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Birth and growth of multicyclic components in random hypergraphs

Vlady Ravelomanana

LIAFA, Université Paris Diderot - Paris 7, UMR CNRS 7089, 75205 Paris Cedex 13, France

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ABSTRACT

Define an ℓ-component to be a connected b-uniform hypergraph with k edges and $k(b-1)-\ell$ vertices. In this paper, we investigate the growth of size and complexity of connected components of a random hypergraph process. We prove that the expected number of creations of ℓ-components during a random hypergraph process tends to 1 as b is fixed and ℓ tends to infinity with the total number of vertices n while remaining $\ell = o(n^{1/3})$. We also show that the expected number of vertices that ever belong to an ℓ-component is $\sim 12^{1/3}\ell^{1/3}n^{2/3}(b-1)^{-1/3}$. We prove that the expected number of times hypertrees are swallowed by ℓ-components is $\sim 2^{1/3}3^{-1/3}n^{1/3}\ell^{-1/3}(b-1)^{-5/3}$. It follows that with high probability the largest ℓ-component during the process is of size of order $O(\ell^{1/3}n^{2/3}(b-1)^{-1/3})$. Our results give insight into the size of giant components inside the phase transition of random hypergraphs and generalize previous results about graphs.

1. Introduction

A *hypergraph* $\mathcal{H}$ is a pair $(\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{1, 2, \ldots, n\}$ denotes the set of vertices of $\mathcal{H}$ and $\mathcal{E}$ is a family of subsets of $\mathcal{V}$ called edges (or hyperedges). For a general treatise on hypergraphs, we refer to Berge [5]. We say that $\mathcal{H}$ is *b-uniform* (or simply *uniform*) if for every edge $e \in \mathcal{E}$, $|e| = b$ (with $b > 1$). In this paper, all considered hypergraphs are b-uniform (with b fixed). We will study the growth of the size and complexity of connected components of a random hypergraph process $\{\mathbb{H}(n,t)\}_{0 \le t \le 1}$ defined as follows. Let K_n be the complete hypergraph built with n vertices and $\binom{n}{b}$ edges (self-loops and multiple edges are not allowed). $\{\mathbb{H}(n,t)\}_{0 \le t \le 1}$ may be constructed by letting each edge e of K_n (amongst the $\binom{n}{b}$ possible edges) appear at random time T_e, with T_e independent and uniformly distributed on $(0, 1)$ and letting $\{\mathbb{H}(n,t)\}_{0 \le t \le 1}$ contain the edges such that $T_e \le t$. For the random graph counterpart of this model, we refer the reader to the seminal paper [17] (see also [27]). This model is closely related to $\{\mathbb{H}(n,M)\}$ where $M \in \left[1, \binom{n}{b}\right]$ represents the number of edges picked uniformly at random amongst the $\binom{n}{b}$ possible edges and which are present in the random hypergraph. The main difference between $\{\mathbb{H}(n,M)\}_{0 \le M \le \binom{n}{b}}$ and $\{\mathbb{H}(n,t)\}_{0 \le t \le 1}$ is that in $\{\mathbb{H}(n,M)\}_{0 \le M \le \binom{n}{b}}$, edges are added at fixed (slotted) times $1, 2, \ldots, \binom{n}{b}$ so at any time M we obtain a random graph with n vertices and M edges, whereas in $\{\mathbb{H}(n,t)\}_{0 \le t \le 1}$ the edges are added at random times. At time $t = 0$, we have a hypergraph with n vertices and 0 edges, and as the time advances all edges e with r.v. T_e such that $T_e \le t$ (where t is the current time), are added to the hypergraph until t reaches 1 in which case, one obtains the complete hypergraph K_n.

We define the *excess* (or the *complexity*) of a connected b-uniform hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ as (see also [21–23,27,29–31]):

$$\text{excess}(\mathcal{H}) = \sum_{e \in \mathcal{E}} (|e| - 1) - |\mathcal{V}| = |\mathcal{E}| \times (b-1) - |\mathcal{V}| . \tag{1}$$

Namely, the complexity (or excess) of connected components ranges from -1 (hypertrees) to $\binom{n}{b}(b-1) - n$ (complete hypergraph). As shown in many research papers [1,17,26,27,29], it is difficult but very useful to decompose the enumeration

E-mail addresses: vlad@liafa.jussieu.fr, vlad@lipn.univ-paris13.fr.

