Forbidden Subgraphs in Connected Graphs [1]

Vlady Ravelomanana [a] Loÿs Thimonier [b]

[a] *LIPN UMR 7030, Université de Paris-Nord*
99, Avenue J. B. Clément. F 93430 Villetaneuse France

[b] *LaRIA EA 2083, Université de Picardie*
5, Rue du Moulin-Neuf. 80000 Amiens France

Abstract

Given a set $\xi = \{H_1, H_2, \cdots\}$ of connected non acyclic graphs, a ξ-free graph is one which does not contain any member of ξ as copy. Define the excess of a graph as the difference between its number of edges and its number of vertices. Let $\widehat{W}_{k,\xi}$ be theexponential generating function (EGF for brief) of connected ξ-free graphs of excess equal to k ($k \geq 1$). For each fixed ξ, a fundamental differential recurrence satisfied by the EGFs $\widehat{W}_{k,\xi}$ is derived. We give methods on how to solve this nonlinear recurrence for the first few values of k by means of graph surgery. We also show that for any finite collection ξ of non-acyclic graphs, the EGFs $\widehat{W}_{k,\xi}$ are always rational functions of the generating function, T, of Cayley's rooted (non-planar) labelled trees. From this, we prove that almost all connected graphs with n nodes and $n+k$ edges are ξ-free, whenever $k = o(n^{1/3})$ and $|\xi| < \infty$ by means of Wright's inequalities and saddle point method. Limiting distributions are derived for sparse connected ξ-free components that are present when a random graph on n nodes has approximately $\frac{n}{2}$ edges. In particular, the probability distribution that it consists of trees, unicyclic components, $\cdots$, $(q+1)$-cyclic components all ξ-free is derived. Similar results are also obtained for multigraphs, which are graphs where self-loops and multiple-edges are allowed.

Key words: Combinatorial problems; enumerative combinatorics; analytic combinatorics; labelled graphs; multivariate generating functions; asymptotic enumeration; random graphs; triangle-free graphs.

[1] First version of this paper appeared in the 4-th Latin American Theoretical INformatics Conference – Punta del Este – Uruguay, April 2000 and some parts of this paper appeared in the 13-th International Conference on Formal Power Series and Algebraic Combinatorics – Arizona – USA, May 2001 (cf. [28, 29]). This research was done while the first author was at LaRIA, Amiens – France.

[2] Corresponding authors. Email: vlad@lipn.univ-paris13.fr, thimon@laria.u-picardie.fr .

1 Introduction

We consider here labelled graphs, i.e., graphs with labelled vertices, undirected edges and without self-loops or multiple edges as well as labelled *multigraphs* which are labelled graphs with self-loops and/or multiple edges. A (n, q) graph (resp. multigraph) is one having n vertices and q edges.

On one hand, classical papers [12, 13, 14, 21] provide algorithms and analysis of algorithms that deal with random graphs or multigraphs generation, estimating relevant characteristics of their evolution. Starting with an initially empty graph of n vertices, we enrich it by successively adding edges. As random graph evolves, it displays a phase transition similar to the typical phenomena observed with percolation process. On the other hand, various authors such as Wright [39, 41] or Bender, Canfield and McKay [4, 5] studied exact enumeration or asymptotic properties of labelled *connected* graphs.

A lot of research is devoted to graphs not containing a prefixed set of subgraphs as copies and various approaches exist for these problems. Most of them, following Erdös and Rényi's seminal papers [12, 13], are probabilistic; moment methods, tail inequalities, or probabilistic inequalities are then essential as well explained in [7]. These approaches take advantage over enumerative ones by allowing treatments under the edges independence assumption [7]. The situation changes radically if we consider connected components, and results relative to connectedness are few. Related works include [39, 40, 41, 4, 5, 6, 14, 21]

Let H be a fixed connected graph; by a *copy of H*, we mean any subgraph, not necessarily induced, isomorphic to H. Let $\mathcal{F}$ be a family of graphs none of which contains a copy of H. In this case, we say that the family $\mathcal{F}$ is *H-free*. Otherwise, a graph containing a copy of H is called a *supergraph* of H. The highly non-trivial task of enumerating *triangle-free* or *quadrilateral-free* components goes back to the book of Harary and Palmer [20].

Mostly forbidden configurations are triangle, quadrilateral, ..., C_p, K_p, $K_{p,q}$ or any combination of them (see [7, Chapter IV], [22, Chapter III]). C_p shall always denote the cycle on p vertices, K_p the complete graph with p vertices and $K_{p,q}$ the complete bipartite graph with p vertices on the first side and q vertices on the second side. For example, we can work with the family of graphs which do not contain a copy of triangle (C_3) or of $K_{3,3}$, i.e., $\{C_3, K_{3,3}\}$-free graphs. Following the authors of [14], we refer as *bicyclic* graphs all connected graphs with n vertices and $(n+1)$ edges and in general $(q+1)$-*cyclic* graphs are connected $(n, n+q)$ graphs. If we define the *excess* of a graph as the difference between its number of edges and its number of vertices, $(q+1)$-*cyclic* graphs are referred also as q-*excess* connected graphs. In general, we refer as *multicyclic* a connected graph which is not acyclic. The same nomenclature holds for multigraphs. More generally, denote by $\xi = \{H_1, H_2, H_3, ...\}$ a set of connected multicyclic graphs (resp. multigraphs); a ξ-*free* graph is then one which does not contain any copy of H_i for all $H_i \in \xi$ as subgraph. Throughout this paper, unless explicitly mentioned, ξ denotes a *finite* set of forbidden configurations.

Our aim in this paper is

1. to study randomly generated graphs with n vertex and approximately $\frac{n}{2}$ edges focusing our attention on the appearance or not of the forbidden configurations,
2. to compute the asymptotic number of ξ-free connected graphs when ξ is finite.

The results obtained here show that some characteristics of random generation as well as asymptotic enumeration of labelled graphs or multigraphs, can be read within the forms of the exponential generating functions (EGF for short) of the sparse components. In fact, denote by $\widehat{W}_k$ $(k \geq -1)$ the EGFs of $(k+1)$-cyclic (connected) graphs. In a series of important papers, [39, 40, 41], E. M. Wright proved that $\widehat{W}_k(z)$ $(k \geq 1)$, where z is the variable marking the number of vertices in the graph, can be expressed as finite sums of power of $1/(1 - T(z))$ where $T(z) = \sum_{n \geq 1} n^{n-1} \frac{z^n}{n!}$ is the EGF for rooted labelled trees [9, 27]. Starting with a functional equation satisfied by our $(k + 1)$-cyclic ξ-free graphs; we will show that their EGF, denoted $\widehat{W}_{k,\xi}$, have the same global forms as those of $(k + 1)$-cyclic graphs, i.e., $\widehat{W}_k$. These forms will allow us to study random graphs without forbidden configurations and also to enumerate asymptotically connected components of these objects under some restrictions. Similar results related to multigraphs will be treated and carried along this paper, in parallel. Since our results concern graphs and multigraphs, we will be frequently assuming throughout this paper that the term *component* is the general term for connected graph as well as for connected multigraph.

1.1 *Asymptotic number of ξ-free $(n, n + o(n^{1/3}))$ components*

In the first part of this paper, we will compute the asymptotic number of triangle-free connected $(n, n + k)$-graphs, whenever $k = o(n^{1/3})$. To do this, we will rely heavily on the results in [41] to prove that the power series $\widehat{W}_{k,C_3}$ satisfy the same inequalities as for $\widehat{W}_k$ which we shall call here "*Wright's inequalities*". Next, we will investigate the asymptotic behavior of the coefficient of z^n in $\frac{1}{(1-T(z))^{k(n)}}$ (where T is the EGF for Cayley's rooted labelled trees) by means of saddle point method. The combination of these computations will permit us to show *almost all* connected $(n, n + o(n^{1/3}))$ graphs, i.e., connected graphs with n vertices and $n + o(n^{1/3})$ edges are triangle-free. These asymptotic results are related to the interesting problems posed by Harary and Palmer in their reference book (see [20, Sect. 10.4, 10.5 and 10.6]). The purpose of this part is also to introduce methods by which the asymptotic number of connected ξ-free $(n, n+k)$ graphs can be computed systematically, whenever $k = o(n^{1/3})$.

1.2 Forbidden subgraphs in random $(n, \frac{n}{2})$ components

The two models of graph evolution, explicitly introduced in [14], are considered in the second part of this note, in order to generate randomly graphs and multigraphs. We will study the structure of evolving graphs and multigraphs when edges are added one at time and at random, mainly looking at the presence or absence of certain configurations. In [21, Theorem 5], the authors proved that the probability that a random graph or multigraph with n vertices and $\frac{n}{2} + O(n^{1/3})$ edges has r_1 bicyclic components, r_2 tricyclic components, ..., r_q $(q+1)$-cyclic components and no components of higher-cyclic order is

$$\left(\frac{4}{3}\right)^r \sqrt{\frac{2}{3}} \frac{b_1^{r_1}}{r_1!} \frac{b_2^{r_2}}{r_2!} \cdots \frac{b_q^{r_q}}{r_q!} \frac{r!}{(2r)!} + O(n^{-\frac{1}{3}}) \tag{1}$$

where $r = r_1 + 2r_2 + \cdots + qr_q$ and the b_i are *Wright's constants* also found by Louchard and Takács ($b_1 = \frac{5}{24}$, $b_2 = \frac{5}{16}$, ...), and are involved in an important series of papers [25, 26, 37, 35, 36, 21, 34, 15].

Given a finite collection $\xi = \{H_1, H_2, H_3, ..., H_q\}$ of multicyclic connected components, with slight modifications of the results in [21], we show that for a random graph or multigraph with n vertices and $m(n) = \frac{n}{2}(1 + \mu n^{-\frac{1}{3}})$ edges, $|\mu| \le n^{1/12}$ (in this paper, we will often choose $\mu = O(n^{-\frac{1}{3}})$ so $m(n) = \frac{n}{2} + O(n^{\frac{1}{3}})$), the probability of finding only acyclic and unicyclic components without copy of H_i, $\forall H_i \in \xi$, is asymptotically the same value as for "general" random graphs *times* $\exp\left(-\sum_{k\in\Theta} \frac{1}{2p}\right)$ where Θ is the subset (possibly empty) of the lengths of all polygons in ξ: $\Theta = \{p, H_i \in \xi$ and H_i is a *p-gon*$\}$. For example, if $\xi = \{C_3, C_4\}$, $\Theta = \{3, 4\}$ and the probability that a random graph or a multigraph with n vertices and $\frac{n}{2} + O(n^{1/3})$ edges has only trees and unicyclic components without *triangles* or *quadrilaterals* as induced subgraphs is

$$\sqrt{\frac{2}{3}} e^{-\frac{1}{6} - \frac{1}{8}} \sim 0.6099 \cdots . \tag{2}$$

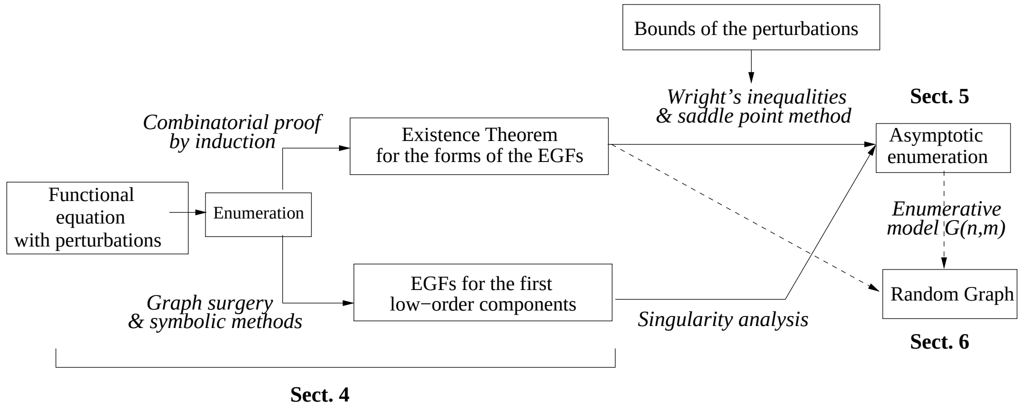


Fig. 1. Summarizing sections 4, 5, 6 and the methods therein.

Recall that an *elementary contraction* of a graph G is obtained by identifying two adjacent points x and y, that is, by the removal of x and y and the addition of a new point z adjacent to those points to which x or y were adjacent. Then a graph G_1 is *contractible* to a graph G_2 if G_2 can be obtained from G_1 by a sequence of elementary contractions. We show that a sufficient condition to change the coefficient b_i, for any $i > 0$, of (1) in this probability is to force ξ to contains the entire family of graphs *contractible* to certain graphs $H_1, H_2, \cdots$ (in this case ξ is *infinite*). We then give the corresponding probability.

The ideas of sections 4, 5 and 6 may be summarized by the figure 1.

1.3 An outline of the paper

The rest of this paper is organized as follows. In section 2, we recall some useful definitions and notations of the stuff we will encounter along this document. In section 3, we will work with the example of bicyclic graphs. The enumeration of these graphs was discovered, as far as we know, independently by Bagaev [1] and by Wright [39]. The purpose of this example is two-fold. First, it brings a simple new combinatorial point of view to the relationship between the generating functions of some *integer partitions*, on one hand, and *graphs*, on the other hand. Next, this example gives us ideas, regarding the *simplest complex components*, i.e., simplest non-acyclic components, of what will happen if we force our graphs to contain some specific configurations (especially the form of the generating functions). In section 4, we start giving the functional equation satisfied by our ξ-free connected graphs involving also the first components containing copies of forbidden configurations. This equation is difficult to solve but leads to the general forms of the EGFs of all $(k+1)$-cyclic ξ-free components. In fact, general combinatorial techniques are presented and used to enumerate the first low-order cyclic triangle-free components. Section 5 presents methods to estimate asymptotically the number of connected components built with n vertices and $n+k$ edges as $n \to \infty$ and $k \to \infty$ but $k = o(n^{1/3})$. The obtained results show that *almost all* $(n, n+o(n^{1/3}))$ connected components are triangle-free and the methods used show that this fact can be generalized to any finite set ξ of forbidden subgraphs. We then turn on the computation of the probability of random graphs/multigraphs without forbidden configurations in section 6. Along this paper, *triangle-free* graphs will be treated as significant example but many results stand for any *finite* set ξ of forbidden multicyclic graphs or multigraphs.

2 Notations

Definitions and tools are given in this section. Because they are mostly well known, they are quickly sketched. Powerful tools in all combinatorial approaches, *generating functions* will be used for our concern. If $F(z)$ is a power series, we write $[z^n]\, F(z)$ for the coefficient of z^n in $F(z)$. We say that $F(z)$ is the *exponential generating function* (EGF for brief) for a collection $\mathcal{F}$ of *labelled* objects if $n!\,[z^n]\, F(z)$ is the number of ways to attach objects in $\mathcal{F}$ that have n elements (see for instance [18] or [38]).

The generating functions for labelled unrooted and labelled rooted trees are nice examples of EGFs. The mathematical theory of labelled trees, as first discussed by Cayley in 1889 [9] was concerned in their enumeration aspect. This study initiated the enumeration of labelled graphs. In fact, a labelled tree is a connected graph with n vertices labelled from 1 to n and $n-1$ edges. It is well known that the number of such structures upon n points is n^{n-2}. Let T be the EGF for labelled rooted trees. A tree consists of a root to which is attached a set of rooted subtrees, thus

$$T(z) = z \left(\sum_{n \geq 0} \frac{T(z)^n}{n!} \right) = \sum_{n \geq 1} n^{n-1} \frac{z^n}{n!} \,. \tag{3}$$

In (3), the exponent of the variable z reflects the number of nodes. One can use *bivariate exponential generating function* to count labelled rooted trees. Throughout this paper, the variable z is the variable recording the number of nodes and w is the variable for the number of edges. For e.g., a tree with n vertices is a connected graph with $n-1$ edges and we have

$$T(w,z) = z \, \exp\left(w \, T(w,z) \right) = \sum_{n > 0} (wn)^{n-1} \frac{z^n}{n!} \,. \tag{4}$$

This bivariate EGF satisfies

$$T(w, z) = \frac{T(wz)}{w} . \tag{5}$$

We will denote by W_k, resp. $\widehat{W}_k$, the EGF for labelled multicyclic connected multigraphs, resp. graphs, with k edges more than vertices. For $k \geq 1$, these EGFs have been computed in [39] and in [21]. A connected graph is of excess k which is always greater than or equal to -1. Let $\widehat{W}_{-1}$ be the EGF of unrooted labelled trees. One can obtain at generating function level the relation

$$\widehat{W}_{-1}(z) = \int_0^z T(x) \frac{dx}{x} , \tag{6}$$

which reflects the fact that any node of an unrooted tree can be taken as the root. The integration of (6) leads to the classical relation

$$\widehat{W}_{-1}(z) = T(z) - \frac{T(z)^2}{2} . \tag{7}$$

It is convenient to work with bivariate EGFs and the bivariate EGFs that enumerate the family $\widehat{\mathcal{W}}_k$ of labelled k-excess graphs, for all $k \geq -1$, can be expressed using the corresponding univariate EGFs as follows

$$\widehat{W}_k(w, z) = w^k \widehat{W}_k(wz) . \tag{8}$$

The factor w^k in the right side of (8) reflects the excess of the component, that is its number of edges minus its number of vertices. The same remark holds between the univariate and bivariate EGFs, W_k, of k-excess multigraphs.

Without ambiguity, one can also associate a given configuration of labelled graph or multigraph with its EGF. For instance, a triangle can be labelled in only one way and we have the following informal relation

$$C_3 \to C_3(w, z) = \frac{1}{3!} w^3 z^3 . \tag{9}$$

For any given multicyclic component H, denote by $W_{k,H}$ (resp. $\widehat{W}_{k,H}$) the EGF of multicyclic H-free multigraphs (resp. graphs) with k edges more than vertices. In these notations, the second index refers to the forbidden configuration(s). Recall that a *smooth* graph or multigraph is one with all vertices of degree ≥ 2 (see [40]). Throughout the rest of this paper, the "*widehat*" notation will be used for EGF of graphs and "*underline*" notation corresponds to the *smoothness* of the species. E.g., $\underline{\widehat{W}_k}$, resp. $\underline{W_k}$, are EGF for connected $(n, n+k)$ smooth graphs, resp. smooth multigraphs.

Remark 1 *We follow the authors of [21] and the widehat notation will be used for graphs generating functions. Although, our main concern is graphs, one can extend the results presented in this paper to multigraphs. In fact, in the giant paper [21], the uniform model of random graphs which allows self-loops and multiple edges is treated and shown to be easier to analyze than the classical model of random graphs due to Erdös and Rényi [13] since the multigraphs EGFs have better expressions.*

We need additional definitions corresponding to the first appearance of the forbidden configurations in some random evolving graphs/multigraphs. For sake of simplicity, we suppose temporarily that $\xi = \{C_3\}$. Consider the random graph process which starts with n initially disconnected nodes. When enriching it by successively adding edges, one at time and at random, the first time a new copy of triangle is created with the last added edge in some connected component, there are two possibilities:

1. the last edge creates *exactly* one and only one triangle,
2. there are many occurrences of triangles but sharing the last added edge which deletion will suppress all copies of triangle in the considered component. We shall call this sort of configuration "*juxtaposition*" of triangles.

The same nomenclature holds when considering a set ξ of forbidden configurations. For example if $\xi = \{C_3, C_4\}$, a "*house*" can appear in some component. More formally, we have the following reformulation related to these kinds of construction:

Definition 2 *Given a subset* $\{H_{i_1}, H_{i_2}, \cdots, H_{i_q}\}$ *of* ξ*, we define the juxtaposition of* $H_{i_1}, H_{i_2}, \cdots, H_{i_q}$ *as a subgraph containing at least one copy of each* H_{i_j} *but such that there exists an edge which deletion will suppress all the occurrences of* $H_{i_1}, H_{i_2}, \cdots, H_{i_q}$*. When there exists* s *shared edges such that the deletion of any of them will suppress all the occurrences of* $H_{i_1}, H_{i_2}, \cdots, H_{i_q}$*, we define this specific configuration as a* s*-juxtaposition.*

Example 3 *We have the figure 2 depicting a* 1*-juxtaposition of* C_3 *and* C_4*, representing a "house". In figure 3, we have a* 1*-juxtaposition and a* 3*-juxtaposition of two* K_4*.*

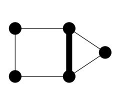


Fig. 2. The "house": 1-juxtaposition of C_3 and C_4 ($\xi = \{C_3, C_4\}$).

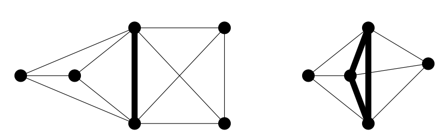


Fig. 3. 1-juxtaposition and 3-juxtaposition of K_4 ($\xi = \{K_4\}$).

