In the way #1, there remains only 1 possibility for tiling the cell 2, and it becomes impossible to tile the cell 3.

- In the way #2, up to symmetry, there are 2 ways to tile the cell 2:
  - In the way #2.1, there remains only 1 possibility for tiling the cell 4, and it becomes impossible to tile the cell 8.
  - In the way #2.2, there remains only 1 possibility for tiling the cell 5, and there are still 2 ways to tile the cell 6:
    - ∗ In the way #2.2.1, it is impossible to tile the cell 8.
    - ∗ In the way #2.2.2, there remains only 1 possibility for tiling the cell 7, then the cell 8, and it becomes impossible to tile the cell 9.

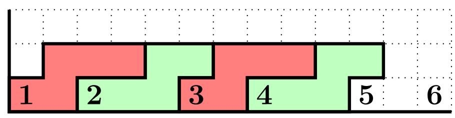


way #1 way #2.1 way #2.2.1 way #2.2.2

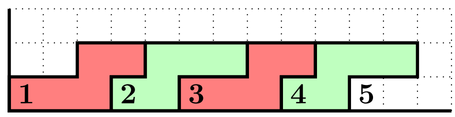
Pentominoes W and N: Up to symmetry, there are respectively 1 (for W) and 2 (for N) ways to tile the cell 1. However, then, an immediate induction shows that our *partial* tilings always have an empty unit square on the bottom row (which we numbered on the figures below by 2, 3, 4, 5, 6, and continuing), and that there is only 1 way to tile the leftmost such square. In particular, it will not be possible to tile entirely the bottom row of any rectangle, hence to tile the rectangle itself:



pentomino W



pentomino N: ways #1 and #2



Overall, we have seen that there were either individual cells, or some row, that we could not tile entirely with our pentominoes. In particular, those pentominoes cannot tile entirely any rectangle. □

4.5 Sub-optimal but good partial tiling or rectangles

To tile a rectangle, we can use the same method as that we used to tile the real plane, then we can keep only those pentominoes that are entirely inside our rectangle. Doing so is not optimal, but is a simple way to place a big number of pentominoes in a rectangle. Indeed, the maximum number of untiled cells is proportional to the perimeter of the rectangle. Therefore, when both dimensions of the rectangle become huge, the perimeter becomes negligible compared to the area.

Proposition 32:
Consider a rectangle of size $m \times n$, with $m \leq n$. Using this method, we can place at least:

- $\left\lceil \frac{(m-2)(n-4)}{5} \right\rceil$ pentominoes C,
- $\left\lceil \frac{(m-4)(n-4)}{5} \right\rceil$ pentominoes T, V, X, F Z or W, and
- $\left\lceil \frac{(m-2)(n-6)}{5} \right\rceil$ pentominoes L, Y or N.

■

Proof of Proposition 32:
We proceed as follows: we place each pentomino in a *box* of minimal size, find sufficient conditions for a box to be entirely inside our rectangle.
First, note that:

- each pentomino C can be placed entirely in a box of size 2×3,
- each pentomino T, V, X, F Z or W can be placed entirely in a box of size 3×3, and
- each pentomino L, Y or N can be placed entirely in a box of size 2×4.

Let $a \times b$ be the dimensions of the box, depending on the pentomino we chose. Using the tiling suggested by the *pretty line* tiling method, each pentomino is placed in a box whose width is smaller than or equal to the height. Then, suppose that we rotate our rectangle to tile so that it has width m and height n.
Consider now a cell, inside the rectangle of size $m \times n$, that is at distance at least $a - 1$ from the left and right sides, and at distance at least $b - 1$ from the top and bottom sides. Then, look at the pentomino that tiles this cell according to the *pretty line* tiling method — regardless of whether this pentomino is entirely in the rectangle or not, and focus on the box that contains this pentomino. The box contains our cell, and therefore cannot cross any size of the rectangle — it can be tangent to one or two sides, but not more. Therefore, the pentomino that tiles this cell is entirely contained in the rectangle, and appears in the tiling mentioned just above.
In particular, our partial tiling tiles at least all the cells just described above, and there are $(m + 2 - 2a)(n + 2 - 2b)$ such cells, which completes the proof. □