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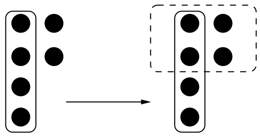

Fig. 1. In this case, $p = \ell = -1$, $q = 2$, $b = 4$. The last added edge is given in dashed lines. The new multicyclic component has excess $2 \times 3 - 6 = 0$.

of (hypergraphs) according to the number of edges and vertices. To this end, a connected component with excess ℓ ($\ell \geq -1$) is called an *ℓ-component*. The notion of excess was first used in [29] where the author obtained substantial enumerative results in the study of connected graphs according to the two parameters, viz. number of vertices and number of edges. A connected component which is not a hypertree (whose excess is -1) is said *multicyclic* (following the terms used by our predecessors in [16,17,19]).

1.1. Related work about connected and random (hyper)graphs

Numerous results have been obtained for random graphs as witnessed by the books [7,20] and the references therein. In comparison, there are very few works about random hypergraphs. One of the most significant results was obtained by Schmidt-Pruznan and Shamir [28] who studied the component structure for random hypergraphs. In particular, they proved that if $b \geq 2$, $M = cn$ with $c < 1/b(b-1)$ then asymptotically almost surely (a.a.s. for short) the largest component of $\mathbb{H}(n, M)$ is of order $\log n$ and for $c = 1/b(b-1)$ it has $\Theta(n^{2/3})$ vertices and as $c > 1/b(b-1)$ a.a.s. $\mathbb{H}(n, M)$ has a unique giant component with $\Theta(n)$ vertices. This result generalizes the seminal papers of Erdös and Rényi who discovered the abrupt change in the structure of the random graph $\mathbb{G}(n, M)$ when $M = cn$ with $c \sim 1/2$ (see [10,11]).

Many approaches lead to beautiful enumerative results about connected graphs. Using different methods and tools Bender et al. [4], Pittel and Wormald [25] and then van der Hofstad and Spencer [15] were able to compute the asymptotic number of connected graphs with n vertices and M edges (for all possible values of M). In contrast, there are only few enumerative results about uniform and connected hypergraphs. As far as we know, the number of connected hypergraphs has been investigated first by Karoński and Łuczak [22] who then used the obtained results to study the phase transition of random uniform hypergraphs [23]. More precisely, the authors of [23] proved limit theorems for the distribution of the size of the largest component of $\mathbb{H}(n, M)$ at the phase transition, i.e. inside the scaling window $M \in \left[n/b(b-1) - O(n^{2/3}), n/b(b-1) + O(n^{2/3})\right]$. In this paper, we follow the probabilistic methods initiated by Janson [16,17] and combine them with the enumerative/analytic methods to study the birth and growth of multicyclic components with respect to their sizes (in terms of number of vertices). In our work, we do not compute the time when such components should appear during the process. Using pure probabilistic approaches Coja-Oghlan et al. [9] were able to obtain the order of magnitude of the number of b-uniform hypergraphs with n vertices and $M = o(n \log n)$ edges. In [2,3], among other results Behrisch et al. established local limit theorems for the maximum order of a component of $\mathbb{H}(n, M)$ (resp. $\mathbb{H}(n, p)$) in the supercritical regimes $M > n/b(b-1)\ (1+\varepsilon)$ (resp. $p = (1+\varepsilon)/\binom{n-1}{b-1}(b-1)$). As remarked by the authors, the results offer alternative approaches to obtain the number of connected hypergraphs. In [1], Andriamampianina and Ravelomanana show how to compute the generating functions of connected hypergraphs which they used with an approach similar to that of Wright [29–31] to compute the asymptotic number of these structures.

1.2. The settings

In this paper, we consider the *continuous time* random hypergraph process described above and will study the creation (or birth) and growth of components of excess ℓ (or ℓ-components) inside the critical window $M = n/b(b-1) + O(n^{2/3})$. These investigations generalize those about random graphs initiated by Janson [17] and continued by Ravelomanana [27].

There are two manners to create a new ($\ell + 1$) component during the $\{\mathbb{H}(n, t)\}_{0 \leq t \leq 1}$ process:

Case (i) either by adding an edge between an existing p-component (with $p \leq \ell$) and $(b - q)$ distinct hypertrees (with $0 \leq q \leq b$) such that the edge encloses q distinct vertices in the p-component,

Case (ii) or by joining with the last added edge many connected components such that the number of multicyclic components of the whole random structure diminishes.