Definition 4 *For any $H \in \xi$, denote by $\widehat{S}_{k,H}$ the EGF of $(k+1)$-cyclic graphs with exactly one copy of H (copies of other graphs of ξ are not allowed). Define by $\widehat{S}_{k,\xi} = \sum_{H\in\xi} \widehat{S}_{k,H}$, the EGF of $(k+1)$-cyclic graphs with one occurrence of a member of ξ. For any subset $\xi^{'} \subseteq \xi$, denote by $\widehat{J}^{(p)}_{k,\xi'}$ the EGF of p-juxtaposition of $\xi^{'}$. We let $\widehat{J}_{k,\xi} = \sum_{\xi'\subseteq\xi} \sum_p p\, \widehat{J}^{(p)}_{k,\xi'}$. Respectively, $S_{k,\xi}$ and $J_{k,\xi}$ are the EGFs for multigraphs with the same characteristics.*

Furthermore, denote by ϑ_w, resp. ϑ_z, the differential operator $w\frac{\partial}{\partial w}$, resp. $z\frac{\partial}{\partial z}$. The operator ϑ_w corresponds to marking an edge of a graph (or a multigraph). Similarly, ϑ_z corresponds to marking a vertex . For the use of pointing and marking, we refer to [19] and for general techniques concerning graphical enumerations we refer to [20].

The following observation will take its importance as we will see later:

Remark 5 *$\widehat{J}_{k,\xi}$ is the EGF of $(k+1)$-cyclic graphs with a shared edge of the juxtaposition marked.*

Remark 6 *Throughout this paper, we will frequently use the following notation when comparing the coefficients of two generating functions. If A and B are two formal power series such that for all $n \geq 0$ we have $[z^n]\, A(z) \leq [z^n]\, B(z)$ then we denote this relation $A \preceq B$ (or $A(z) \preceq B(z)$).*

3 The link between the EGF of bicyclic graphs and integer partitions

At least in 1967, there were 10 different proofs for the EGF for trees according to the paper of Moon [27] and 16 proofs related in [23]. Then, Rényi [30] found the formula to enumerate unicyclic graphs which can be expressed in terms of the generating function of rooted labelled trees, namely

$$\widehat{W}_0(z) = \frac{1}{2} \ln \frac{1}{1-T(z)} - \frac{T(z)}{2} - \frac{T(z)^2}{4} \,. \tag{10}$$

We refer here to the symbolic methods developed in [32] for modern computation of formulae like (10). The formula for unicyclic multigraphs is very similar and there are terms due to self-loops and multiple edges

$$W_0(z) = \frac{1}{2} \ln \frac{1}{1-T(z)} \,. \tag{11}$$

It may be noted that in some connected graphs, as well as multigraphs the number of edges exceeding the number of vertices can be seen as useful enumerating parameter. The term *bicyclic* graphs, appeared first in the seminal paper of Flajolet *et al.* [14] followed few years later by the huge one of Janson *et al.* [21] and was concerned with all connected graphs with $(n+1)$ edges and n vertices. The authors of these documents choose then the word *bicyclic* for connected component which is constructed by adding a random edge to a unicyclic component. Bagaev [1] first found a method to count such graphs. His method of *shrinking-and-expanding* graphs is well explained in [2]. Wright [39] found a recurrent formula well adapted for formal calculation to compute the number of all connected graphs of excess k (for all $k \geq 1$). Our aim in this section is to show that the problem of the enumeration of *bicyclic graphs* can also be solved with techniques involving integer partitions. We present here a simple treatment very close to the Wright's method as a warm-up for the forthcoming results in the next sections.

Given a fixed set of n vertices, there exist two types of graphs which are connected and have $(n+1)$ edges as described in the figure 4.

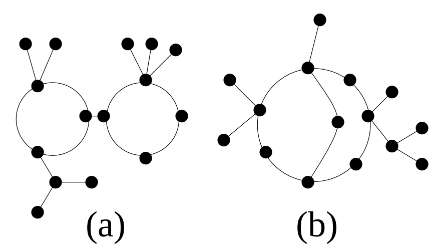


Fig. 4. Examples of bicyclic components.

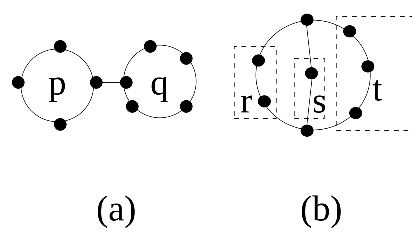


Fig. 5. Smooth bicyclic components.

Wright [39] showed with his *reduction* method that the EGF of all multicyclic graphs, namely bicyclic graphs, can be expressed in terms of the EGF of labelled rooted trees. In order to count the number of ways to label a graph, we can repeatedly *prune* it by suppressing recursively any vertex of degree 1. We then remove as many vertices as edges. As these structures present many symmetries, our experiences suggest us so far that we ought to look at our previously described object without symmetry and without the possible rooted subtrees. There are

$$\binom{n}{p}\binom{n-p}{q}\frac{(p-1)!}{2}p\frac{(q-1)!}{2}q(n-p-q)! = \frac{n!}{4}$$

manners to label the graph represented by the figure 5 (a) whenever $p \neq q$. In the graph of figure 5 (b), if $r \neq s$, $s \neq t$, $t \neq r$, there are $\frac{n!}{2}$ ways to label the graph. Note that these results are independent from the size of the subcycles. One can obtain all smooth bicyclic graphs after considering possible symmetry criterions. In figure 5 (a), if the subcycles have the same length, $p = q$, a factor $\frac{1}{2}$ must be considered and we have $n!/8$ ways to label the graph. Similarly, the graph of figure 5 (b) can have the 3 arcs with the same number of vertices. In this case, a factor 1/6 is introduced. If only two arcs have the same number of vertices, we need a symmetrical factor 1/2. Thus, the enumeration of smooth bicyclic graphs can be viewed as specific problem of integer partitioning into 2 or 3 parts following the dictates of the basic graphs in figure 6.

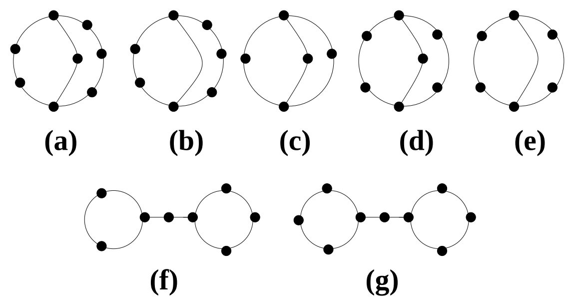


Fig. 6. The different basic smooth bicyclic graphs.

With the same notations as in [11], denote by $P_i(z)$, respectively $Q_i(z)$, the generating functions of the number of partitions of an integer in i parts, respectively in i different parts. Let $\underline{\widehat{W_1}}(z)$ be the univariate EGF for smooth bicyclic graphs, then we have $\underline{\widehat{W_1}}(z) = f(P_2(z), P_3(z), Q_2(z), Q_3(z))$, i.e.,

$$\begin{aligned}\underline{\widehat{W_1}}(z) = \quad & \underbrace{\frac{1}{2} z^2 (Q_3(z) + Q_2(z))}_{\text{figures 6 (a), 6 (b)}} + \underbrace{\frac{1}{12} \frac{z^5}{1-z^3}}_{\text{6 (c)}} \\ & + \underbrace{\frac{1}{4}\left(\frac{z^4}{1-z^2} + \frac{z^5}{(1-z)(1-z^2)} - \frac{z^5}{(1-z^3)}\right)}_{\text{6 (d), 6 (e)}} \\ & + \underbrace{\frac{1}{4}\frac{z^6}{(1-z)^2(1-z^2)}}_{\text{6 (f)}} + \underbrace{\frac{1}{8}\frac{z^5}{(1-z)(1-z^2)}}_{\text{6 (g)}} . \end{aligned} \tag{12}$$

In formula (12) or equivalently $\underline{\widehat{W_1}}(z) = \frac{z^4}{24}\frac{(6-z)}{(1-z)^3}$, the denominator $\frac{1}{(1-z)^3}$ denotes the fact that there is at most 3 arcs or 3 *degrees of liberty* of integer partitions of the vertices in a bicyclic graph. The same remark holds for the denominators $\frac{1}{(1-T(z))^{3k}}$ in Wright's formulae [39] for all $(k+1)$-cyclic connected labelled graphs. To get the whole EGF for bicyclic graphs, we have to substitute z by $T(z)$ in $\underline{\widehat{W_1}}(z)$ in order to replace all (shrinked) vertices of the smooth graphs by labelled rooted trees. The form of these EGF takes its importance when studying the asymptotic behavior of random graphs or multigraphs with a given excess. In fact, the known expansion of the Cayley's function, T, at its singularity $z = \frac{1}{e}$ is (see [24, 16, 17])

$$T(z) = 1 - \sqrt{2}\delta + \frac{2}{3}\delta^2 - \frac{11}{36}\sqrt{2}\delta^3 + \cdots\,, \; (\delta = \sqrt{1 - ez})\,. \tag{13}$$

As the EGFs of multicyclic components can be expressed in terms of T, the key point of their characteristics corresponds directly to the analytical properties of *tree polynomial* $t_n(y)$ defined as follow

$$\frac{1}{(1 - T(z))^y} = \sum_{n \geq 0} t_n(y) \frac{z^n}{n!}\,. \tag{14}$$

($t_n(y)$ is a polynomial of degree n in y.) Knuth and Pittel [24] studied their properties. For *fixed* y as $n \to \infty$, we have (see [24, lemma 2])

$$t_n(y) = \frac{\sqrt{2\pi} n^{(n-1/2+y/2)}}{2^{y/2}\Gamma(y/2)} + O(n^{n-1+y/2})\,. \tag{15}$$

This equation tells us that in the EGF, $\widehat{W}_1$ of bicyclic graphs, expressed here as a sum of powers of $1/(1 - T(z))$

$$\begin{aligned} \widehat{W}_1(z) &= \frac{T(z)^4}{24} \frac{(6 - T(z))}{(1 - T(z))^3} \\ &= \frac{5}{24} \frac{1}{(1 - T(z))^3} - \frac{19}{24} \frac{1}{(1 - T(z))^2} + \cdots\,, \end{aligned} \tag{16}$$

only the coefficient $\frac{5}{24}$ of $t_n(3)$ is asymptotically significant.

4 Functional equation for ξ-free graphs/multigraphs and the forms of their EGFs

4.1 Differential recurrence for ξ-free components

EGFs of triangle-free unicyclic components can be easily obtained when avoiding cycle of length 3 in the general formulae for unicyclic graphs (10), resp. multigraphs (11). Denote respectively by W_{0,C_3} and $\widehat{W}_{0,C_3}$ the EGFs for unicyclic multigraphs and graphs without triangle (C_3), we have

$$W_{0,C_3}(z) = \frac{1}{2} \ln \frac{1}{1 - T(z)} - \frac{T(z)^3}{6}\,, \tag{17}$$

$$\widehat{W}_{0,C_3}(z) = \frac{1}{2}\ln\frac{1}{1-T(z)} - \frac{T(z)}{2} - \frac{T(z)^2}{4} - \frac{T(z)^3}{6}\,. \tag{18}$$

Enumerating components of higher cyclic order without triangle is much more difficult. However, we have the following lemma:

Lemma 7 *For all $i \geq -1$, denote by $\widehat{W}_{i,C_3}$ the EGF for triangle-free $(i+1)$-cyclic graphs. Let $\widehat{S}_{i,C_3}$ and $\widehat{J}_{i,C_3}$ be the EGFs described as in definition 4. Then, the bivariate EGFs $\widehat{W}_{k+1,C_3}$, $\widehat{S}_{k+1,C_3}$, $\widehat{J}_{k+1,C_3}$ and $\widehat{W}_{p,C_3}$ for $-1 \leq p \leq k$ are related by the differential recurrence:*

$$\begin{aligned}\vartheta_w \widehat{W}_{k+1,C_3} \;+\; 3\widehat{S}_{k+1,C_3} \;+\; \widehat{J}_{k+1,C_3} \;&=\; w\Big(\frac{{\vartheta_z}^2 - \vartheta_z}{2} - \vartheta_w\Big)\widehat{W}_{k,C_3} \\ &+\; w\left(\sum_{-1\leq p\leq q\leq k+1,\, p+q=k} \frac{1}{1+\delta_{p,q}}(\vartheta_z \widehat{W}_{p,C_3})(\vartheta_z \widehat{W}_{q,C_3})\right)\end{aligned} \tag{19}$$

where $\delta_{p,q} = 1$ iff $p = q$, otherwise $\delta_{p,q} = 0$. Similarly, we have for multigraphs (with the same parameters):

$$\begin{aligned}\vartheta_w W_{k+1,C_3} \;+\; 3S_{k+1,C_3} \;+\; J_{k+1,C_3} \;&=\; w\Big(\frac{{\vartheta_z}^2}{2} W_{k,C_3}\Big) \\ &+\; w\left(\sum_{-1\leq p\leq q\leq k+1,\, p+q=k} \frac{1}{1+\delta_{p,q}}(\vartheta_z W_{p,C_3})(\vartheta_z W_{q,C_3})\right)\,.\end{aligned} \tag{20}$$

Proof. There are two ways to obtain a $(k+2)$-cyclic component from components of lower cyclic order, which are in the right part of (19) and are assumed to be triangle-free. For multigraphs, we have to employ the combinatorial operation $\frac{{\vartheta_z}^2}{2}$.

First of all, consider a triangle-free $(k+1)$-cyclic component. To add a new edge to this component, we have to choose two vertices, different and already not adjacent for graphs, and not necessarily different for multigraphs. For graphs, the combinatorial operator used to choose two different vertices is $\frac{{\vartheta_z}^2 - \vartheta_z}{2}$. Then, we have to avoid the adjacent vertices by means of the operator $-\vartheta_w$ (see [21, Section 10] or [19] for the use of marking and pointing). If the new $(k+2)$-cyclic component contains a triangle, the triangle can only occur in the following cases:

1. The new edge creates exactly a triangle. In this case, the last added edge is necessarily one of the 3 edges of the new triangle.
2. The last edge creates many triangles but necessarily juxtaposed as defined above (definition 2), and in this latter case, the last edge is necessarily the

one which is shared between all the occurrences of triangle.

Thus, the left side of (19), resp. of (20), distinguishes the last added edge in the new $(k+2)$-cyclic component.

Next, a $(k+2)$-cyclic triangle-free component can be built when creating an edge between a $(p+1)$-cyclic and a $(q+1)$-cyclic triangle-free components such that $p+q=k$ and $-1 \leq p \leq q \leq k+1$ (note that the case $p=-1$ and $q=k+1$ corresponds to the case where a tree is attached to a $(k+1)$-cyclic triangle-free component). This construction is done by choosing one vertex belonging to the $(p+1)$-cyclic component and another vertex from the $(q+1)$-cyclic component. A symmetry factor, $\frac{1}{2!}$, occurs when $p=q$.

The right side of (19) simply reflects the constructions used to build a $(k+2)$-cyclic connected graph In (20), the term $\frac{\vartheta_z{}^2}{2} W_{k,C_3}$ represents all $(k+1)$-cyclic multigraphs with an ordered pair $\langle x, y \rangle$ of marked vertices (see also [21, Sect. 4, Eq. (4.2) and following]). □

When considering a finite set ξ of forbidden configurations, we have the following generalization of lemma 7:

Lemma 8 *Suppose that $\xi = \{H_1, \cdots, H_p\}$, $|\xi| < \infty$. Let $\widehat{W}_{k+1,\xi}$, $\widehat{S}_{k+1,H_i}$, $\widehat{J}_{k+1,\xi}$ and $\widehat{W}_{k+1,\xi}$ be the EGFs defined as in above (definition 4). Let ρ_s be the finite set of all s-juxtapositions of member(s) of ξ and denote by $e(H_i)$ the number of edges of H_i. Then, we have for graphs*

$$\vartheta_w \widehat{W}_{k+1,\xi} + \sum_{H_i \in \xi} e(H_i) \widehat{S}_{k+1,H_i} + \widehat{J}_{k+1,\xi} = w\Big(\frac{\vartheta_z{}^2 - \vartheta_z}{2} - \vartheta_w\Big)\widehat{W}_{k,\xi} + w\left(\sum_{-1 \leq p \leq q \leq k+1,\, p+q=k} \frac{1}{1+\delta_{p,q}} (\vartheta_z \widehat{W}_{p,\xi})(\vartheta_z \widehat{W}_{q,\xi})\right). \tag{21}$$

For the EGFs of connected ξ-free multigraphs, we have

$$\vartheta_w W_{k+1,\xi} + \sum_{H_i \in \xi} e(H_i) S_{k+1,H_i} + J_{k+1,\xi} = w\Big(\frac{\vartheta_z{}^2}{2} W_{k,\xi}\Big) + w\left(\sum_{-1 \leq p \leq q \leq k+1,\, p+q=k} \frac{1}{1+\delta_{p,q}} (\vartheta_z W_{p,\xi})(\vartheta_z W_{q,\xi})\right). \tag{22}$$

(21) and (22) are simply generalization of (19) and (20).

EGFs for respectively bicyclic graphs with one triangle and with exactly one juxtaposition of triangles can be obtained using the method developed in section 3, with the help of figures 7 and 8.

Remark 9 *Since Wright's reduction method*[3] *suggests us to work with labelled smooth components, figures such as 7 and 8 represent the situation after smoothing. Also for any family $\mathcal{F}_k$ of $(k+1)$-cyclic components with EGF $F_k(z)$, the EGF of smooth species of $\mathcal{F}_k$ is simply obtained by means of substitutions of all occurrences of $T(z)$ in $F_k(z)$ by z. Conversely, if $\underline{F_k}(z)$ is the EGF of smooth species of $\mathcal{F}_k$, then $F_k(z) = \underline{F_k}(T(z))$ gives the EGF associated to the whole family $\mathcal{F}_k$.*

Remark 10 *Since all EGFs we deal with can be expressed in terms of $T(z)$ in the univariate case, and of w and $T(wz)$ in the bivariate case, we assume that $T \equiv T(z)$ to express univariate EGFs. In the case of bivariate EGFs, we let $T \equiv T(wz)$. These notations should not induce ambiguity to the reader who can read the meaning within the context.*

The following figures can be used to compute the EGFs $\widehat{S}_{1,C_3}$ and $\widehat{J}_{1,C_3}$

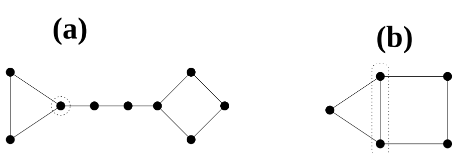


Fig. 7. Smooth bicyclic graphs with one occurrence of triangle.

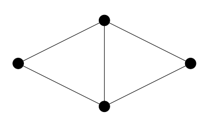


Fig. 8. Smooth bicyclic graph with a 1-juxtaposition of 2 triangles.

Using similar techniques as for (12) with the help of the previous figures, we have for $\underline{\widehat{S}_{1,C_3}}$ and $\underline{\widehat{J}_{1,C_3}}$

$$\underline{\widehat{S}_{1,C_3}}(z) = \underbrace{\frac{1}{2} z^5 \frac{1}{1-z}}_{\text{figure 7 (b)}} + \underbrace{\frac{z^6}{4} \frac{1}{(1-z)^2}}_{\text{figure 7 (a)}} \qquad (23)$$

[3] the second method in [39], see also the proof of lemma 15 in §4.4

and

$$\underline{\widehat{J}_{1,C_3}}(z) = \frac{z^4}{4}\,. \tag{24}$$

Again, to obtain the whole EGFs we have to substitute z by $T \equiv T(z)$, replacing all shrinked vertices of the smooth graphs by labelled rooted trees.

$$\widehat{S}_{1,C_3}(z) = \frac{T^5}{4}\frac{(2-T)}{(1-T)^2}\,,\ \widehat{J}_{1,C_3}(z) = \frac{T^4}{4}\,. \tag{25}$$

Thus, using (25) and (19) we have

$$\widehat{W}_{1,C_3}(z) = \frac{T^5}{24}\frac{(2+6T-3T^2)}{(1-T)^3}\,. \tag{26}$$

We know from (15) that the decomposition of formula such as (26) into sums of powers of $\frac{1}{1-T}$, are useful in order to study the asymptotic behavior of the number of such objects. We have

$$\begin{aligned}\widehat{W}_{1,C_3}(z) = \textstyle\sum_{n\geq 0} \Big(& \tfrac{5}{24}t_n(3) - \tfrac{25}{24}t_n(2) + \tfrac{47}{24}t_n(1) - \tfrac{35}{24} - \tfrac{5}{24}t_n(-1) \\ & + \tfrac{25}{24}t_n(-2) - \tfrac{5}{8}t_n(-3) + \tfrac{1}{8}t_n(-4)\Big)\tfrac{z^n}{n!}\,.\end{aligned} \tag{27}$$

In order to enumerate the first multicyclic ξ-free components for general ξ, we introduce some more techniques in the next paragraphs.

4.3 General techniques for first multicyclic components and instantiations

In this paragraph, we give methods that can be applied to enumerate first low-order cyclic components, i.e., with excess 1 and 2 for a forbidden p-gon and in general for an excess up to $l+1$ and $l+2$ for all forbidden components of excess l. For e.g., the EGF of C_3-free tricyclic graphs are given as instantiation of these methods and follows the formula (26) given above. Also, we will see later that these techniques are useful to obtain the forms of the EGFs $\widehat{W}_{k,\xi}$ and $W_{k,\xi}$ by induction (see §4.4). We consider here only connected graphs with exactly one occurrence of H since if $H^{'}$ represents any juxtaposition of H, we can work directly in the same manner with a single occurrence of $H^{'}$.

First of all, we have to prune recursively all vertices of degree 1. The obtained graphs are smooth. We can subdivide these graphs containing an occurrence of H in 3 types: types (a) and (b) are such as those represented by figure 7 and type (c) is as in the figure 9 below where H represents a triangle.

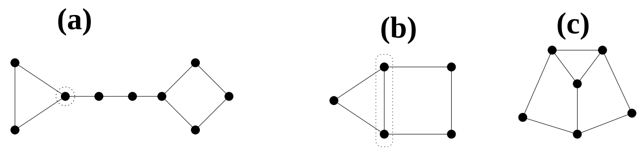


Fig. 9.

The first two types (a) and (b) of figure 7 can be described as follows:

- (a) represents the concatenation of two components H and F (respectively non H-free and H-free) by a common vertex or more generally by a path between the two components. In the figure, H is simply a triangle. Note that a cutpoint (a vertex whose removal increases the number of connected components) belongs to the triangle after the recursive deletions of vertices of degree 1. This is referred here as a *serial composition* of components.
- (b) is the concatenation of the same components but by a common edge. This construction is referred as a *parallel composition* of components.
- Figure 9 (c) represents components which are not in figure 7 (a) nor in figure 7 (b).

4.3.1 The serial composition or concatenation by a vertex

Since a graph with one cutpoint belonging to a forbidden configuration may be considered to be rooted at this cutpoint, the number of connected graphs with one cutpoint can be expressed in terms of the EGFs of the different subgraphs rooted at the same cutpoint (cf. [20] or [33]). This construction may be interpreted combinatorially as follows.

Lemma 11 *Let $\mathcal{F}$ be a family of connected H-free graph. Denote by $\underline{F}$ the EGF of the graphs obtained when smoothing a graph of $\mathcal{F}$. Let A_1 be the EGF of connected graphs containing possibly many copies of H and obtained as the concatenation of graphs of $\mathcal{F}$ and of H by a vertex belonging to H. Then, A_1 satisfies*

$$A_1 \preceq \left[\frac{1}{z} \, \left(z\frac{\partial}{\partial z}\underline{F}(z) \right) \, \left(z\frac{\partial}{\partial z}H(z) \right) \right]_{|z=T(z)} \tag{28}$$

and let A_2 be the EGF of all connected graphs obtained when allowing a path starting at a vertex belonging to H and joining any graph of $\mathcal{F}$. A_2 satisfies

$$A_2 \preceq \left[\frac{1}{z} \, \left(\frac{1}{1-z} \right) \, \left(z\frac{\partial}{\partial z}\underline{F}(z) \right) \, \left(z\frac{\partial}{\partial z}H(z) \right) \right]_{|z=T(z)} . \tag{29}$$

In (28) and (29), equalities hold when H is two-connected.

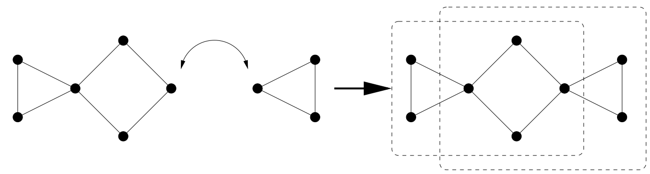


Fig. 10. Serial composition with symmetric factor $\frac{1}{2}$.

Proof. Recall that for two EGFs A and B, $A \preceq B$ means that $\forall n$, $[z^n]\,A(z) \leq [z^n]\,B(z)$ (cf. remark 6). First, let us consider the case where H is two-connected. In this case, the concatenation of H with a graph of $\mathcal{F}$, by a vertex of H, leads to a graph with a single copy of H in the resulting graph. Thus, the fact that there is *exactly* one occurrence of copy of H in the concatenation insures the *uniqueness of the decomposition* into two graphs such that one belongs to $\mathcal{F}$ and the other is (necessarily) H. The lemma is a combination of the approach presented in [33] and Wright's reduction method [39]. We have to introduce a factor $\frac{1}{z}$ to relabel the common cutpoint considered here as shared between the smooth components. $\vartheta_z \underline{F}(z) = z\frac{\partial}{\partial z}\underline{F}(z)$ and $\vartheta_z H(z) = z\frac{\partial}{\partial z}H(z)$ are used to distinguish the vertex to be shared between pruned components of $\mathcal{F}$ and of H. In (29) to represent a possible *path*, we insert the term $\frac{1}{1-z}$ i.e., a sequence of vertices of degree 2 except the two extremal nodes, between the two sides. When substituting z by $T(z)$, we reverse the *vertexectomy* process starting with a smooth graph and sprout rooted trees from each node. Hence, in the case where H is two-connected, we have the equalities in (28) and (29). The situation changes a bit for more general configurations. Typically, we can have concatenations of H and graphs of $\mathcal{F}$ which can lead to a new graph with two (or more) occurrences of H. This is the case depicted by figure 10 where H is made with a triangle and a square attached by a vertex and the graph of $\mathcal{F}$ is simply a triangle. In this special case, we just have to introduce a symmetry factor $\frac{1}{2!}$ and then the upper bound of (28) is valid. In fact, the upper bound enumerates graphs where the concatenation such as the one obtained in figure 10 are counted twice or more. $\square$

4.3.2 The parallel composition or concatenation by an edge

Graphs of the type represented by the figure 7 (b) can be enumerated in a very close way.

Lemma 12 *Let $\mathcal{F}$ and $\underline{F}$ be defined as in lemma 11 above. Let B be the EGF associated to the graphs containing copies of H and obtained as the concatenation of two graphs of $\mathcal{F}$ and of H sharing a common edge. B satisfies*

$$B \preceq \left[\frac{2}{wz^2} \left(w\frac{\partial}{\partial w}\underline{F}(w,z) \right) \left(w\frac{\partial}{\partial w}H(w,z) \right) \right]_{|wz=T(wz)} . \qquad (30)$$

Proof. The formula (30) differs slightly from the one in (29). The factor $\frac{2}{wz^2}$ comes from the fact that we have here, as in the figure 7 (b), a common edge which is defined by his two common vertices and can be seen as a root-edge. A graph such as those represented by the figure 7 (b) can be considered as pendant to this edge. Also, we have the equality whenever H is two-connected. Otherwise symmetries can arise but the upper bound of (30) remains valid for the same reasons as for (28) and (29). □

Unfortunately, equation likes (21) of lemma 8 are much easier to propose than to really solve. However, we can derive the EGF of the first multicyclic H-free components by applying the techniques presented above.

4.3.3 The example of triangle-free graphs

The EGFs of unicyclic and bicyclic graphs without triangles are given by formulae (18) and (26). For graphs having 2 excesses, the removal of all edges and vertices by the Wright's reduction method leads to the set of graphs represented by figure 11 for graphs containing 1 triangle and figure 12 for graphs with a juxtaposition of triangles.

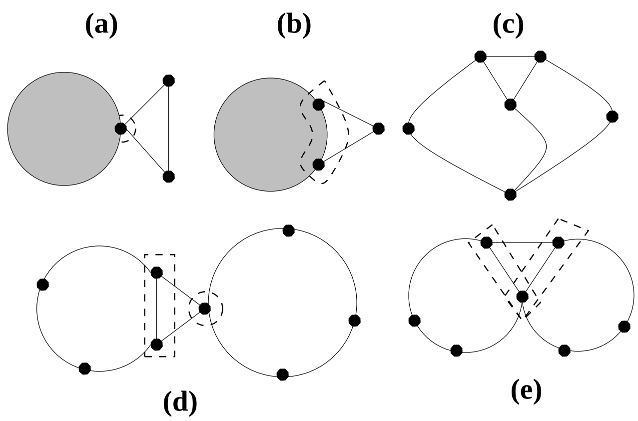


Fig. 11. Basic tricyclic graphs with exactly one triangle. The subgraph in grey represent bicyclic triangle-free components.

As before, given a family $\mathcal{F}$ of graphs, we denote by $\underline{F}$ the EGF of *smooth* elements of $\mathcal{F}$, i.e., graphs without endvertices (vertices of degree 1). The bivariate EGF of bicyclic triangle-free smooth graphs, $\underline{\widehat{W}_{1,C_3}}$ is obtained from (26), namely

$$\underline{\widehat{W}_{1,C_3}}(w,z) = w\frac{w^5z^5}{24}\frac{(2+6wz-3w^2z^2)}{(1-wz)^3} \tag{31}$$

Note that $\vartheta_w C_3(w,z) = \vartheta_z C_3(w,z) = \frac{w^3z^3}{2}$. Thus, the application of the lemmas 11 and 12 to the smooth graphs depicted by figures 11 (a) and 11 (b) gives

$$\frac{w^3 z^2}{2(1-wz)}\ \vartheta_z(\underline{\widehat{W}_{1,C_3}}(w,z)) + w^2 z\ \vartheta_w(\underline{\widehat{W}_{1,C_3}}(w,z))\,. \tag{32}$$

Similarly, we have for smooth graphs represented by the figure 11 (d)

$$\frac{1}{z(1-wz)}\Big(\frac{2}{wz^2}\,(\vartheta_w\underline{\widehat{W}_{0,C_3}}(w,z))\,(\frac{w^3z^3}{2})\Big)\Big(\vartheta_z\underline{\widehat{W}_{0,C_3}}(w,z)\Big) \tag{33}$$

and for figure 11 (e), we find

$$\frac{2}{wz^2}\left(\frac{2}{wz^2}\Big(\frac{w^3z^3}{2}\Big)\Big(\vartheta_w\underline{\widehat{W}_{0,C_3}}(w,z)\Big)^2\right)\,. \tag{34}$$

A simple way to enumerate the smooth graphs represented by the figure 11 (c) is to consider that the three paths between the triangle and the vertex v are symmetric. Taking into account the fact that only one of these three paths can be reduced to a simple edge (to avoid another triangle), we have the following EGF associated to these smooth graphs

$$\frac{z^7}{3!(1-z)^3} + \frac{z^6}{2!(1-z)^2}\,. \tag{35}$$

In total, the bivariate EGF for all graphs such that smooth species are depicted by the figures 11 (c), 11 (d) and 11 (e) is given by

$$w^2\frac{T^6}{6}\frac{(3-2T)}{(1-T)^3} + \frac{w^2T^7}{2(1-T)^2} + \frac{w^2T^8}{4(1-T)^3}\,. \tag{36}$$

Summing (32) and (36), one can deduce the bivariate EGF for tricyclic graphs containing exactly a triangle

$$\widehat{S}_{2,C_3}(w,z) = \frac{w^2T^6}{48}\frac{(48+18T-140T^2+119T^3-30T^4)}{(1-T)^5}\,. \tag{37}$$

We turn now to the enumeration of tricyclic graphs with one occurrence of juxtaposition of triangles. The figure 12 represents the 2-excess smooth graphs with juxtapositions of triangles.

We observe that figures 12 (b) and 12 (c) can be handled with the techniques of lemma 12 using the EGF $\underline{\widehat{W}_{0,C_3}}$ and $\frac{w^5z^4}{2!2!}$ (which is the EGF of the smooth juxtaposition of 2 triangles). Similarly, we can use lemma 11 for the figures

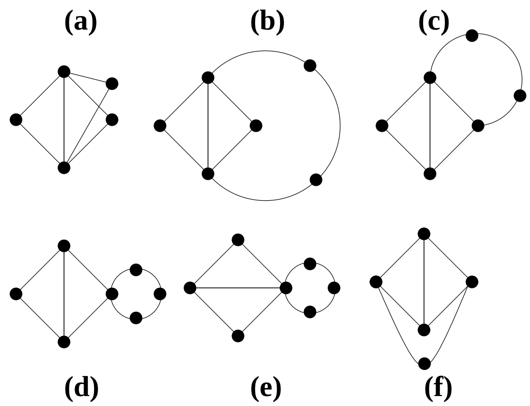


Fig. 12. Basic tricyclic graphs with juxtapositions of triangles.

12 (d) and 12 (e). The EGF associated to the smooth graph of figure 12 (a) is simply $\frac{w^7z^5}{2!3!}$, and the one for smooth graphs depicted by the figure 12 (f) is $\frac{w^7z^5}{4(1-wz)}$. In fact, graphs such as the one drawn in figure 12 (f) can be obtained by replacing an edge of the complete graph K_4 with a path of length at least 2. The EGF that corresponds to the figure 12 is then

$$\underline{\widehat{J}_{2,C_3}}(w,z) = \frac{w}{z(1-wz)}\vartheta_z(\frac{w^5z^4}{4})\vartheta_z(\underline{\widehat{W}_{0,C_3}}(w,z))$$
$$+\frac{2}{wz^2}\vartheta_w(\frac{w^5z^4}{4})\vartheta_w(\underline{\widehat{W}_{0,C_3}}(w,z)) + w^2\frac{(wz)^5}{2!3!} + w^2\frac{(wz)^5}{4(1-wz)}\,. \qquad (38)$$

Thus, the bivariate EGF of tricyclic graphs containing exactly a juxtaposition of triangles is

$$\widehat{J}_{2,C_3}(w,z) = \frac{w^2T^5}{6}\frac{(2+5T-4T^2)}{(1-T)^2}\,. \qquad (39)$$

The bivariate EGF of tricyclic triangle-free graphs is then obtained using (37), (39) and (19), namely,

$$\widehat{W}_{2,C_3}(w,z) = w^2\frac{T^6}{48}\frac{(7+36T-18T^2-40T^3+40T^4-10T^5)}{(1-T)^6}\,. \qquad (40)$$

4.4 General forms of the EGFs of ξ-free components

Although lemmas 7 and 8 do not allow us to solve completely the problems of enumerating ξ-free connected graphs with a given number of vertices and edges, the combination of these lemmas with subtle combinatorial constructions provides alternative solutions to get the general forms of the EGFs $\widehat{W}_{k,\xi}$ and $W_{k,\xi}$. Recall the following theorem due to Wright

Theorem 13 (Wright 1977) *For $k \geq 1$, the EGFs, $\widehat{W}_k$, of $(k+1)$-cyclic graphs can be expressed as a finite sum of powers of $\frac{1}{1-T(z)}$ with rational coefficients and we have*

$$\widehat{W}_k(z) = \frac{b_k}{(1-T(z))^{3k}} - \frac{c_k}{(1-T(z))^{3k-1}} + \sum_{2 \leq s \leq 3k-2} \frac{\omega_{k,s}}{(1-T(z))^s}\,. \quad (41)$$

The $(b_k)_{k\geq 1}$ are called the Wright's constants of first order (also called Wright-Louchard-Takács constants, see for e.g. [34]). $b_1 = \frac{5}{24}$ and for $k \geq 1$, b_k is defined recursively by

$$2(k+1)b_{k+1} = 3k(k+1)b_k + 3\sum_{t=1}^{k-1} t(k-t)b_t b_{k-t}\,. \quad (42)$$

The $(c_k)_{k\geq 1}$ are the Wright's constants of second order and are defined recursively, using (42), by $c_1 = \frac{19}{24}$ and for $k \geq 1$

$$\begin{aligned} 2(3k+2)c_{k+1} = {} & 8(k+1)b_{k+1} + 3kb_k + (3k+2)(3k-1)c_k \\ & + 6\sum_{t=1}^{k-1} t(3k-3t-1)b_t c_{k-t}\,. \end{aligned} \quad (43)$$

The proof of theorem 13 is an interesting combinatorial exercise involving essentially the pointing operators ϑ_w and ϑ_z (see [39, 21]). Note that formulae (41), (42) and (43) are obtained with Wright's fundamental differential recurrence (well explained in [21, section 6]) and which is written here with the notations of this paper

$$\begin{aligned} \vartheta_w \widehat{W}_{k+1} = {} & w\Big(\frac{{\vartheta_z}^2 - \vartheta_z}{2} - \vartheta_w\Big)\widehat{W}_k \\ & + w\left(\sum_{-1\leq p\leq q\leq k+1,\, p+q=k} \frac{1}{1+\delta_{p,q}}(\vartheta_z \widehat{W}_p)(\vartheta_z \widehat{W}_q)\right)\,. \end{aligned} \quad (44)$$

For our connected $(k+1)$-cyclic triangle-free graphs, we have the following existence theorem on the forms of their EGFs:

Theorem 14 *There exists rational $\omega_{k,i}^{(C_3)}$ such that for all $k \geq 2$, the univariate EGF, $\widehat{W}_{k,C_3}$, associated to $(k+1)$-cyclic triangle-free graphs, is of the form:*

$$\widehat{W}_{k,C_3}(z) = \frac{b_k}{(1-T)^{3k}} - \frac{c_k^{(C_3)}}{(1-T)^{3k-1}} + \sum_{i\leq 3k-2} \frac{\omega_{k,i}^{(C_3)}}{(1-T)^i} \quad (45)$$

where $T \equiv T(z)$, the summation is finite and the coefficients $c_k^{(C_3)}$ are defined, for all $k \geq 1$, by

$$\begin{cases} c_1^{(C_3)} = \frac{25}{24}\,, \\ c_{k+1}^{(C_3)} = c_{k+1} + \frac{3}{2} k b_k\,. \end{cases} \tag{46}$$

Before proving theorem 14, the connected components with one occurrence of triangle are subdivided into 3 kinds of constructions, according to the degrees of the vertices of the unique triangle (after smoothing). Let us define these classifications. A smooth graph containing a triangle is of three kinds:

- exactly one vertex of the triangle is of degree ≥ 3,
- exactly two vertices of the triangle are of degree ≥ 3,
- the 3 vertices of the triangle are all of degree ≥ 3.

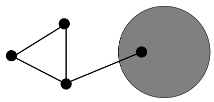


Fig. 13. One vertex of the triangle is of degree ≥ 3.

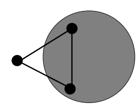


Fig. 14. Two vertices of the triangle are of degree ≥ 3.

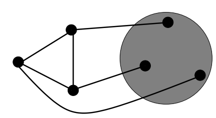


Fig. 15. Other smooth components.

Graphs whose situations after smoothing are depicted by figures 13 and 14 can be handled by the techniques of lemmas 11 and 12, and will be considered more precisely later. Note that in the figures, the right parts (in grey) of the constructions correspond to multicyclic structures without triangle. The lemma 15 gives the form of the EGF of the connected component with exactly one occurrence of triangle depicted by the figure 15.

Lemma 15 *The EGF of $(k+1)$-cyclic graphs containing one occurrence of triangle with all of its vertices of degree at least 3 has the following form*

$$\sum_{s \leq 3k-3} \frac{\epsilon_{k,s}}{(1-T(z))^s} \tag{47}$$

where the summation is finite and the coefficients $\epsilon_{k,s}$ are rational numbers.

Proof. Our idea is to apply Wright's reduction method on our specific configuration. Since this method is known but is not that familiar, we repeat here the main steps. Suppose that we have a connected graph with k edges more than vertices containing one triangle and suppose that the recursive suppressions of vertices of degree 1 lead to a graph of the type depicted by figure 15. That is, the obtained smooth graph has t vertices of degree at least 2 and $t+k$ edges (here, t is less that or equal to the number of vertices of the original graph). This way, we get a smooth graph with r vertices of degree at least 3, $r \leq 2k$. These vertices of degree ≥ 3 are called *special vertices* and let us *color* the edges of the triangle in order to distinguish them. The paths between these points, except the colored edges of the triangle, are of four kinds and we apply the following special operations on them (see [39, Sect. 6]):

1. An α-path begins and ends with the same special point and so must have at least two interior points. We elide all its interior points except two of them.
2. A β-path joins two different special vertices and we elide all its interior points.
3. If two different special vertices are joined by more than one special path, at most one of these paths is reduced to a single edge which we call a $\delta - path$.
4. The remaining paths, or all the paths if there is no δ-path, are called γ-paths and for each γ-path, we elide all its interior points except one of them.

The obtained graph is called *Wright's basic graph.* Denote respectively by a, b, c and d the number of α-, β-, γ- and δ- paths. Since each elision has removed exactly one edge and one vertex, the number of vertices of the basic graph is exactly $r + 2a + c$. Taking into account, the colored edges of the triangle and the operations made upon the special paths, the number of edges in the basic graph is $r + 2a + c + k = 3a + b + 2c + d + 3$. Thus, we have $a + b + c + d + 3 = r + k \leq 3k$. We find

$$a + b + c \leq 3k - 3\,. \tag{48}$$

To obtain any of the original graphs without vertices of degree 1, we distribute the previously $t - r - 2a - c$ elided nodes on the α-, β- and γ- paths. (48) gives us ideas on the number of ways to redistribute these points: suppose that $f(n)$ is the number of labelings of the $(n, n+k)$-graphs which can produce the considered basic graph. Let $F(z)$ be their EGF:

$$F(z) = \sum_n f(n) \frac{z^n}{n!}\,. \tag{49}$$

To obtain each of the original $(t, t+k)$ graphs without endvertices, the distribution of the $(t - r - 2a - c)$ nodes on the $(a + b + c)$ α-, β- and γ-paths can be done in y ways where y is the number of partitions of $(t - r - 2a - c)$ into $(a + b + c)$ parts. Relabel the obtained graph and replace the t vertices with t rooted and labelled trees. All the graphs are enumerated but they are not all different. In fact, they are enumerated g times where g is the order of the automorphisms of the current Wright's basic graph. Thus, we have

$$gF(z) = \sum_t yT(z)^t = \frac{T(z)^{r+2a+c}}{(1-T(z))^{a+b+c}}\,. \tag{50}$$

Summing over all the finitely many possible basic graphs, we obtain the lemma.
□

Proof of theorem 14. Denote by $(\mathcal{P}_{k,\widehat{W}})$, $(\mathcal{P}_{k,\widehat{S}})$ and $(\mathcal{P}_{k,\widehat{J}})$ the following properties:

- $(\mathcal{P}_{k,\widehat{W}})$: $\widehat{W}_{k,C_3}$ is of the form given by the equation (45).
- $(\mathcal{P}_{k,\widehat{S}})$:
  If $k = 1$,

$$\widehat{S}_{1,C_3}(z) = \frac{1}{4\,(1-T)^2} - \frac{1}{(1-T)} - \frac{1}{4}T^4 + \frac{1}{4}T^2 + \frac{1}{2}T + \frac{3}{4} \tag{51}$$

  and for all $k \geq 2$, $\widehat{S}_{k,C3}$ is of the form

$$\widehat{S}_{k,C_3}(z) = \frac{3(k-1)b_{k-1}}{2\,(1-T(z))^{3k-1}} + \sum_{i\leq 3k-2} \frac{\sigma_{k,i}^{(C_3)}}{(1-T(z))^i}\,. \tag{52}$$

- $(\mathcal{P}_{k,\widehat{J}})$:
  If $k = 1$

$$\widehat{J}_{1,C_3}(z) = \frac{T^4}{4} \tag{53}$$

  and if $k = 2$, we have

$$\widehat{J}_{2,C_3}(z) = \frac{1}{2(1-T)^2} - \frac{2}{(1-T)} + \frac{3}{2} + T + \frac{T^2}{2} - \frac{T^4}{2} - \frac{2T^5}{3}\,. \tag{54}$$

  For all $k \geq 3$, $\widehat{J}_{k,C_3}$ is of the form

$$\widehat{J}_{k,C_3}(z) = \frac{3(k-2)b_{k-2}}{(1-T(z))^{3k-4}} + \sum_{i\leq 3k-5} \frac{\upsilon_{k,i}^{(C_3)}}{(1-T(z))^i}\,. \tag{55}$$

where the coefficients $(\omega_{k,i}^{(C_3)})$, $(\sigma_{k,i}^{(C_3)})$ and $(\upsilon_{k,i}^{(C_3)})$ are rational numbers and the summations in (45), (52) and (55) are **finite**.

We will show by *induction* on k, that for all $k \geq 1$, the properties $(\mathcal{P}_{k,\widehat{W}})$, $(\mathcal{P}_{k,\widehat{S}})$ and $(\mathcal{P}_{k,\widehat{J}})$ described above are simultaneously verified. To do this, we have $(\mathcal{P}_{1,\widehat{W}})$, $(\mathcal{P}_{1,\widehat{S}})$, $(\mathcal{P}_{1,\widehat{J}})$ and $(\mathcal{P}_{2,\widehat{J}})$ and we have to check that if $(\mathcal{P}_{i,\widehat{W}})$, $(\mathcal{P}_{i,\widehat{S}})$ and $(\mathcal{P}_{i,\widehat{J}})$ are true for all i such that $1 \leq i \leq k-1$ then $(\mathcal{P}_{k,\widehat{W}})$, $(\mathcal{P}_{k,\widehat{S}})$

and $(\mathcal{P}_{k,\widehat{J}})$ are also satisfied. Note that due to the presence of the factor $(k-1)$ in (52), resp. $(k-2)$ in (55), we have to give $\widehat{S}_{1,C_3}$, $\widehat{J}_{1,C_3}$ and $\widehat{J}_{2,C_3}$. Rewriting (40) and (37) as sums of powers of $\frac{1}{1-T}$, we have

$$
\begin{aligned}
\widehat{W}_{2,C_3}(z) &= \frac{5}{16(1-T)^6} - \frac{5}{3(1-T)^5} + \frac{167}{48(1-T)^4} - \frac{91}{24(1-T)^3} \\
&+ \frac{55}{16(1-T)^2} - \frac{35}{8(1-T)} + \frac{125}{48} + \frac{17T}{12} + \frac{11T^2}{24} - \frac{5T^3}{24} - \frac{5T^4}{12} - \frac{5T^5}{24}, \\
\widehat{S}_{2,C_3}(z) &= \frac{5}{16(1-T)^5} - \frac{65}{48(1-T)^4} + \frac{7}{3(1-T)^3} - \frac{73}{24(1-T)^2} \\
&+ \frac{61}{12(1-T)} - \frac{10}{3} - \frac{103T}{48} - \frac{53T^2}{48} - \frac{5T^3}{48} + \frac{31T^4}{48} + \frac{5T^5}{8}. \qquad (56)
\end{aligned}
$$

Thus, $\widehat{S}_{2,C_3}(z)$, $\widehat{J}_{2,C_3}(z)$, and $\widehat{W}_{2,C_3}(z)$ can be formulated as finite sums of power of $\frac{1}{(1-T)}$ and properties $(\mathcal{P}_{2,\widehat{W}})$, $(\mathcal{P}_{2,\widehat{S}})$ and $(\mathcal{P}_{2,\widehat{J}})$ are satisfied. Note that we let $b_0 = \frac{1}{2}$, due to the fact that $\vartheta_z \widehat{W}_{0,C_3}(z) = \frac{1}{2}\frac{T^4}{(1-T)}$. Now, suppose that $(\mathcal{P}_{i,\widehat{W}})$, $(\mathcal{P}_{i,\widehat{S}})$ and $(\mathcal{P}_{i,\widehat{J}})$ are true for $i \in [1,\, k-1]$. If we want to compute directly $\widehat{W}_{k,C_3}$, the differential recurrence relation (19) of lemma 7 is not useful except if we know the EGFs $\widehat{S}_{k,C_3}$ and $\widehat{J}_{k,C_3}$. However, assuming that $(\mathcal{P}_{i,\widehat{W}})$, $(\mathcal{P}_{i,\widehat{S}})$ and $(\mathcal{P}_{i,\widehat{J}})$ are true for $i \in [2,\, k-1]$, we can compute the forms of $\widehat{S}_{k,C_3}$ and $\widehat{J}_{k,C_3}$ using combinatorial decompositions of these graphs. In the rest of this proof, our attention will be focused on the terms involving $\frac{1}{(1-T(z))^{3k}}$ and $\frac{1}{(1-T(z))^{3k-1}}$ for $\widehat{W}_{k,C_3}$ and $\frac{1}{(1-T(z))^{3k-1}}$ for $\widehat{S}_{k,C_3}$. Under the hypothesis of the induction, let us compute the forms of $\widehat{S}_{k,C_3}$ and $\widehat{J}_{k,C_3}$. More specifically, the components represented by figures 13 and 14 can be decomposed and the forms of their EGFs can be computed using the EGF of the triangle (eq. (9)), the operator ϑ_z (to distinguish the common point) and the form of the EGF $\widehat{W}_{k-1,C_3}$ which is assumed by the induction hypothesis. Recall that $\underline{\widehat{W}_{k-1,C_3}}$ denotes the EGF of *k-cyclic smooth graphs* without triangle obtained when deleting recursively all vertices of degree 1. Using lemma 11, we obtain the univariate EGF of all the graphs such that the situation after smoothing is depicted by figure 13, namely

$$\left[\frac{1}{z}\,\frac{1}{1-z}\,\vartheta_z(\frac{z^3}{3!})\vartheta_z\underline{\widehat{W}_{k-1,C_3}}(z)\right]_{|z=T(z)} \qquad (57)$$

Similarly, the smooth graph represented by figure 14 can be enumerated using the operator ϑ_w. We obtain the following bivariate EGF

$$\left[\frac{2}{wz^2}\vartheta_w(\frac{w^3z^3}{3!})\vartheta_w(\underline{\widehat{W}_{k-1,C_3}}(w,z))\right]_{|wz=T(wz)} \qquad (58)$$

Using the form of the EGF of $(k+1)$-cyclic components given by lemma 15, we find the form of the bivariate EGF of smooth graphs of $\underline{\widehat{S}_{k,C_3}}$,

$$\underline{\widehat{S}_{k,C_3}}(w,z) = \left(\frac{w}{z(1-wz)}\vartheta_z(\frac{w^3z^3}{3!})\vartheta_z(\underline{\widehat{W}_{k-1,C_3}}(w,z))\right) + \left(\frac{2}{wz^2}\vartheta_w(\frac{w^3z^3}{3!})\vartheta_w(\underline{\widehat{W}_{k-1,C_3}}(w,z))\right) + w^k \sum_{i\leq 3k-2} \frac{s_{k,i}^{(C_3)}}{(1-wz)^i}\,. \tag{59}$$

Remark that the constants $s_{k,i}^{(C_3)}$ are not those described by eq. (52) because we have to take into account the terms from $\frac{2}{wz^2}\vartheta_w(\frac{w^3z^3}{3!})\vartheta_w(\underline{\widehat{W}_{k-1,C_3}}(w,z))$. Thus, we find

$$\begin{aligned}\underline{\widehat{S}_{k,C_3}}(w,z) &= \frac{w^3z^2}{2(1-wz)} \times w^{k-1}\,\vartheta_z\left(\frac{b_{k-1}}{(1-wz)^{3k-3}} + \sum_{i\leq 3k-4}\frac{s_{k-1,i}^{(C_3)}}{(1-wz)^i}\right)\\ &+ w^2z\vartheta_w\left(\frac{w^{k-1}b_{k-1}}{(1-wz)^{3k-3}} + \sum_{i\leq 3k-4}\frac{w^{k-1}s_{k-1,i}^{(C_3)}}{(1-wz)^i}\right) + w^k\sum_{i\leq 3k-2}\frac{s_{k,i}^{(C_3)}}{(1-wz)^i}\,. \end{aligned}\tag{60}$$

A bit of calculus leads to the EGF of $(k+1)$-cyclic components with exactly one triangle

$$\widehat{S}_{k,C_3}(w,z) = w^k\left(\frac{3(k-1)b_{k-1}}{2(1-T)^{3k-1}} + \sum_{i\leq 3k-2}\frac{\sigma_{k,i}^{(C_3)}}{(1-T)^i}\right)\,. \tag{61}$$

and $(\mathcal{P}_{k,\widehat{S}})$ is verified. Similarly, the same principles can be used to compute the form of $\widehat{J}_{k,C_3}$ when replacing the single occurrence of triangle by a single occurrence of juxtaposition of triangles which can be considered in its turn as a single subgraph. For this purpose, we have to replace the EGF $\frac{w^3z^3}{3!}$ of the triangle by EGFs of juxtapositions of triangles, viz. $\frac{w^5z^4}{2!2!}$ (EGF of the smooth graph depicted by figure 8), $\frac{w^7z^5}{2!3!}$, $\cdots$, $\frac{w^{2i+1}z^{i+2}}{2!i!}$, $\cdots$. We find

$$\begin{aligned}\underline{\widehat{J}_{k,C_3}}(w,z) &= \frac{w}{z(1-wz)}\vartheta_z(\frac{w^5z^4}{4})\,\vartheta_z(\underline{\widehat{W}_{k-2,C_3}}(w,z))\\ &+ \frac{w}{z(1-wz)}\vartheta_z(\frac{w^7z^5}{12})\vartheta_z(\underline{\widehat{W}_{k-3,C_3}}(w,z))\\ &+ \frac{2}{wz^2}\vartheta_w(\frac{w^5z^4}{4})\vartheta_w(\underline{\widehat{W}_{k-2,C_3}}(w,z)) + w^k\sum_{i\leq 3k-3}\frac{\iota_{k,i}^{(C_3)}}{(1-wz))^i}\,. \end{aligned}\tag{62}$$

Hence, we have the form of $3\widehat{S}_{k,C_3} + \widehat{J}_{k,C_3}$ which starts with $\frac{9(k-1)b_{k-1}}{2(1-T)^{3k-1}}$. We need some useful notations, mainly related to those of Wright [39, 41]. Denote by $\mathbb{X}$ the following EGF

$$\mathbb{X} \equiv 1 - T\,. \tag{63}$$

Let $\Lambda_1^{(C_3)} = 0$ and for all $k \geq 2$, let $\Lambda_k^{(C_3)}$ be the following formal power series

$$\Lambda_k^{(C_3)} :\ \Lambda_k^{(C_3)}(z) = \sum_{t=1}^{k-1} \Big(\vartheta_z \widehat{W}_{t,C_3}(z)\Big)\Big(\vartheta_z \widehat{W}_{k-t,C_3}(z)\Big)\,. \tag{64}$$

Let F be an EGF. For all $k \geq 1$, we denote by Δ and $\Omega_k^{(C_3)}$ the following operators

$$\Delta_{k+1} :\ \Delta_{k+1}\,(F) = 2\Big(k + 1 - T\frac{\partial}{\partial T}\Big)\,(F) \tag{65}$$

and

$$\Omega_k^{(C_3)} :\ \Omega_k^{(C_3)}\,(F) = \Big((\vartheta_z^2 - 3\vartheta_z - 2k) + 2(\vartheta_z \widehat{W}_{0,C_3}(z))\vartheta_z\Big)\,(F)\,. \tag{66}$$

Using these notations, we remark that the functional equation (19) of lemma 7 can be reformulated as follows

$$\begin{aligned}&\Delta_{k+1}\widehat{W}_{k+1,C_3} + 6\widehat{S}_{k+1,C_3} + 2\widehat{J}_{k+1,C_3} = \\ &\quad \Omega_k^{(C_3)}\widehat{W}_{k,C_3} + \Lambda_k^{(C_3)}\,,\ (k \geq 1)\,.\end{aligned} \tag{67}$$

Then, we remark that

$$\Delta_k \mathbb{X}^{-t} = \Delta_k \frac{1}{(1-T)^t} = 2\mathbb{X}^{-t}(t\mathbb{X}^{-1} + k - t)\,. \tag{68}$$

We also have

$$\vartheta_z \widehat{W}_{0,C_3}(z) = \frac{T^4}{2(1-T)^2} = \frac{\mathbb{X}^{-2}}{2} - 2\mathbb{X}^{-1} + 3 - 2\mathbb{X} + \frac{\mathbb{X}^2}{2}\,. \tag{69}$$

$$\begin{aligned}&(\vartheta_z^2 - \vartheta_z - 2(k-1))\mathbb{X}^{-t} + 2(\vartheta_z \widehat{W}_{0,C_3})(\vartheta_z \mathbb{X}^{-t}) = \\ &\quad t(t+3)\mathbb{X}^{-t-4} - t(2t+8)\mathbb{X}^{-t-3} + \cdots\,.\end{aligned} \tag{70}$$

Using these formulae, the induction hypothesis, the form of the generating function $6\widehat{S}_{k,C_3} + 2\widehat{J}_{k,C_3}$ and the formula (19) of lemma 7, when looking after the coefficients of $\mathbb{X}^{-3k+1}$ and $\mathbb{X}^{-3k}$, we find

$$\widehat{W}_{k,C_3} = b_k \mathbb{X}^{-3k} - c_k^{(C_3)} \mathbb{X}^{-3k+1} + \cdots$$

where the sequences (b_k) and $(c_k^{(C_3)})$ satisfy exactly the recurrences given by (42) and

$$\begin{cases} c_1^{(C_3)} = \frac{25}{24}\,, \\ 2(3k+2)c_{k+1}^{(C_3)} = 8(k+1)b_{k+1} \\ +\, 6kb_k + (3k-1)(3k+2)c_k^{(C_3)} \\ +\, 6\sum_{t=1}^{k-1} t(3k-3t-1)b_t c_{k-t}^{(C_3)}\,. \end{cases} \tag{71}$$

Now, we can show (46) by induction. We have $c_1^{(C_3)} = \frac{25}{24}$, $b_1 = \frac{5}{24}$ and $c_2 = \frac{65}{48}$ and we can check $c_2^{(C_3)} = \frac{5}{3} = c_2 + \frac{3}{2}b_1$. Suppose that for i from 1 to $k-1$, $c_i^{(C_3)}$ verifies

$$c_{i+1}^{(C_3)} = c_{i+1} + \frac{3}{2} i b_i\,.$$

Using (71) and the induction hypothesis, we have for $i = k$ (we have to be careful with $c_1^{(C_3)} = c_1 + \frac{1}{4}$)

$$\begin{aligned} 2(3k+2)c_{k+1}^{(C_3)} &= 8(k+1)b_{k+1} + 6kb_k \\ &+ (3k-1)(3k+2)c_k + \frac{3}{2}(3k-1)(3k+2)(k-1)b_{k-1} \\ &+ 12(k-1)b_{k-1}c_1 + 3(k-1)b_{k-1} \\ &+ 6\sum_{t=1}^{k-2} t(3k-3t-1)b_t c_{k-t} \\ &+ 9\sum_{t=1}^{k-2} t(3k-3t-1)(k-t-1)b_t b_{k-t-1}\,. \end{aligned} \tag{72}$$

And as already remarked by Wright, [41, eq. (3.5)], for any given sequence (α_k) we have

$$\sum_{t=1}^{k-1} t\alpha_t \alpha_{k-t} = \frac{k}{2}\sum_{t=1}^{k-1} \alpha_t \alpha_{k-t}\,. \tag{73}$$

Rearranging, we find using the definition of c_{k+1} given by (43) and (73)

$$\begin{aligned} 2(3k+2)c_{k+1}^{(C_3)} &= 2(3k+2)c_{k+1} + 3kb_k \\ &+ (3 + \frac{3}{2}(3k-1)(3k+2))(k-1)b_{k-1} \end{aligned}$$

$$+\frac{9}{2}(3k+1)\sum_{t=1}^{k-2} tb_t(k-t-1)b_{k-t-1}\,. \tag{74}$$

Since $3\sum_{t=1}^{k-2} tb_t(k-t-1)b_{k-t-1} = 2kb_k - 3(k-1)kb_{k-1}$, we obtain

$$\begin{aligned} 2(3k+2)c_{k+1}^{(C_3)} &= 2(3k+2)c_{k+1} + 3kb_k \\ &+(3+\frac{3}{2}(3k-1)(3k+2))(k-1)b_{k-1} \\ &+3(3k+1)kb_k - \frac{9}{2}(k-1)k(3k+1)b_{k-1}\,. \end{aligned} \tag{75}$$

Finally, we find $2(3k+2)c_{k+1}^{(C_3)} = 2(3k+2)c_{k+1} + 3(3k+2)kb_k$. $\square$

As a consequence, if we want to work with a forbidden subgraph H which is not unicyclic (e.g. K_4), the decomposition of $\widehat{W}_{k,H}$ into sums of negative powers of $\mathbb{X}$ (i.e. tree polynomials) starts

$$\widehat{W}_{k,H} = b_k\mathbb{X}^{-3k} - c_k\mathbb{X}^{-3k+1} + \cdots\,.$$

The same remark holds for any finite collection of forbidden subgraphs which are not unicyclic.

In the next theorem, we will generalize the case $\xi = \{C_3\}$.

Theorem 16 *Let $\xi = \{H_1, H_2, \cdots, H_p\}$ a finite collection of multicyclic components. Suppose that ξ contains r, $r > 0$, distinct polygons (unicyclic smooth graphs). Denote by $\widehat{W}_{k,\xi}$ the EGF of $(k+1)$-cyclic ξ-free labelled graphs. For all $k \geq 2$, $\widehat{W}_{k,\xi}$ can be expressed as a finite sum of powers of $\frac{1}{1-T}$ and has the following form: For $k = 1$, we have*

$$\widehat{W}_{1,\xi}(z) = \frac{5}{24(1-T(z))^3} - \frac{(19/24 + r/4)}{(1-T(z))^2} + \sum_{i\leq 1} \frac{{\psi_{i,1}}^{(\xi)}}{(1-T(z))^i} \tag{76}$$

and for $k > 1$

$$\widehat{W}_{k,\xi}(z) = \frac{b_k}{(1-T(z))^{3k}} - \frac{c_k^{(\xi)}}{(1-T(z))^{3k-1}} + \sum_{i\leq 3k-2} \frac{{\psi_{i,k}}^{(\xi)}}{(1-T(z))^i} \tag{77}$$

where b_k is Wright's coefficient of first order given by (42) and $c_k^{(\xi)}$ is given recursively by $c_1^{(\xi)} = \frac{19+6r}{24}$ and for $k \geq 1$

$$c_{k+1}^{(\xi)} = c_{k+1} + \frac{3}{2}rkb_k\,. \tag{78}$$

Proof. The proof of this theorem is very close to that of theorem 14. Suppose that ξ contains r polygons ($r > 0$). Furthermore, suppose that C_q is the greatest polygon of ξ. That is

$$\widehat{W}_{0,\xi} = \frac{1}{2}\ln\frac{1}{1-T} - \frac{T}{2} - \frac{T^2}{4} - \sum_i \frac{T^i}{2i}$$

where in the summation i describes all lengths (less than or equal to q) of the forbidden polygons. Then, since

$$\frac{T^q}{(1-T)^2} = \mathbb{X}^{-2} - (q+1)\mathbb{X}^{-1} + \sum_{j=1}^{q} T^{q-j} \,,$$

we have

$$2\vartheta_z \widehat{W}_{0,\xi}(z) = \frac{T^{q+1}}{(1-T)^2} + \sum_j \frac{T^j}{1-T}$$

where the summation is over all lengths of the $q-r-2$ authorized (distinct) polygons. So,

$$2\vartheta_z \widehat{W}_{0,\xi}(z) = \mathbb{X}^{-2} - (r+3)\mathbb{X}^{-1} + \text{Polynomial}_\xi(T) \tag{79}$$

and $2(\vartheta_z \widehat{W}_{0,\xi}(z))(\vartheta_z \mathbb{X}^{-t})$ starts with

$$t\mathbb{X}^{-t-4} - (r+4)t\mathbb{X}^{-t-3} + \cdots . \tag{80}$$

Defining the operator $\Omega_k^{(\xi)}$ as

$$\Omega_k^{(\xi)} : \; \Omega_k^{(\xi)} = \Big((\vartheta_z^2 - 3\vartheta_z - 2k) + 2(\vartheta_z \widehat{W}_{0,\xi}(z))\vartheta_z\Big) \tag{81}$$

and $\Lambda_k^{(\xi)}$ as the formal power seriers

$$\Lambda_k^{(\xi)} : \; \Lambda_k^{(\xi)}(z) = \sum_{t=1}^{k-1} \Big(\vartheta_z \widehat{W}_{t,\xi}(z)\Big)\Big(\vartheta_z \widehat{W}_{k-t,\xi}(z)\Big) \,, \tag{82}$$

we can generalize (67)

$$\Delta_{k+1}\widehat{W}_{k+1,\xi} + 2\sum_{\mathcal{H}\in\xi} e(\mathcal{H})\widehat{S}_{k+1,\mathcal{H}} + 2\widehat{J}_{k+1,\xi} =$$

$$\Omega_k^{(\xi)} \widehat{W}_{k,\xi} + \Lambda_k^{(\xi)}\,,\ (k \geq 1)\,. \tag{83}$$

Then, we find

$$\Omega_k^{(\xi)} \mathbb{X}^{-t} = t(t+3)\mathbb{X}^{-t-4} - t(2t+r+7)\mathbb{X}^{-t-3} + \cdots \tag{84}$$

As for theorem 14, we find that $c_{k+1}^{(\xi)}$ satisfies $c_1^{(\xi)} = c_1 + \frac{r}{4}$ and for $k \geq 1$

$$\begin{aligned} 2(3k+2)c_{k+1}^{(\xi)} &= 8(k+1)b_{k+1} + 3k(r+1)b_k + \\ &(3k-1)(3k+2)c_k^{(\xi)} + 6\sum_{t=1}^{k-1} t(3k-3t-1)b_t c_{k-t}^{(\xi)}\,. \end{aligned} \tag{85}$$

We can now argue as for the proof of theorem 14 to verify that the sequence $(c_k^{(\xi)})$ satisfies (78). □

In the next section, we will determine the asymptotic number of triangle-free labelled components when the number of exceeding edges satisfies $k = o(n^{1/3})$.

5 Asymptotic number of sparsely connected labelled triangle-free components

The methods we give are based on the fundamental work of Wright in [41] with some ingredients from analytic combinatorics.

First of all, we will study the behavior of

$$t_n(a\,n+\beta) = n!\ [z^n]\,\frac{1}{(1-T(z))^{a\,n+\beta}}$$

where $a \equiv a(n)$ tends to 0 as $n \to \infty$ and β is fixed. Then, we will show that if $\beta_1 < \beta_2$, $a \equiv a(n) \to 0$ as $n \to \infty$ but $\frac{a\,n}{\ln n^3} \to \infty$, then $\frac{t_n(a\,n+\beta_1)}{t_n(a\,n+\beta_2)} \to 0$.

Next, we will give a general framework analogous to that of Wright in [41]. More precisely, let (b_k) and $(c_k^{(C_3)})$ be the coefficients given by (42) and (46). We will show that the coefficients of the EGFs $\widehat{W}_{k,C_3}$ satisfy the following inequalities

$$n!\,[z^n]\,\widehat{W}_{k,C_3}(z) \leq n!\,[z^n]\,\frac{b_k}{(1-T(z))^{3k}} \quad \textit{and}$$

$$n!\,[z^n]\left(\frac{b_k}{(1-T(z))^{3k}}-\frac{c_k^{(C_3)}}{(1-T(z))^{3k-1}}\right)\leq n!\,[z^n]\,\widehat{W}_{k,C_3}(z) \tag{86}$$

which we shall call *Wright's inequalities* for triangle-free graphs. Thus, the inequalities in (86) and the fact that $\frac{t_n(a\,n-1)}{t_n(a\,n)} \to 0$ imply that almost all connected components with n vertices and $n+o(n^{1/3})$ edges are ξ-free whenever $k = o(n^{1/3})$. Equivalently, we will show that the number $c_{C_3}(n, n+k)$ of triangle-free $(k+1)$-cyclic graphs is asymptotically the same as the number $c(n, n+k)$ of $(k+1)$-cyclic general graphs computed by Wright in [41] (see [4] for the extension of Wright's asymptotic results).

5.1 Saddle point method for tree polynomials

In [24], Knuth and Pittel studied combinatorially and analytically the polynomial $t_n(y)$ defined as follows

$$t_n(y) = n!\,[z^n]\,\frac{1}{(1-T(z))^y} \tag{87}$$

which they call *tree polynomial.* In fact, the authors of [24] observed that the analysis of these polynomials can also be used to study random graphs.

The lemma below is an application of the saddle point method [8, 17] to study the asymptotic behavior of the coefficients $n!\,[z^n]\,(1-T(z))^{-m(n)}$ as m, n tend to infinity but $m = o(n)$.

Lemma 17 *Let $a \equiv a(n)$ such that $a \to 0$ but $\frac{a\,n}{\ln n^3} \to \infty$, and β a fixed number. Then, the tree polynomial $t_n(a\,n+\beta)$ defined in (87) satisfies*

$$t_n(a\,n+\beta) = \frac{n!}{2\sqrt{\pi n}}\,\frac{\exp{(nu_0)}(1-u_0)^{(1-\beta)}}{{u_0}^n(1-u_0)^{a\,n}}\left(1+O(\sqrt{a})+O(\frac{1}{\sqrt{a\,n}})\right) \tag{88}$$

where $u_0 = 1+\frac{a}{2}-\sqrt{a(1+\frac{a}{4})}$.

Proof. Cauchy's integral formula gives

$$\begin{aligned} t_n(a\,n+\beta) &= n!\,[z^n]\,\frac{1}{(1-T(z))^{a\,n+\beta}} \\ &= \frac{n!}{2\pi i}\oint\frac{1}{(1-T(z))^{a\,n+\beta}}\frac{dz}{z^{n+1}} \end{aligned} \tag{89}$$

where we integrate around a small circle enclosing the origin and whose radius is smaller than $1/e$ (since $1/e$ is the radius of convergence of the formal power series $T(z) = \sum_{n\geq 1} n^{(n-1)} \frac{z^n}{n!}$). We make the substitution $u = T(z)$ and get $dz = e^{-u}(1-u)du$. Thus,

$$t_n(a\,n+\beta) = \frac{n!}{2\pi i} \oint \frac{e^{nu}\,du}{(1-u)^{a\,n+\beta-1}\,u^{n+1}}\,. \tag{90}$$

The power $(\exp(u)/(1-u)^a)^n$ suggests us to use the saddle point method. We will describe briefly this method for our case and refer to de Bruijn [8, Chap. 5], Flajolet and Sedgewick [17] or Bender [3] for more details on general asymptotic methods.

We set $h(u) = u - \ln(u) - a\ln(1-u)$. Starting with (90), we now have

$$t_n(a\,n+\beta) = \frac{n!}{2\pi i} \oint (1-u)^{1-\beta} \exp(nh(u)) \frac{du}{u}\,. \tag{91}$$

Let $F(r,\theta)$ be the integrand of

$$\begin{aligned} &\frac{1}{2\pi r^n} \int_{-\pi}^{\pi} (1-re^{i\theta})^{1-\beta} \exp(nh(re^{i\theta}))d\theta \\ &= \frac{1}{2\pi r^n} \int_{-\pi}^{\pi} F(r,\theta)d\theta\,. \end{aligned} \tag{92}$$

The saddle point method consists to remark that $F(r,\theta)$ turns very quickly as $n\to\infty$ such that the essential of the integral is captured by only few values of θ, say $\theta\in[-\theta_0,\,\theta_0]$ (with $\theta_0\to 0$). Then, we have to choose the radius r in order to concentrate the main contribution of the integral, viz. for $\theta\in[-\theta_0,\,\theta_0]$, $|F(r,\theta)|$ represents the essential of the integral. In other words, we have to find a vicinity of $\theta=0$ where $|F(r,\theta)|$ takes its maximum. Hence, we investigate the roots of $h'(u)=0$ and we find two saddle points, at $u_0 = 1+\frac{a}{2}-\sqrt{a(1+\frac{a}{4})}$ and $u_1 = 1+\frac{a}{2}+\sqrt{a(1+\frac{a}{4})}$. We notice that $h''(u) = \frac{1-2u+(1+a)u^2}{u^2(1-u)^2}$, $h''(u_0) = 2+3\sqrt{a}+O(a)$ and $h''(u_1) = 2-3\sqrt{a}+O(a)$. The main point of the application of the saddle point method here is that $h'(u_0)=0$ and $h''(u_0)>0$, hence $nh(u_0\exp(i\theta))$ is approximately $nh(u_0) - n{u_0}^2 h''(u_0)\frac{\theta^2}{2}$ in the vicinity of $\theta=0$. If we integrate (91) around a circle passing vertically through $u=u_0$, we obtain:

$$t_n(a\,n+\beta) = \frac{n!}{2\pi i} \int_{-\pi}^{\pi} (1-u_0e^{i\theta})^{1-\beta} \exp(nh(u_0e^{i\theta}))d\theta \tag{93}$$

where

$$h(u_0 e^{i\theta}) = u_0 \cos\theta + i u_0 \sin\theta - \ln u_0 - i\theta - a\ln(1 - u_0 e^{i\theta}) \; . \tag{94}$$

Denote by $\mathfrak{Re}(z)$ the real part of z, we have

$$\begin{aligned} f(\theta) &= \mathfrak{Re}(h(u_0 e^{i\theta})) \\ &= u_0\cos\theta - \ln u_0 - a\ln(|1 - u_0 e^{i\theta}|) \\ &= u_0\cos\theta - \ln u_0 - a\ln u_0 - \frac{a}{2}\ln\big(1 + \frac{1}{u_0^2} - \frac{2}{u_0}\cos\theta\big) \, . \end{aligned} \tag{95}$$

It comes

$$f^{'}(\theta) = \frac{d}{d\theta}\mathfrak{Re}(h(u_0 e^{i\theta})) = -u_0\sin\theta - \frac{\frac{a}{2}\big(\frac{2}{u_0}\sin\theta\big)^2}{\big(1 + \frac{1}{u_0^2} - \frac{2}{u_0}\cos\theta\big)} \tag{96}$$

and $f^{'}(\theta) = 0$ if $\theta = 0$. Also, $f(\theta)$ is a symmetric function of θ and in $[-\pi, -\theta_0] \cup [\theta_0, \pi]$, for a given θ_0, $0 < \theta_0 < \pi$, it takes it maximum value for $\theta = \theta_0$. Since $|\exp(h(u))| = \exp(\mathfrak{Re}(h(u)))$, when splitting the integral in (93) into three parts, viz. "$\int_{-\pi}^{-\theta_0} + \int_{-\theta_0}^{\theta_0} + \int_{\theta_0}^{\pi}$", we know that it suffices to integrate from $-\theta_0$ to θ_0, for a convenient value of θ_0, because the others can be bounded by the magnitude of the integrand at θ_0. In fact, we have

$$\begin{aligned} h(u_0 e^{i\theta}) &= h(u_0) + \frac{{u_0}^2 (e^{i\theta} - 1)^2}{2!} h^{''}(u_0) + \frac{{u_0}^3 (e^{i\theta} - 1)^3}{3!} h^{(3)}(u_0) \\ &\quad + \frac{{u_0}^4 (e^{i\theta} - 1)^4}{4!} h^{(4)}(u_0) + \sum_{p \ge 5} \frac{{u_0}^p (e^{i\theta} - 1)^p}{p!} h^{(p)}(u_0) \\ &= h(u_0) + \sum_{p \ge 2} \alpha_p (e^{i\theta} - 1)^p \, , \end{aligned} \tag{97}$$

where $\alpha_p = \frac{{u_0}^p}{p!} h^{(p)}(u_0)$. We compute $h^{(p)}(u_0) = (-1)^p (p-1)! \Big(\frac{1}{{u_0}^p} - \frac{a}{(1-u_0)^p}\Big)$, for $p \ge 2$. Then, on first hand we obtain

$$\begin{aligned} \alpha_p &= \frac{(-1)^p}{p}\Big(1 - \frac{a{u_0}^p}{(1-u_0)^p}\Big) \\ &= \frac{(-1)^p}{p} + \frac{(-1)^{p+1}}{p} \frac{a(1 + \frac{a}{2} - \sqrt{a(1+\frac{a}{4})})^p}{a^{\frac{p}{2}}(\sqrt{1+\frac{a}{4}} - \frac{\sqrt{a}}{2})^p} \\ &= \frac{(-1)^p}{p} + \frac{(-1)^{p+1}}{p} \frac{2^p}{a^{\frac{p}{2}-1}} \frac{(1 + \frac{a}{2} - \sqrt{a(1+\frac{a}{4})})^p}{(\sqrt{1+\frac{a}{4}} - \frac{\sqrt{a}}{2})^p} \, . \end{aligned} \tag{98}$$

Hence,

$$|\alpha_p| \le O\Big(\frac{2^p}{a^{\frac{p}{2}-1}}\Big)\,,\ \ (a \to 0)\,. \tag{99}$$

On the other hand,

$$|e^{i\theta} - 1| = \sqrt{2(1-\cos\theta)} < \theta\,,\ (\theta > 0)\,. \tag{100}$$

Thus, the summation in (97) can be bounded for values of θ and a such that $\theta \to 0$, $a \to 0$ but $\frac{\theta}{\sqrt{a}} \to 0$ and we have

$$\begin{aligned}|\sum_{p\ge 4}\alpha_p(e^{i\theta}-1)^p| &\le \sum_{p\ge 4}|\alpha_p\theta^p| \\ &\le \sum_{p\ge 4} O\Big(\frac{2^p\theta^p}{a^{\frac{p}{2}-1}}\Big) = O\Big(\frac{\theta^4}{a}\Big)\,.\end{aligned} \tag{101}$$

It follows that for $\theta \to 0$, $a \to 0$ and $\frac{\theta}{\sqrt{a}} \to 0$

$$\begin{aligned}h(u_0e^{i\theta}) = h(u_0) &- \frac{1}{2}\,\frac{u_0}{(1-u_0)^2}(1+a-2u_0+{u_0}^2)\theta^2 \\ &+ i\frac{u_0}{6(1-u_0)^3}\,(1+a+(a-3)u_0+3{u_0}^2-{u_0}^3)\theta^3 + O\Big(\frac{\theta^4}{a}\Big)\,,\end{aligned} \tag{102}$$

where the term in the big-oh takes into account the terms from $(e^{i\theta}-1)^2$ and $(e^{i\theta}-1)^3$ of (97) which we can neglect since $(e^{i\theta}-1)^2 = -\theta^2 - i\theta^3 + O(\theta^4)$ and $(e^{i\theta}-1)^3 = -i\theta^3 + \frac{3}{2}\theta^4 + iO(\theta^5)$. Therefore, if $a \to 0$ but $\frac{an}{(\ln n)^2} \to \infty$, if we let $\theta_0 = \frac{\ln n}{\sqrt{n\rho}}$ with $\rho = \frac{u_0(1+a-2u_0+{u_0}^2)}{(1-u_0)^2} = 2 - \sqrt{a} + O(a)$, we can remark (as already said) that it suffices to integrate (93) from $-\theta_0$ to θ_0, using the magnitude of the integrand at θ_0 to bound the resulting error. Hence,

$$\begin{aligned}&|(1-u_0e^{i\theta_0})^{(1-\beta)}\Big(\exp\left(nh(u_0e^{i\theta_0})\right) - nu_0 + n\ln u_0 + a\ln(1-u_0)\Big)| = \\ &|1-u_0e^{i\theta_0}|^{(1-\beta)}\exp\Big(-\frac{n}{2}\rho\,{\theta_0}^2 + +O(n\frac{{\theta_0}^4}{a})\Big) = O\Big(e^{-\frac{(\ln n)^2}{2}}\Big)\,.\end{aligned} \tag{103}$$

To estimate $t_n(a\,n+\beta)$, it proves convenient to compute

$$J_n = \int_{-\theta_0}^{\theta_0}(1-u_0e^{i\theta})^{(1-\beta)}\exp\left(nh(u_0e^{i\theta})\right)d\theta\,. \tag{104}$$

If we make the substitution $\theta = \frac{t}{\sqrt{n\rho}}$, we have (recall that $\theta_0 = \frac{\ln n}{\sqrt{n\rho}}$)

$$J_n = \frac{1}{\sqrt{n\rho}} \int_{-\ln n}^{\ln n} \left(1 - u_0 e^{\frac{it}{\sqrt{n\rho}}}\right)^{(1-\beta)} \exp\left(nh(u_0 e^{\frac{it}{\sqrt{n\rho}}})\right) dt\,. \tag{105}$$

Since $(1 - u_0 e^{\frac{it}{\sqrt{n\rho}}})^{(1-\beta)} = (1-u_0)^{(1-\beta)}(1 + O(t/\sqrt{na}))$, J_n becomes

$$J_n = \frac{1}{\sqrt{n\rho}} \lambda_n$$

where $\lambda_n = \int_{-\ln n}^{\ln n} (1-u_0)^{(1-\beta)} \exp\left(nh(u_0) - \frac{t^2}{2} + i f_3 \frac{t^3}{\sqrt{na}} + O(\frac{t^4}{na})\right)\left(1 + O(\frac{t}{\sqrt{na}})\right) dt$ and $f_3 = -\frac{\sqrt{a}(1+a+(a-3)u_0+3u_0^2-u_0^3)}{\sqrt{u_0}(1+a-2u_0+u_0^2)^{\frac{3}{2}}} = -\frac{\sqrt{2}}{12} - \frac{5}{48}\sqrt{a} + O(a)$. We obtain

$$\begin{aligned}
J_n =& \frac{(1-u_0)^{(1-\beta)}}{\sqrt{n\rho}} e^{(nh(u_0))} \times \\
& \left[\int_{-\ln n}^{\ln n} e^{-\frac{t^2}{2}} \cos\left(f_3 \frac{t^3}{\sqrt{na}}\right)\left(1 + O(\frac{t}{\sqrt{na}}) + O(\frac{t^4}{na})\right) dt \right] \\
=& \frac{(1-u_0)^{(1-\beta)}}{\sqrt{n\rho}} e^{(nh(u_0))} \times \\
& \left[\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} \left(1 + O(\frac{t}{\sqrt{na}}) + O(\frac{t^6}{na})\right) dt \;+\; O(e^{-\frac{(\ln n)^2}{2}}) \right] \\
=& \frac{\sqrt{2\pi}(1-u_0)^{(1-\beta)} e^{(nh(u_0))}}{\sqrt{n\rho}} \left(1 + O(\frac{1}{\sqrt{na}})\right) \\
=& \sqrt{\frac{\pi}{n}} (1-u_0)^{(1-\beta)} e^{(nh(u_0))} \left(1 + O(\sqrt{a}) + O(\frac{1}{\sqrt{na}})\right).
\end{aligned} \tag{106}$$

We used $\cos(x) = 1 + O(x^2)$ and $\exp(O(x)) = 1 + O(x)$ when $x = O(1)$. Since $t_n(a\,n + \beta) = \frac{n!}{2\pi} J_n$, the proof of lemma 17 is now complete. □

5.2 Wright's inequalities

In order to adapt the techniques of Wright to our ξ-free components, we need to bound the *perturbative terms*, i.e., the EGFs containing the first apparitions of the forbidden configurations $\widehat{S}_{k,\xi}$ and $\widehat{J}_{k,\xi}$.

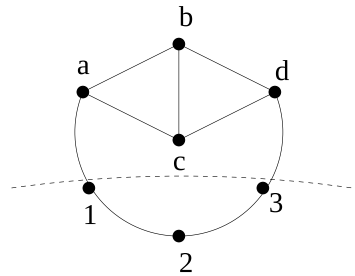


Fig. 16. Illustration of Bagaev's method.

5.2.1 Upper bounds of $\widehat{S}_{k,\xi}$ and $\widehat{J}_{k,\xi}$

To take control on these EGFs, let us recall briefly the *shrinking-and-expanding* Bagaev's method [2]: In order to enumerate graphs of a given type, an induced subgraph with special properties should be chosen and shrunk to a marked vertex. Separately, we have to calculate:

- the number of the obtained graphs, rooted at a fixed vertex of degree d,
- the number of the shrunk subgraphs,
- the number of ways to reconstruct the initial graphs.

We note that this technique generalizes the methods of lemmas 11 and 12. As an illustration of this method, consider the graph depicted by figure 16 where H is represented by the juxtaposition of triangles. The number of ways to label this graph can be computed easily using Bagaev's techniques. In fact, we have

$$\binom{7}{3} \times \underbrace{2 \times 1}_{reconstruction} \times 3 \times 6 = \frac{7!}{4}$$

manners to label the graph of figure 16 (3 manners to label the path with 3 vertices and 6 manners to label the juxtaposition of triangles). This method is very useful to bound graph typified by the one in figure 16 (where our interest is focused on the juxtaposition of triangles). The difficulties arise mainly from the number of possible reconstructions. In the current example, we have to rely the vertices 1 and 3 to 2 vertices belonging to $\{a, b, c, d\}$. Thus, the number of reconstructions is at most 4^2 (including graphs different from the one in figure 16).

Consider now $\widehat{S}_{k,\xi}$ with the special case $\xi = \{C_3\}$.

Lemma 18 *For all $k \geq 1$ and $\forall \varepsilon > 0$*

$$\widehat{S}_{k+1,C_3} \preceq \left(\frac{3}{2} + \varepsilon\right) \frac{k b_k}{\mathbb{X}^{3k+2}} . \tag{107}$$

Proof. The bound of (107) is inspired by the forms of the EGF $\widehat{S}_{k+1,C_3}$. We will prove (107) by induction. We can verify that $\widehat{S}_{2,C_3} \preceq \frac{5}{12\mathbb{X}^5}$, using (37). Suppose that $\widehat{S}_{i,C_3} \preceq \frac{2(i-1)b_{i-1}}{\mathbb{X}^{3i-1}}$, for $i \in [2,\, k-1]$ and let us prove that $\widehat{S}_{k,C_3} \preceq \frac{2(k-1)b_{k-1}}{\mathbb{X}^{3k-1}}$.

Split the set of $(k+1)$-cyclic graphs with exactly one occurrence of triangle into three subsets as follows :

1- the first subset Σ_1 contains all graphs whose situations after smoothing are characterized by the fact that exactly one vertex of the triangle is of degree ≥ 3,
2- similarly, the second subset Σ_2 is built with all graphs whose situations after smoothing are characterized by the fact that exactly two vertices of the triangle are of degree ≥ 3,
3- Σ_3 contains all other graphs of $\widehat{\mathcal{S}}_{k,C_3}$ not in $\Sigma_1 \cup \Sigma_2$.

We can bound the number of the graphs of the subsets Σ_1 and Σ_2, using lemmas 11, 12, $\underline{\widehat{W}_{k-1,C_3}} \preceq \frac{b_{k-1}}{(1-z)^{3k-3}}$ (since Wright showed $\widehat{W}_{k-1} \preceq \frac{b_{k-1}}{\mathbb{X}^{3k-3}}$ [41]) and the fact that $\vartheta_z(\frac{1}{\mathbb{X}^t}) \preceq \frac{t}{\mathbb{X}^{t+2}}$ for $t \geq 0$. In fact,

$$\begin{aligned}\Sigma_1(z) + \Sigma_2(z) = & \left[\frac{1}{z(1-z)}\left(\vartheta_z \frac{b_{k-1}}{(1-z)^{3k-3}}\right)\left(\vartheta_z \frac{z^3}{3!}\right)\right]_{|z=T} + \\ & \left[\frac{2}{wz^2}\left(\vartheta_w \frac{w^{k-1}b_{k-1}}{(1-wz)^{3k-3}}\right)\left(\vartheta_w \frac{w^3z^3}{3!}\right)\right]_{|wz=T} \\ & \preceq \frac{3}{2}\,\frac{(k-1)b_{k-1}}{\mathbb{X}^{3k-3}}(\mathbb{X}^{-2} - \mathbb{X}^{-1} + \frac{5}{3} - \mathbb{X}) \\ & \preceq \frac{3}{2}\,\frac{(k-1)b_{k-1}}{\mathbb{X}^{3k-1}}\,. \end{aligned} \tag{108}$$

For graphs of Σ_3, we have two subcases. Denote by $\Sigma_3^{'}$, resp. $\Sigma_3^{''}$, the graphs of Σ_3 such that the deletion of the 3 vertices and the 3 edges of the triangle will leave a connected graph, resp. disconnected graphs. The figures 11 (c) and 11 (e) illustrate these 2 classifications. In the first case, i.e. $\Sigma_3^{'}$, we will not use the induction hypothesis. In fact, to build a graph of $\Sigma_3^{'}$, we have to rely d vertices ($d \geq 3$) of a graph of $\widehat{\mathcal{W}}_{k-d,C_3}$ to the triangle. Thus, the number of manners to construct a graph of $\Sigma_3^{'}$ of order n this way is at most

$$\begin{aligned} & 3^d \binom{n}{3}\binom{n-3}{d}(n-3)!\left[z^{n-3}\right]\widehat{W}_{k-d,C_3}(z) \\ & \leq \frac{3^d}{6}\binom{n-3}{d} n!\left[z^{n-3}\right]\widehat{W}_{k-d,C_3}(z) \\ & \leq \frac{3^d}{6\,d!} n^d\, n!\,[z^n]\,\widehat{W}_{k-d,C_3}(z) \end{aligned}$$

$$\leq \frac{3^d}{6\,d!} n!\,[z^n]\,\vartheta_z^d\,\widehat{W}_{k-d,C_3}(z)\,,(3 \leq d \leq k+1)\,. \tag{109}$$

In terms of generating function, we then have (summing over d)

$$\Sigma_3'(z) \preceq \sum_{d\geq 3} \frac{3^d}{6\,d!} \vartheta_z^d\,\widehat{W}_{k-d,C_3}(z)\,. \tag{110}$$

First, let us treat the cases $d = k+1$ and $d = k$. We have

$$\vartheta_z^{(k+1)}\,\widehat{W}_{-1} = \vartheta_z^k\,T = \vartheta_z^{k-1}\,\frac{T}{\mathbb{X}}$$

and

$$\vartheta_z^k\,\widehat{W}_{0,C_3} = \vartheta_z^{k-1}\left(\frac{T^4}{2\mathbb{X}^2}\right).$$

Since $\frac{T}{X} \preceq \frac{1}{\mathbb{X}^2}$, we have

$$\frac{3^k}{6k!}\Big(1+\frac{3}{k+1}\Big)\Big(\vartheta_z^{(k+1)}\,\widehat{W}_{-1} + \vartheta_z^k\,\widehat{W}_{0,C_3}\Big) \preceq \frac{3^k}{k!}\vartheta_z^{(k-1)}\Big(\frac{1}{\mathbb{X}^2}\Big)\,.$$

Similarly

$$\vartheta_z^{(k-1)}\frac{1}{\mathbb{X}^2} \preceq \vartheta_z^{(k-2)}\frac{2}{\mathbb{X}^4}\cdots \preceq \frac{2\times 4\times\cdots\times 2(k-1)}{\mathbb{X}^{2k}} = \frac{2^k(k-1)}{\mathbb{X}^{2k}}$$

and we obtain for $d = k+1$ and $d = k$ in (110)

$$\frac{3^{k+1}}{6(k+1)!}\vartheta_z^{(k+1)}\,\widehat{W}_{-1} + \frac{3^k}{6k!}\vartheta_z^k\,\widehat{W}_{0,C_3} \preceq \frac{6^k}{6k!}\frac{(k-1)!}{\mathbb{X}^{2k}}\,. \tag{111}$$

Next, we have

$$\frac{b_{k+1}}{b_k} \geq \frac{3}{2}k$$

since $b_k = (\frac{3}{2})^k(k-1)!d_k$ and (d_k) is an increasing sequence (cf. [41, eq. (1.4)]). Thus,

$$b_k \geq \frac{3}{2}(k-1)b_{k-1} \geq \Big(\frac{3}{2}\Big)^2(k-1)(k-2)b_{k-2} \geq \cdots \geq \Big(\frac{3}{2}\Big)^{k-1}(k-1)!\;b_1$$

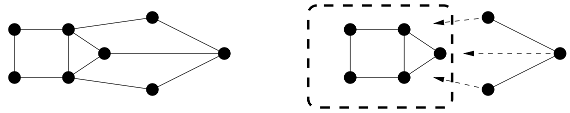


Fig. 17. A representative graph of Σ_3'' and its reconstruction.

and

$$(k-1)! \leq 6(k-1)b_{k-1}\,. \tag{112}$$

Finally,

$$\frac{3^{k+1}}{6(k+1)!}\vartheta_z^{(k+1)}\,\widehat{W}_{-1} + \frac{3^k}{6k!}\vartheta_z^k\,\widehat{W}_{0,C_3} \preceq \frac{6^k}{6k!}\frac{(k-1)b_{k-1}}{\mathbb{X}^{2k}}\,. \tag{113}$$

Summing (110) over d for $d \in [3,\, k-2]$, we obtain

$$\begin{aligned}
&\sum_{d=3}^{k-2}\frac{3^d}{6\,d!}\vartheta_z^d\,\widehat{W}_{k-d,C_3}(z) \preceq \sum_{d=3}^{k-2}\frac{3^d}{6\,d!}\vartheta_z^d\,\frac{b_{k-d}}{\mathbb{X}^{3k-3d}}\\
&\preceq \sum_{d=3}^{k-2}\frac{3^d}{6\,d!}\frac{(3k-3d)(3k-3d+2)\cdots(3k-3d+2(d-1))b_{k-d}}{\mathbb{X}^{3k-d}}\\
&\preceq \sum_{d=3}^{k-2}\frac{3^d}{6\,d!}\frac{3^d(k-d)(k-d+\frac{2}{3})(k-d+\frac{4}{3})\cdots(k-\frac{1}{3}d-\frac{2}{3})b_{k-d}}{\mathbb{X}^{3k-d}}\\
&\preceq \sum_{d=3}^{k-2}\frac{3^d}{6\,d!}\;3^d\frac{(k-d)(k-d+1)(k-d+2)\cdots(k-1)b_{k-d}}{\mathbb{X}^{3k-3}}
\end{aligned} \tag{114}$$

So using (112) and (113), we get after a bit of algebra

$$\Sigma_3' \preceq \sum_{d=3}^{k+1}\frac{6^{d-1}}{d!}\;\frac{(k-1)b_{k-1}}{\mathbb{X}^{3k-3}} \preceq 379\,\frac{(k-1)b_{k-1}}{\mathbb{X}^{3k-3}}\,. \tag{115}$$

We can apply the same techniques as above for graphs of Σ_3''. However, we need here the help of the induction hypothesis where we will choose $\varepsilon = \frac{1}{2}$ for sake of simplicity. In fact, a graph from Σ_3'' can be seen as the composition of two graphs: the first from $\widehat{\mathcal{S}}_{e_1,C_3}$ and the second from $\widehat{\mathcal{W}}_{e_2,C_3}$ (e.g. the graph in the dashed box of figure 17). Furthermore, suppose that the first graph is of order p, the second $n-p$ and that we have to rely d vertices of the second to the triangle (e.g. in the figure 17, $d=3$, $p=5$ and $n=8$). The number of manners to label such composition is less than or equal to

$$3^d\binom{n}{p}\binom{n-p}{d}p!\,[z^p]\,\widehat{S}_{e_1,C_3}\,(n-p)!\,\left[z^{n-p}\right]\widehat{W}_{e_2,C_3}$$

$$
\begin{aligned}
&\le 3^d \binom{n-p}{d} n!\,[z^n]\,\widehat{S}_{e_1,C_3}\;\times\;\widehat{W}_{e_2,C_3} \\
&\le \frac{3^d}{d!}\,n!\,[z^n]\,\vartheta_z^d\,(\widehat{S}_{e_1,C_3}\;\times\;\widehat{W}_{e_2,C_3})\,. \qquad (116)
\end{aligned}
$$

We have $d+e_1+e_2=k$ and using the induction hypothesis on $\widehat{S}_{e_1,C_3}$ with the fact that $\widehat{W}_{e_2,C_3} \preceq \widehat{W}_{e_2} \preceq \frac{b_{e_2}}{\mathbb{X}^{3e_2}}$, we obtain

$$
\begin{aligned}
\Sigma_3'' &\preceq \sum_{d+e_1+e_2=k} \frac{3^d}{d!}\vartheta_z^d(\widehat{S}_{e_1,C_3}\;\times\;\widehat{W}_{e_2,C_3}) \\
&\preceq \sum_{d+e_1+e_2=k} \frac{3^d}{d!}\vartheta_z^d \frac{2(e_1-1)b_{e_1-1}b_{e_2}}{\mathbb{X}^{3e_1+3e_2-1}} \\
&\preceq 2 \sum_{d+e_1+e_2=k} \frac{3^d}{d!}(3k-3d-1)\cdots(3k-d-3)\frac{(e_1-1)b_{e_1-1}b_{e_2}}{\mathbb{X}^{3k-d-1}} \\
&\preceq 2 \sum_{d+e_1+e_2=k} \frac{3^d}{d!}3^d(k-d-\frac{1}{3})\cdots(k-\frac{d}{3}-1)\frac{(e_1-1)b_{e_1-1}b_{e_2}}{\mathbb{X}^{3k-d-1}} \\
&\preceq 2 \sum_{d+e_1+e_2=k} \frac{3^d}{d!}3^d(k-d)(k-d+1)\cdots(k-1)\frac{b_{k-d}}{\mathbb{X}^{3k-d-1}}\,, \qquad (117)
\end{aligned}
$$

because we have

$$
(e_1-1)b_{e_1-1}b_{e_2} \le b_{k-d} \qquad (118)
$$

since

$$
\begin{aligned}
(e_1-1)b_{e_1-1}\,b_{e_2} &= \Big(\frac{3}{2}\Big)^{e_1+e_2-1}(e_1-1)!\,(e_2-1)!\,d_{e_1-1}\,d_{e_2} \\
&\le \Big(\frac{3}{2}\Big)^{k-d-1}(e_1-1)!\,e_2!\,d_{e_1-1}\,d_{e_2} \\
&\le \Big(\frac{3}{2}\Big)^{k-d-1}(e_1+e_2)!\,d_{e_1+e_2-1} \\
&\le \Big(\frac{3}{2}\Big)^{k-d}(e_1+e_2)!\,d_{k-d} \\
&= b_{k-d}\,. \qquad (119)
\end{aligned}
$$

(We used $(k+1)!d_{k+1} = (k+1)!d_k + \sum_{h=1}^{k-1} h!(k-h)!d_h d_{k-h}$ [41, eq. (1.4)].) Hence,

$$
\Sigma_3'' \preceq 2\sum_{d\ge 1}\frac{6^d}{d!}\,\frac{kb_k}{\mathbb{X}^{3k-2}} \preceq 805\frac{(k-1)b_{k-1}}{\mathbb{X}^{3k-2}}\,. \qquad (120)
$$

We have $[z^n]\,\frac{1}{\mathbb{X}^{3k-3}}\Big(\frac{\varepsilon}{\mathbb{X}^2} - \frac{805}{\mathbb{X}} - 379\Big) \geq 0$, $\forall n \geq 1$ since $\forall n > 0$, $[z^n]\,(\varepsilon - 805\mathbb{X} - 379\mathbb{X}^2) \geq 0$ and $[z^n]\,\widehat{S}_{k,C_3} = 0$ for $0 \leq n \leq 2$. (In fact, $\forall a,\, b,\, c > 0$, we have $0 \preceq a - b\mathbb{X} - c\mathbb{X}^2 = (a-b-c) + bT + 2c(T - \frac{T^2}{2})$.) Finally, we obtain $\widehat{S}_{k,C_3} \preceq \Big(\frac{3}{2} + \varepsilon\Big)\frac{(k-1)b_{k-1}}{\mathbb{X}^{3k-1}}$. □

By similar methods, one can prove

Lemma 19 *For $\varepsilon > 0$ and $k \geq 2$,*

$$\widehat{J}_{k+1,C_3} \preceq \Big(6 + \varepsilon\Big)\frac{(k-1)b_{k-1}}{\mathbb{X}^{3k-1}}\,. \tag{121}$$

Before proving lemma 19, we notice that working with juxtaposition of t triangles as subgraph is much easier.

Definition 20 *Denote by $\widehat{J}^{(t)}_{k,C_3}$ the EGF that counts k-excess graphs with a juxtaposition of exactly t triangles sharing a common edge.*

(For instance, the graph of figure 16 belongs to the family $\widehat{\mathcal{J}}^{(2)}_{2,C_3}$.)

Lemma 21 $\forall \varepsilon > 0$*, $k > t > 1$, $k \geq 3$, we have*

$$\widehat{J}^{(t)}_{k,C_3} \preceq (3+\varepsilon)\ \frac{(t+2)}{2!t!}\ \frac{(k-t)b_{k-t}}{\mathbb{X}^{3k-3t+2}}\,. \tag{122}$$

Proof (sketch). Smooth members of $\widehat{\mathcal{J}}^{(t)}_{t-1,C_3}$ are counted by

$$\underline{\widehat{\mathcal{J}}_{t-1,C_3}}^{(t)}(w,z) = \frac{w^{2t+1}z^{t+2}}{2!t!}\,. \tag{123}$$

Thus, the reader can remark that the bound in (122) is suggested by serial concatenation of graphs of $\widehat{\mathcal{J}}^{(t)}_{t-1,C_3}$ and of $\widehat{\mathcal{W}}_{k-t,C_3}$. At this stage, (122) can be proved as it was be done for the bound of $\widehat{S}_{k,C_3}$ in lemma (18). The main change is that the "unique occurrence of triangle" has been replaced by a "unique occurrence of juxtaposition of t triangles" with EGF $\frac{w^{2t+1}z^{t+2}}{2!t!}$. □

Proof of lemma 19. It suffices to sum over all possible values of t. We have

$$\begin{aligned}\widehat{J}_{k,C_3} &\preceq \frac{(3+\frac{\varepsilon}{2})}{2\mathbb{X}^{3k-4}}\sum_{t=2}^{k}\frac{(t+2)(k-t)b_{k-t}}{t!} \quad \left(\text{we use } \frac{1}{\mathbb{X}^{3k-3t+2}} \preceq \frac{1}{\mathbb{X}^{3k-4}}\right), \\ &\preceq \frac{(3+\frac{\varepsilon}{2})(k-2)b_{k-2}}{\mathbb{X}^{3k-4}}\sum_{t=2}^{k}\frac{t}{t!} \preceq \frac{(6+\varepsilon)(k-2)b_{k-2}}{\mathbb{X}^{3k-4}}\,. \end{aligned} \tag{124}$$

□

Lemmas 18 and 19 suggest themselves for generalization for any finite set ξ. Although, we do not intend to present such generalization here, we are convinced that this can be done practically in the same ways as we did for (107) and (121).

5.2.2 Bounds of $\widehat{W}_{k,\xi}$

In this paragraph, we present results that are strongly related to those of Wright. In fact, the Wright's seminal paper contains general techniques that are well suited for our triangle-free graphs. In paragraph §4.4, we obtained the general forms of the EGFs $\widehat{W}_{k,\xi}$ (see theorem 16). Recall that (b_k) and (c_k) are given respectively by (42) and (43). The lemmas 22 – 28 stated below will serve us to show by induction the inequalities (86). Before, let us specify some useful notations.

Notations.

For all $k \geq 1$, define by $\mathcal{L}_k$ and $\mathcal{R}_k$ the generating functions given by (recall that $\mathbb{X} = 1 - T$)

$$\mathcal{L}_k(z) = \widehat{W}_{k,C_3}(z) - \frac{b_k}{\mathbb{X}^{3k}} + \frac{c_k^{(C_3)}}{\mathbb{X}^{3k-1}} \tag{125}$$

and

$$\mathcal{R}_k(z) = \frac{b_k}{\mathbb{X}^{3k}} - W_{k,C_3}(z) \,. \tag{126}$$

Recall that we just have to prove that $\mathcal{L}_\mathbf{k} \succeq \mathbf{0}$ for all $k \geq 1$ since $\mathcal{R}_\mathbf{k} \succeq \mathbf{0}$ was proved by Wright [41].

First of all, the following lemma gives bounds of $c_k^{(\xi)}$ by means of b_k:

Lemma 22 *For all $k \geq 1$, we have $kb_k \leq c_k^{(\xi)} \leq \frac{19+6r}{5} kb_k$, where r is the number of polygons of ξ.*

Proof. We let $c_k^{(\xi)} = kb_k(1+\beta_k^{(\xi)})$. Hence, $\beta_1^{(\xi)} = \frac{14+6r}{5}$ (where r is the number of the forbidden polygons of distinct lengths). After a bit of algebra, we find

$$\begin{aligned} 2(3k+2)(k+1)b_{k+1}\,(1+\beta_{k+1}^{(\xi)}) &= 8(k+1)b_{k+1} + 3k(r+1)b_k \\ &+(3k-1)(3k+2)kb_k(1+\beta_k^{(\xi)}) \\ &+6\sum_{t=1}^{k-1} t(k-t)(3k-3t-1)b_t b_{k-t}(1+\beta_{k-t}^{(\xi)}) \,. \end{aligned} \tag{127}$$

Let $\mathcal{B}_k$ and $\mathcal{C}_k^{(\xi)}$ be the rational numbers defined with the help of (b_k) and $(c_k^{(\xi)})$ by

$$\mathcal{B}_k = \sum_{t=1}^{k-1} t(k-t) b_t b_{k-t}\,. \tag{128}$$

$$\mathcal{C}_k^{(\xi)} = \sum_{t=1}^{k-1} t(3k-3t-1) b_t c_{k-t}^{(\xi)}\,. \tag{129}$$

Using (73), we find

$$\begin{aligned} 6\sum_{t=1}^{k-1} t(k-t)(3k-3t-1)b_t b_{k-t} &= 3(3k-2)\mathcal{B}_k \\ &= 2(k+1)(3k-2)b_{k+1} - 3k(k+1)(3k-2)b_k\,. \end{aligned} \tag{130}$$

Thus,

$$\begin{aligned} 2(3k+2)(k+1)b_{k+1}\beta_{k+1}^{(\xi)} &= (3r+7)kb_k + k(3k+2)(3k-1)b_k\beta_k^{(\xi)} \\ &+6\sum_{t=1}^{k-1} t(k-t)(3k-3t-1)b_t b_{k-t}\beta_{k-t}^{(\xi)} \end{aligned} \tag{131}$$

and we have $\beta_k^{(\xi)} > 0$ for all ξ and $k > 0$. We let $\rho_k^{(\xi)} = \max_{1\le t\le k}\beta_t^{(\xi)} \geq \frac{14+6r}{5}$. Then, (131), (73) and (42) give

$$\begin{aligned} 2(3k+2)(k+1)b_{k+1}\beta_{k+1}^{(\xi)} \le\, &(3r+7)kb_k + \\ &\Big(k(3k+2)(3k-1)b_k \\ &+6\sum_{t=1}^{k-1} t(k-t)(3k-3t-1)b_t b_{k-t}\Big)\rho_k^{(\xi)} \\ \le\, &(3r+7)kb_k + \\ &\Big(k(3k+2)(3k-1)b_k + 3(3k-2)\mathcal{B}_k\Big)\rho_k^{(\xi)} \\ \le\, &(3r+7)kb_k \\ &+ (2(3k-2)(k+1)b_{k+1} + 4kb_k)\rho_k^{(\xi)}\,. \end{aligned} \tag{132}$$

Now, if we suppose that $\beta_{k+1}^{(\xi)} > \rho_k^{(\xi)}$, we will have

$$\begin{aligned} 8(k+1)b_{k+1}\beta_{k+1}^{(\xi)} &\le 4kb_k\beta_{k+1}^{(\xi)} + (3r+7)kb_k \\ 12k(k+1)b_k\beta_{k+1}^{(\xi)} + 12\mathcal{B}_k\beta_{k+1}^{(\xi)} &\le 4kb_k\beta_{k+1}^{(\xi)} + (3r+7)kb_k \end{aligned} \tag{133}$$

so that

$$12(k+1)b_k\beta_{k+1}^{(\xi)} \leq 4b_k\beta_{k+1}^{(\xi)} + (3r+7)b_k \tag{134}$$

and

$$\beta_{k+1}^{(\xi)} \leq \frac{3r+7}{4(3k+2)} \tag{135}$$

which is in contradiction with the fact that $\beta_{k+1}^{(\xi)} > \rho_k^{(\xi)} \geq \frac{14+6r}{5}$ (this will lead us to $3r+7 > 18kr + 42k$). So, $(\beta_k^{(\xi)})$ is a nonincreasing sequence and $\rho_k^{(\xi)} = \frac{14+6r}{5}$ for all $k > 0$. □

Next, we have the lemmas 23, 24, 25, 26, 27 stated below, corresponding to the lemmas 6, 7, 8, 9 and 10 of [41] but adapted for our ξ-free graphs. Lemmas 3 and 4 of [41] are contained in lemma 28.

Lemma 23 *If* $\left(\widehat{W}_{t,\xi} - \frac{b_t}{\mathbb{X}^{3t}} + \frac{c_t^{(\xi)}}{\mathbb{X}^{3t-1}}\right) \succeq 0$, *for* t *such that* $1 \leq t \leq k-1$ *then*

$$\Lambda_k^{(\xi)} \succeq \frac{T^2}{\mathbb{X}^{3k+4}} \left(9\mathcal{B}_k - 6\mathcal{C}_k^{(\xi)}\,\mathbb{X}\right) \tag{136}$$

where $\mathcal{C}_k^{(\xi)}$ *is given by (129) and* $\Lambda_k^{(\xi)}$ *is given by (82).*

Proof. If $x_1, \cdots, x_6$ are positive real numbers and $x_1 \geq x_2 - x_3$, $x_4 \geq x_5 - x_6$ then

$$x_1 x_4 \geq x_2 x_5 - x_2 x_6 - x_5 x_3\,. \tag{137}$$

In fact, if $x_2 < x_3$ and/or $x_5 < x_6$, the right side of the above inequality is negative. Otherwise, if $x_2 \geq x_3$ and $x_5 \geq x_6$, we have:

$$x_1 x_4 \geq (x_2 - x_3)(x_5 - x_6) \geq x_2 x_5 - x_2 x_6 - x_5 x_3\,.$$

Assume now that $1 \leq t \leq k-1$. We have $\widehat{W}_{t,\xi} \succeq 0$, $b_t/\mathbb{X}^{3t} \succeq 0$, $(c_t + 3/2r(t-1)b_{t-1})/\mathbb{X}^{3t-1} \succeq 0$ and $\mathcal{L}_t \succeq 0$ for $1 \leq t \leq k-1$. Consequently, the coefficients of $\vartheta_z \widehat{W}_{t,\xi}(z)$ are positive for the same value of t. Setting

$$x_1 = s!\,[z^s]\,\vartheta_z \widehat{W}_{t,\xi}(z)\,,$$
$$x_2 = b_t s!\,[z^s]\,\vartheta_z \frac{1}{\mathbb{X}^{3t}}\,,$$

$$
\begin{aligned}
x_3 &= c_t^{(\xi)} s! \, [z^s] \, \vartheta_z \frac{1}{\mathbb{X}^{3t-1}} \, , \\
x_4 &= (n-s)! \left[z^{n-s}\right] \vartheta_z \widehat{W}_{k-t,\xi}(z) \, , \\
x_5 &= b_{k-t} (n-s)! \left[z^{n-s}\right] \vartheta_z \frac{1}{\mathbb{X}^{3k-3t}} \text{ and} \\
x_6 &= c_{k-t}^{(\xi)} (n-s)! \left[z^{n-s}\right] \vartheta_z \frac{1}{\mathbb{X}^{3k-3t-1}}
\end{aligned}
\tag{138}
$$

where $s \in [0, n]$, after substituting the values of x_i, $i \in [1, 6]$ in (137) and summing over s and t, $t \in [1, k-1]$, we obtain (136). □

Similarly, we have

Lemma 24 *If* $\left(\frac{b_t}{\mathbb{X}^{3t}} - \widehat{W}_{t,\xi}\right) \succeq 0$ *for* $1 \le t \le k-1$ *then*

$$
\Lambda_k^{(\xi)} \preceq 9 \mathcal{B}_k \frac{T^2}{\mathbb{X}^{3k+4}} \, . \tag{139}
$$

In the following lemmas, we work again with the special case $\xi = \{C_3\}$ for sake of clarity.

Lemma 25 *Define by* $Y_k^{(C_3)}$ *and* $Z_k^{(C_3)}$ *the formal power series*

$$
Y_k^{(C_3)}(z) = \Delta_{k+1} \frac{b_{k+1}}{\mathbb{X}^{3k+3}} - \Omega_k^{(C_3)} \frac{b_k}{\mathbb{X}^{3k}} - 9 \mathcal{B}_k \frac{T^2}{\mathbb{X}^{3k+4}} \tag{140}
$$

$$
Z_k^{(C_3)}(z) = \Delta_{k+1} \frac{c_{k+1}^{(C_3)}}{\mathbb{X}^{3k+2}} - \Omega_k^{(C_3)} \frac{c_k^{(C_3)}}{\mathbb{X}^{3k-1}} - 6 \mathcal{C}_k^{(C_3)} \frac{T^2}{\mathbb{X}^{3k+3}} \, . \tag{141}
$$

For all $k \ge 1$, *we have* $Z_k^{(C_3)} \succeq Y_k^{(C_3)} + 6 \widehat{S}_{k+1,C_3} + 2 \widehat{J}_{k+1,C_3} \succeq 0$.

Proof. First, we remark that

$$
\begin{aligned}
\Omega_k^{(C_3)}(\mathbb{X}^{-t}) &= t(t+3)\mathbb{X}^{-t-4} - t(2t+8)\mathbb{X}^{-t-3} \\
&\quad + t(t+8)\mathbb{X}^{-t-2} - 7t\mathbb{X}^{-t-1} + (5t-2k)\mathbb{X}^{-t} - t\mathbb{X}^{-t+1} \, .
\end{aligned}
\tag{142}
$$

Thus, using this (68) and (128), we have

$$
\begin{aligned}
Y_k^{(C_3)}(z) &= \Big(6kb_k + 8(k+1)b_{k+1}\Big)\mathbb{X}^{-3k-3} \\
&\quad - \Big(15kb_k + 6(k+1)b_{k+1}\Big)\mathbb{X}^{-3k-2} \\
&\quad + 21kb_k\mathbb{X}^{-3k-1} - 13kb_k\mathbb{X}^{-3k} + 3kb_k\mathbb{X}^{-3k+1} \, .
\end{aligned}
\tag{143}
$$

Similarly, we find

$$\begin{aligned}
Z_k^{(C_3)}(z) &= \Big(6kb_k + 8(k+1)b_{k+1}\Big)\mathbb{X}^{-3k-3} \\
&+\Big(2(4k+3)c_{k+1}^{(C_3)} + 2(3k-1)c_k^{(C_3)} - 16(k+1)b_{k+1} - 12kb_k\Big)\mathbb{X}^{-3k-2} \\
&+\Big(8(k+1)b_{k+1} + 6kb_k - 2(3k+2)c_{k+1}^{(C_3)} - 5(3k-1)c_k^{(C_3)}\Big)\mathbb{X}^{-3k-1} \\
&+7(3k-1)c_k^{(C_3)} - (13k-5)c_k^{(C_3)}\mathbb{X} + (3k-1)c_k^{(C_3)}\mathbb{X}^2 . \qquad (144)
\end{aligned}$$

Rearranging (143), we obtain

$$\begin{aligned}
Y_k^{(C_3)}(z) &= 3kb_k\mathbb{X}^{-3k-2}(2\mathbb{X}^{-1} - 5) \\
&+2(k+1)b_{k+1}\mathbb{X}^{-3k-3}(4\mathbb{X}^{-1} - 3) \\
&+kb_k\mathbb{X}^{-3k}(21\mathbb{X}^{-1} - 13) + 3kb_k\mathbb{X}^{-3k+1} \qquad (145)
\end{aligned}$$

and so $Y_k^{(C_3)} \succeq 0$. By (107) and (121), we have $\widehat{S}_{k+1,C_3} + \widehat{J}_{k+1,C_3} \preceq \frac{2kb_k}{\mathbb{X}^{3k+2}}$ Hence,

$$Z_k^{(C_3)} - Y_k^{(C_3)} - 6\widehat{S}_{k+1,C_3} - 2\widehat{J}_{k+1,C_3} \succeq Z_k^{(C_3)} - Y_k^{(C_3)} - 12kb_k\mathbb{X}^{-3k-2} \qquad (146)$$

and

$$\begin{aligned}
&Z_k^{(C_3)} - Y_k^{(C_3)} - 6\widehat{S}_{k+1,C_3} - 2\widehat{J}_{k+1,C_3} \quad \succeq \\
&\Big(2(4k+3)c_{k+1}^{(C_3)} + 2(3k-1)c_k^{(C_3)} - 9kb_k - 10(k+1)b_{k+1}\Big)\mathbb{X}^{-3k-2} \\
&+\Big(8(k+1)b_{k+1} - 15kb_k - 2(3k+2)c_{k+1}^{(C_3)} - 5(3k-1)c_k^{(C_3)}\Big)\mathbb{X}^{-3k-1} \\
&+\Big(7(3k-1)c_k^{(C_3)} + 13kb_k\Big)\mathbb{X}^{-3k} - \Big((13k-5)c_k^{(C_3)} + 3kb_k\Big)\mathbb{X}^{-3k+1} \\
&+(3k-1)c_k^{(C_3)}\mathbb{X}^{-3k+2} . \qquad (147)
\end{aligned}$$

Rewriting, we have

$$\begin{aligned}
&Z_k^{(C_3)} - Y_k^{(C_3)} - 6\widehat{S}_{k+1,C_3} - 2\widehat{J}_{k+1,C_3} \quad \succeq \\
&\Big(2(4k+3)c_{k+1}^{(C_3)} + 2(3k-1)c_k^{(C_3)} - 9kb_k - 10(k+1)b_{k+1}\Big)(\mathbb{X}^{-1} - 2)^2 \\
&+\Big(2(13k+10)c_{k+1}^{(C_3)} + 3(3k-1)c_k^{(C_3)} - 51kb_k - 32(k+1)b_{k+1}\Big)(\mathbb{X}^{-1} - 2) \\
&+\Big((44k+28)c_{k+1}^{(C_3)} + 9(3k-1)c_k^{(C_3)} - 69kb_k - 40(k+1)b_{k+1}\Big) \\
&+\Big((13k-5)c_k^{(C_3)} + 3kb_k\Big)(T-1) \\
&+(3k-1)c_k^{(C_3)}\mathbb{X}^{-2} \qquad (148)
\end{aligned}$$

and by lemma 7, (46) and (42) after some calculations we find $Z_k^{(C_3)} - Y_k^{(C_3)} - 6\widehat{S}_{k+1,C_3} - 2\widehat{J}_{k+1,C_3} \succeq 0$ □

Lemma 26 *For all $t \in [1,\, k-1]$, if*

$$\left(\widehat{W}_{t,C_3} - \frac{b_t}{\mathbb{X}^{3t}} + \frac{c_t^{(C_3)}}{\mathbb{X}^{3t-1}}\right) \succeq 0 \tag{149}$$

then

$$\Delta_{k+1}\left[\widehat{W}_{k+1,C_3} - \frac{b_{k+1}}{\mathbb{X}^{3k+3}} + \frac{c_{k+1}^{(C_3)}}{\mathbb{X}^{3k+2}}\right] \succeq \Omega_k^{(C_3)}\left[\widehat{W}_{k,C_3} - \frac{b_k}{\mathbb{X}^{3k}} + \frac{c_k^{(C_3)}}{\mathbb{X}^{3k-1}}\right]. \tag{150}$$

Proof. Using lemmas 23, 25 and (83), we infer that

$$\begin{aligned} \Delta_{k+1}\mathfrak{L}_{k+1} - \Omega_k^{(C_3)}\mathfrak{L}_k &= -6\widehat{S}_{k+1,C_3} - 2\widehat{J}_{k+1,C_3} + \Lambda_k^{(C_3)} - Y_k + Z_k \\ -\mathbb{X}^{-3k-2}T^2/\mathbb{X}^2(9\mathcal{B}_k - 6\mathcal{C}_k^{(C_3)}/\mathbb{X}) &\succeq (Z_k - Y_k) \succeq 0\,. \end{aligned} \tag{151}$$

□

Lemma 27 *Let $n_0 = n_0(k) = \frac{3}{2} + \sqrt{(2k + \frac{9}{4})}$. If $k \geq 2$ and $0 \leq n \leq n_0(k)$ then the coefficients of $\mathfrak{L}_k$.*

Proof. If $n < n_0(k)$ then $\binom{n}{2} < n + k$ and *a fortiori* there is no $(k+1)$-cyclic connected graphs. Let $k \geq 2$ and $n < n_0(k)$. Since $n!\,[z^n]\,\widehat{W}_{k,\xi}(z) = 0$, we have to prove only that

$$n!\,[z^n]\left(\frac{c_k^{(C_3)}}{\mathbb{X}^{3k-1}} - \frac{b_k}{\mathbb{X}^{3k}}\right) \geq 0\,, \tag{152}$$

because $n!\,[z^n]\,\frac{b_k}{\mathbb{X}^{3k}} \geq 0$. As $c_k^{(C_3)} \geq c_k$, it suffices to show that

$$n!\,[z^n]\left(\frac{c_k}{\mathbb{X}^{3k-1}} - \frac{b_k}{\mathbb{X}^{3k}}\right) \geq 0\,.$$

Let

$$M(z) = 1 + \sum_{n\geq 1} n^n \frac{z^n}{n!}\,, \tag{153}$$

i.e., $M = \frac{1}{\mathbb{X}} = \frac{1}{1-T}$. Note that if $n < j$ and $t < n_0$ then $t < 3k-1$ and lemma 23 tells us that $(3k-t)c_k \geq 3kb_k$ and $\binom{3k-1}{t}c_k \geq \binom{3k}{t}b_k$. Thus,

$$
\begin{aligned}
&n!\,[z^n]\left[\frac{c_k}{\mathbb{X}^{3k-1}} - \frac{b_k}{\mathbb{X}^{3k}}\right] = \\
&\quad n!\,[z^n]\left[c_k(1+M(z))^{3k-1} - b_k(1+M(z))^{3k}\right] = \\
&\quad \sum_{t=0}^{n_0} n!\,[z^n]\left[\binom{3k-1}{t}c_k - \binom{3k}{t}b_k\right] M(z)^t\,.
\end{aligned} \tag{154}
$$

□

We are now ready to prove (86).

Lemma 28 *For all $k \geq 1$, the formal power series $\mathfrak{L}_k$ satisfies*

$$
\mathfrak{L}_k(z) = \widehat{W}_{k,C_3}(z) - \frac{b_k}{\mathbb{X}^{3k}} + \frac{c_k^{(C_3)}}{\mathbb{X}^{3k-1}} \succeq 0\,.
$$

Proof. First, $\mathfrak{L}_1 \succeq 0$ by (26). Suppose that $\mathfrak{L}_i \succeq 0$ for all $i \in [1, k-1]$ and we have to show that $\mathfrak{L}_k \succeq 0$. Hence, we can use lemma 26. By definition,

$$
\Omega_{k-1}^{(C_3)}\mathfrak{L}_{k-1}(z) = (\vartheta_z^2 - 3\vartheta_z - 2(k-1))\mathfrak{L}_{k-1}(z) + 2(\vartheta_z W_{0,C_3}(z))(\vartheta_z\,\mathfrak{L}_{k-1}(z))\,.
$$

If $n > n_0(k)$ then $n^2 - 3n - 2k > 0$ and we have

$$
\begin{aligned}
[z^n]\,\Omega_{k-1}^{(C_3)}\mathfrak{L}_{k-1}(z) = &\quad (n^2 - 3n - 2k + 2)\,[z^n]\,\mathfrak{L}_{k-1}(z) \\
&+ 2\,[z^n]\,(\vartheta_z W_{0,C_3}(z))(\vartheta_z\,\mathfrak{L}_{k-1}(z)) \geq 0
\end{aligned}
$$

Lemma 26 tells us that $\Delta_k\mathfrak{L}_k \geq 0$. Taking into account the definition of Δ given by (65), we obtain for $n \geq n_0(k-1)$:

$$
2(n+k)\,[z^n]\,\mathfrak{L}_k \geq 2\,[z^n]\,T\vartheta_z\mathfrak{L}_k = 2\sum_{s=1}^{n-1}\binom{n}{s}s(n-s)^{n-s-1}\,[z^s]\,\mathfrak{L}_k(z)\,. \tag{155}
$$

And lemma 27 leads to $[z^n]\,\mathfrak{L}_k(z) \geq 0$, if $n < n_0(k)$. Since $n_0(k-1) < n_0(k)$ we can infer by induction on n using (155) that $\mathfrak{L}_k \succeq 0$. □

5.3 Asymptotic results

Denote by $c(n, n+k)$ the number of connected graphs having n vertices and $n+k$ edges. Our aim of this paragraph is to establish that the number $c_\xi(n, n+k)$ of ξ-free connected graphs with n vertices and $n+k$ edges is asymptotically the same as $c(n, n+k)$ whenever $k = o(n^{1/3})$. Combining lemmas 17, 23 and 28, we obtain the following important results:

Theorem 29 *Almost all graphs having n vertices and n+k edges are triangle-free when n, $k \to \infty$ but $k = o(n^{1/3})$.*

Proof. On one hand, lemma 17 shows that if $a \equiv a(n) \to 0$ as $n \to \infty$, and if b_1 and b_2 are two fixed numbers such that $b_1 < b_2$, then we have $t_n(an + \beta_1) \ll t_n(an + \beta_2)$ since in (88) we obtain a factor $(1 - u_0)^{(1-\beta)} = (\sqrt{a(1+\frac{a}{4})} - \frac{a}{2})^{(1-\beta)} = a^{\frac{1-\beta}{2}} + O(a)$. On the other hand, we have

$$kb_k \le c_k^{(C_3)} \le \frac{25}{5} kb_k$$

and

$$\frac{b_k}{\mathbb{X}^{3k}} - \frac{c_k^{(C_3)}}{\mathbb{X}^{3k-1}} \preceq \widehat{W}_{k,C_3} \preceq \frac{b_k}{\mathbb{X}^{3k}} \; (k \ge 1)\,.$$

Since $c_k^{(C_3)} = c_k + O((k-1)b_{k-1}) = O(kb_k)$, we have to find the values of k for which

$$kb_k t_n(3k-1) \ll b_k t_n(3k)\,.$$

We will use formula (88) of lemma 17 to estimate $t_n(a\,n + \beta_1)$ and $t_n(a\,n + \beta_2)$, with $an = 3k$, $\beta_1 = -1$, resp. $\beta_2 = 0$. It proves convenient to compute $\frac{kt_n(a\,n+\beta_1)}{t_n(a\,n+\beta_2)}$ and we have

$$\begin{aligned} \frac{kt_n(a\,n + \beta_1)}{t_n(a\,n + \beta_2)} &= \frac{kt_n(3k-1)}{t_n(3k)} \\ &= k(1 - u_0) = k(\sqrt{a} + O(a)) \\ &= \frac{n}{3}(a^{\frac{3}{2}} + O(a^2))\,. \end{aligned} \tag{156}$$

Consequently, if $k = o(n^{1/3})$ the number $c_\xi(n, n+k)$ is asymptotically the same as $c(n, n+k)$. □

Also, we have

Theorem 30 (Wright 1980) *As n, $k \to \infty$ but $k = o(n^{1/3})$, we have*

$$\begin{aligned} c(n, n+k) = d_k\,(3\pi)^{1/2}(e/12k)^{k/2} n^{n+1/2(3k-1)} \\ \times\Big(1 + O(k^{-1}) + O(k^{3/2}/n^{1/2})\Big) \end{aligned} \tag{157}$$

where $d_k = \frac{1}{2\pi} + O(1/k)$.

Note that the value $d = \frac{1}{2\pi} = \lim_{k\to\infty} d_k$ was independently found by Voblyi [37] and by Meertens [4].

As a corollary of theorems 29 and 30, we obtain

Corollary 31 *If n, $k \to \infty$ but $k = o(n^{1/3})$ the asymptotic number of $(n, n+k)$ triangle-free connected graphs is given by*

$$d_k\,(3\pi)^{1/2}(e/12k)^{k/2}n^{n+1/2(3k-1)}\Big(1 + O(k^{-1}) + O(k^{3/2}/n^{1/2})\Big)\,. \qquad (158)$$

6 Random graphs and forbidden subgraphs

As shown in [14, 21], the machinery of generating functions permits to study the limit distribution of random graphs and multigraphs with great precision. In this section, we will show that probabilistic results on random ξ-free graphs and multigraphs can be obtained when looking at the form of their generating functions, mainly looking at the so-called *leading coefficients* of their decompositions into tree polynomials, i.e., using the results of the previous sections and some analytical facts contained in [21].

We consider here two models of random graphs, namely the *permutation model* and the *multigraph process*. The idea is to start with n totally disconnected vertices and to add successive edges one at time and at random [12, 13]. In the first model, also called *graph process*, we consider all $N = \binom{n}{2}$ possible edges x — y with $x < y$ which are introduced in random order, allowing all $N!$ permutations with the same probability.

In the second model, also called *uniform model*, ordered pairs $\langle x, y\rangle$ are generated repeatedly $(1 \le x, y \le n)$ and the edge x — y is added to the multigraph. Thus, this process can generate self-loops and multiple edges. Remark that we follow Janson *et al.* and for purposes of analysis, we assign a *compensation factor* to a multigraph M, viz. a multigraph M on n labelled vertices can be defined by a symmetric $n \times n$ matrix of nonnegative integers m_{xy}, where $m_{xy} = m_{yx}$ is the number of undirected edges x — y in G. The *compensation factor* associated to M is given by

$$\kappa(M) = 1 \Bigg/ \prod_{x=1}^{n}\left(2^{m_{xx}}\prod_{y=x}^{n} m_{xy}!\right) \qquad (159)$$

Thus, if $m = \sum_{x=1}^{n}\sum_{y=x}^{n} m_{xy}$ is the total number of edges, the number of sequences $\langle x_1, y_1\rangle\langle x_2, y_2\rangle \ldots \langle x_m, y_m\rangle$ that lead to M is then exactly

$$2^m\, m!\,\kappa(M)\,. \qquad (160)$$

(We refer to [21, Sect. 1] for more details about κ.)

At generating function level, it follows that after adding m edges, the uniform model on n vertices will produce a multigraph in a family $\mathcal{F}$ with probability

$$\frac{2^m\, m!\, n!}{n^{2m}}\; [w^m z^n]\; F(w,z)\,. \tag{161}$$

Similarly, if $\mathcal{F}$ is a family of graphs with labelled vertices, the probability that m steps of the permutation model will produce a graph in $\mathcal{F}$ is

$$\frac{n!}{\binom{N}{m}}\,[w^m z^n]\, F(w,z)\,, \qquad N = \binom{n}{2}\,. \tag{162}$$

In [21, Theorem 5], the authors proved that only leading coefficients of $t_n(3k)$ are relevant to compute the probability that randomly generated graphs or multigraphs will produce r_1 bicyclic components, r_2 tricyclic components, $\cdots$ We have the following results about ξ-free components and random graphs:

Theorem 32 *The probability that a random graph or multigraph with n vertices and $\frac{n}{2}$ edges has only acyclic, unicyclic, bicyclic components all triangle-free is*

$$\sqrt{\frac{2}{3}}\cosh\left(\sqrt{\frac{5}{18}}\right)\, e^{-\frac{1}{6}} + O(n^{-1/3}) \approx 0.789\ldots\,. \tag{163}$$

More generally, let $\Theta = \{p \in \mathbb{N}, p \geq 3$ and $C_p \in \xi\}$. The probability that a random graph or multigraph with n vertices and $\frac{n}{2}$ edges has only acyclic, unicyclic, bicyclic components all C_p-free, $p \in \Theta$, is

$$\sqrt{\frac{2}{3}}\cosh\left(\sqrt{\frac{5}{18}}\right)\, e^{-\sum_{p\in\Theta}\frac{1}{2p}} + O(n^{-1/3})\,. \tag{164}$$

Proof. This is a corollary of [21, eq (11.7)] using the formulae (17), (18) and (27). Incidentally, random graphs and multigraphs have the same asymptotic behavior as shown by the proof of [21, Theorem 4]. As multigraphs graphs without cycles of length 1 and 2, the forbidden cycles of length 1 and 2 bring a factor $e^{-3/4}$ which is cancelled by a factor $e^{+3/4}$ because of the ratio between weighting functions that convert the EGF of graphs and multigraphs into probabilities. Indeed, formulae (161) and (162) are asymptotically related by the formula

$$\binom{\binom{n}{2}}{m} = \left(\frac{n^{2m}}{2^m m!}\right)\exp\big(-\frac{m}{n} - \frac{m^2}{n^2} + O(\frac{m}{n^2}) + O(\frac{m^3}{n^4})\big),\; m \leq \binom{n}{2}\,. \tag{165}$$

The situation changes radically when cycles of length greater to or less than 3 are forbidden. Equations (17), (18) and the "significant coefficient" $\frac{5}{24}$ of $t_n(3)$ in (27) and the demonstration of [21, Lemma 3] show us that the term $-\frac{T(z)^p}{2p}$, introduced in (17) and (18) for each forbidden p-gon, simply changes the result by a factor of $e^{-1/2p} + O(n^{-1/3})$. $\square$

The example of forbidden p-gon suggests itself for a generalization.

Theorem 33 *Let $\xi = \{H_1, H_2, H_3, ...H_q\}$ be a finite collection of multicyclic connected graphs or multigraphs. Then the probability that a random graph with n vertices and $\frac{1}{2}n + O(n^{\frac{1}{3}})$ edges has r_1 bicyclic components, r_2 tricyclic components,$\cdots$, $(k+1)$-cyclic components, all components $\{H_1, H_2, H_3, ...H_q\}$-free and no components of higher cyclic order is*

$$(\frac{4}{3})^r \exp\big(-\sum_{p\in\Theta} \frac{1}{2p}\big) \sqrt{\frac{2}{3}}\, \frac{b_1^{r_1}}{r_1!}\, \frac{b_2^{r_2}}{r_2!} \;\cdots\; \frac{b_k^{r_k}}{r_k!} \frac{r!}{(2r)!} + O(n^{-1/3}) \tag{166}$$

where $\Theta = \{p \geq 3\ ,\ \exists i \in [1, q]$ such that H_i is a p-gon$\}$.

Theorem 33 raised a natural question. Under what conditions on the forbidden configurations of graphs will the coefficients (b_i) change? The theorem 34 below shows that a sufficient condition to change a coefficient b_i of (166) is that ξ must contain all graphs *contractible* to a certain i-excess graph H.

Theorem 34 *Let H be a k-excess multicyclic graph (resp. multigraph) with $k > 0$. Suppose that $c(H)\, n!$ is the number of ways to label H (for example $c(K_4) = 1/24$). Denote by $\mathcal{A}_k^{(H)}$ the set of all k-excess graphs contractible to H. Then the probability that a random graph (resp. multigraph) with n vertices and $m(n) = \frac{n}{2} + O(n^{1/3})$ edges has r_1 bicyclic, r_2 tricyclic, ..., r_p $(p+1)$-cyclic components, all without component isomorphic to any member of the set $\mathcal{A}_k^{(H)}$ and with $r = r_1 + 2r_2 + \cdots + pr_p$ is*

$$(\frac{4}{3})^r \sqrt{\frac{2}{3}}\, \frac{b_1^{r_1}}{r_1!} \;\cdots\; \frac{b_{k-1}^{r_{k-1}}}{r_{k-1}!}\, \frac{(b_k - c(H))^{r_k}}{r_k!}\, \frac{b_{k+1}^{r_{k+1}}}{r_{k+1}!} \;\cdots\; \frac{b_p^{r_p}}{r_p!} \frac{r!}{(2r)!} + O(n^{-1/3})\,. \tag{167}$$

Proof. The EGF associated to $\mathcal{A}_k^{(H)}$ is simply

$$A_k^{(H)}(w, z) = w^k\, c(H)\, \frac{T(wz)^n}{(1 - T(wz))^{3k}}\,. \tag{168}$$

Thus in (166) if we want to avoid all graphs *contractible* to H, we have to subtract (168) from the EGF of connected k-excess graphs. $\square$

Note that in [21, lemma 3], theorems 32, 33 and 34, the number of edges $m = m(n)$ varies from $\frac{n}{2}$ to $\frac{n}{2} + O(n^{1/3})$. The discrepancy in the windows is a consequence of the parameter μ in [21, lemma 3], where $m(n) = \frac{1}{2}n(1+\mu n^{-1/3})$ and $|\mu| \leq n^{1/12}$. Hence, when choosing very small μ, such as $\mu = O(n^{-1/3})$, one can get results like theorems 4-5 in [21] or theorems 32, 33 and 34 here.

Acknowledgements

The authors wish to thank C. Banderier, P. Flajolet and G. Schaeffer for helpful discussions and encouragements relating to this research and also all the anonymous referees for their efforts reading and improving the quality of this paper.

References

[1] G. N. Bagaev, Random graphs with degree of connectedness equal 2, *Discrete Analysis* **22** (1973) 3–14, (in Russian).

[2] G. N. Bagaev and V. A. Voblyi, The shrinking-and-expanding method for the graph enumeration, *Discrete Mathematics and Applications* **8** (1998) 493–498.

[3] E. A. Bender, Asymptotic Methods in Enumeration, *SIAM Review* **16** (1974) 485–515.

[4] E. A. Bender, E. R. Canfield and B. D. McKay, The asymptotic number of labeled connected graphs with a given number of vertices and edges, *Random Structures and Algorithms* **1** (1990) 127–169.

[5] E. A. Bender, E. R. Canfield and B. D. McKay, Asymptotic Properties of Labeled Connected Graphs, *Random Structures and Algorithms* **3** (1992) 183–202.

[6] E. A. Bender, E. R. Canfield and B. D. McKay, The asymptotic number of labeled graphs with n vertices, q edges, and no isolated vertices, *Journal of Combinatorial Theory, Series A* **80** (1997) 124–150.

[7] B. Bollobas, *Random graphs* (Academic Press, 1985).

[8] N. G. de Bruijn, *Asymptotic Methods in Analysis* (Dover Publications, New York 1981).

[9] A. Cayley, *A Theorem on Trees*, *Quart. J. Math. Oxford Ser.* **23** (1889) 376–378.

[10] F. R. K. Chung, Open problems of Paul Erdös in Graph Theory, *Journal of Graph Theory* **25** (1997) 3–36.

[11] L. Comtet, *Analyse Combinatoire* (Presses Universitaires de France, Paris 1970).

[12] P. Erdös and A. Rényi, On random graphs, *Publ. Math. Debrecen* **6** (1959) 290–297.

[13] P. Erdös and A. Rényi, On the evolution of random graphs, *Magyar Tud. Akad. Mat. Kut. Int. Kzl.* **5** (1960) 17–61.

[14] P. Flajolet, D. E. Knuth and B. Pittel, The First Cycles in an Evolving Graph, *Discrete Mathematics* **75** (1989) 167–215.

[15] P. Flajolet, P. Poblete and A. Viola, On the Analysis of Linear Probing Hashing, *Algorithmica* **22** (1998) 490–515.

[16] P. Flajolet and A. Odlyzko, Singularity analysis of generating functions, *SIAM J. Discrete Math.* **3** (1990) 216–240.

[17] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Book to appear. Chapters are available electronically at http://algo.inria.fr/flajolet/Publications/books.html.

[18] P. Flajolet, P. Zimmerman and B. Van Cutsem, A calculus for the random generation of labelled combinatorial structures, *Theoretical Computer Sciences* **132** (1994) 1–35.

[19] I. P. Goulden and D. M. Jackson, *Combinatorial Enumeration* (Wiley, New York, 1983).

[20] F. Harary and E. Palmer, *Graphical Enumeration* (Academic Press, 1973).

[21] S. Janson, D. E. Knuth, T. Łuczak and B. Pittel, The Birth of the Giant Component, *Random Structures and Algorithms* **4** (1993) 233–358.

[22] S. Janson, T. Łuczak and A. Rucinski, *Random Graphs* (Wiley-Interscience Series in Discrete Mathematics and Optimization, 2000).

[23] D. E. Knuth, *The Art Of Computing Programming, v.1, Fundamental Algorithms* (2nd Edition, Addition-Wesley, Reading 1973).

[24] D. E. Knuth and B. Pittel, A recurrence related to trees, *Proc. Am. Math. Soc.* **105** (1989) 335–349.

[25] G. Louchard, The Brownian excursion: a numerical analysis, *Computers and Mathematics with Applications* **10** (1984) 413–417.

[26] G. Louchard, Kac's formula, Lévy's local time and Brownian excursion, *Journal of Applied Probability* **21** (1984) 479–499.

[27] J. W. Moon, Various proofs of Cayley's formula for counting trees, in: F. Harary, ed., *A seminar on graph theory*, Holt, Rinehart and Winston, New York, 1967, 70–78.

[28] V. Ravelomanana and L. Thimonier, Some Remarks on Sparsely Connected Isomorphism-Free Labeled Graphs, in: G. H. Gonnet, D. Panario and A. Viola, eds., *4th Latin American Theoretical INformatics – Punta del Este – Uruguay, Lecture Notes in Computer Science* **1776** (2000) 28–37.

[29] V. Ravelomanana and L. Thimonier, Enumeration of the First Multicyclic Isomorphism-Free Labelled Graphs (Extended Abstract), in: H. Barcelo and V. Welker, eds., *Proc. 13th International Conf. Formal Power Series and Algebraic Combinatorics – ASU Tempe – USA* (2001) 411–420.

[30] A. Rényi, On connected graphs I, *Publ. Math. Inst. Hungarian Acad. Sci.*

4 (1959) 385–388.

[31] J. Riordan, The numbers of labeled colored and chromatic trees, *Acta Mathematica* **97** (1958) 211–225.

[32] R. Sedgewick and P. Flajolet, *An introduction to the Analysis of Algorithms* (Addison-Wesley, 1996).

[33] S. M. Selkow, The Enumeration of Labeled Graphs by Number of Cutpoints, *Discrete Mathematics* **185** (1998) 183–191.

[34] J. Spencer, Enumerating Graphs and Brownian Motion, *Communications on Pure and Applied Mathematics* **50** (1997) 293–296.

[35] L. Takács, Conditional limit theorems for branching processes, *Journal of Applied Mathematics and Stochastic Analysis* **4** (1991) 263–292.

[36] L. Takács, On a probability problem connected with railway traffic, *Journal of Applied Mathematics and Stochastic Analysis* **4** (1991) 1–27.

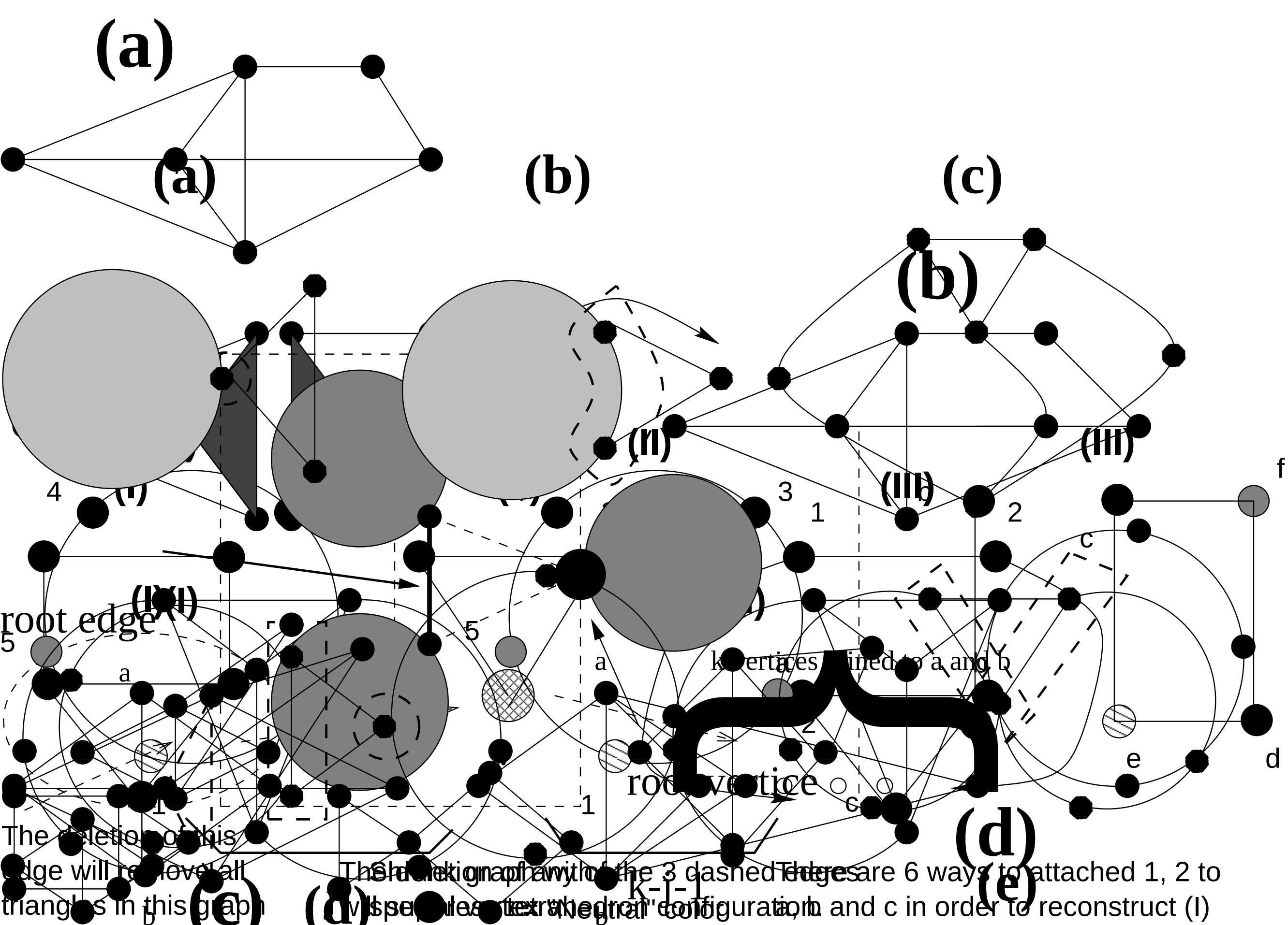
[37] V. A. Voblyi, Wright and Stepanov-Wright coefficients (Russian), *Mat. Zametki* **42** (1987) 854–862, *Trans. Math. Notes* **42** (1987) 969–974.

[38] H. S. Wilf, *Generatingfunctionology* (Academic Press, New-York, 1990).

[39] E. M. Wright, The Number of Connected Sparsely Edged Graphs, *Journal of Graph Theory* **1** (1977) 317–330.

[40] E. M. Wright, The Number of Connected Sparsely Edged Graphs. II. Smooth graphs and blocks, *Journal of Graph Theory* **2** (1978) 299–305.

[41] E. M. Wright, The Number of Connected Sparsely Edged Graphs. III. Asymptotic results, *Journal of Graph Theory* **4** (1980) 393–407.


(a)
(b)
(c)
(d)
(e)
root edge
root vertice
There are 6 ways to attached 1, 2 to