Observe that in the first case, to create an $(\ell+1)$-component, we must have $(b-1)+p-q = \ell+1$. In this case, it is also important to note that if $p \geq 0$ the number of multicyclic components remains the same after the addition of the last edge. In the following figures, we depict two possible ways to create multicyclic components. Note that in Fig. 1, the number of multicyclic component increases by 1 whereas in Fig. 2, it remains the same.

The first transition – case (i) – described above will be denoted $p \to \ell$. For example, Fig. 1 (resp. Fig. 2) depicts a transition $-1 \to 0$ (resp. $0 \to 2$).

Similarly, the second transition described by case (ii) is denoted $\oplus_i p_i \to \ell$. Fig. 3 exemplifies such a transition ($-1 \oplus 0 \oplus 0 \to 2$). Note that in case (ii), at least two of the former components are necessarily multicyclic, i.e. of excess > -1, otherwise the last edge encloses hypertrees and a multicyclic component as in case (i).

We say that an ℓ-component is *created* by a transition $p \to \ell$ with $p < \ell$ or by a transition $\oplus_i p_i \to \ell$. For $\ell \geq 0$, we say that an ℓ-component *grows* when it swallows some hypertrees without increasing its own complexity (transition $\ell \to \ell$).

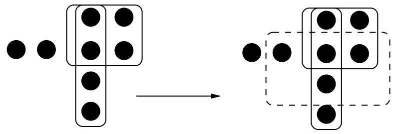

Fig. 2. Here, $p = 0, q = 1, b = 4$. The multicyclic component on the left grows in complexity passing through 0 to $3 \times 3 - 7 = 2$.

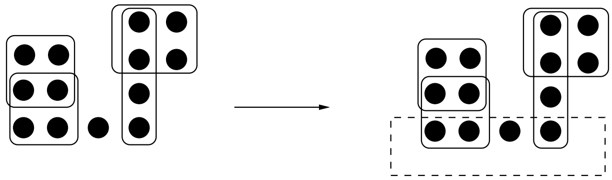

Fig. 3. A transition $-1 \oplus 0 \oplus 0 \to 2$.

Following Janson in [17], we have two points of view:

- *The static view.* Let $\mathcal{C}_\ell(m)$ denote the collection of all ℓ-components in $\{\mathbb{H}(n, t)\}_{0\le t\le 1}$. Consider the family $\mathcal{C}_\ell^\star = \bigcup_m \mathcal{C}_\ell(m)$ for every ℓ-component that appears at some stage of the continuous process, ignoring when it appears: the elements of $\mathcal{C}_\ell^\star$ are called *static ℓ-components*.
- *The dynamic view.* A connected component can be viewed as "the same" according to its excess even after it has grown by swallowing some hypertrees (transition $\ell \to \ell$). Such component whose excess remains the same can be viewed as a *dynamic ℓ-component* as its size evolves.

We define $V_\ell = |\mathcal{V}_\ell|$ as the number of vertices that at some stage of the process belong to an ℓ-component and ${V_\ell}^{\max} = \max\{|V(C)| : C \in \mathcal{C}_\ell^\star\}$ to be the size of the largest ℓ-component that ever appears. We have ${V_\ell}^{\max} \le V_\ell$ and each ℓ-component has at most ${V_\ell}^{\max}$ vertices while the union of all ℓ-components has at most V_ℓ vertices.

1.3. Our results and outline of the paper

We combine analytic combinatorics [12] and probabilistic theory [20] to study the extremal characteristics of the components of a random hypergraph process inside its phase transition [23] and find that the size of the largest ℓ-component with k edges and $k(b-1) - \ell$ vertices is of order $O(\ell^{1/3} n^{2/3} (b-1)^{-1/3})$ when b is fixed and $\ell \to \infty$ with n but $\ell = o(n^{1/3})$. Under the same conditions, we prove that the expected number of creations of ℓ-component is ~ 1, $\mathbb{E}V_\ell \sim 12^{1/3} \ell^{1/3} n^{2/3} (b-1)^{-1/3}$ and the expected number of static ℓ-components (i.e., the number of different ℓ-components evolving during the process) is $2^{1/3} 3^{-1/3} n^{1/3} (b-1)^{-5/3} \ell^{-1/3}$. Similar results are also computed for components of fixed excess.

This paper is organized as follows. In the next section, we introduce the general expression of the expectations of several random variables of our interest. In Section 3, the computations of the expectations are developed focusing on the particular and instructive case of components with fixed complexities. The last paragraph provides several technical lemmas useful in order to study the extremal case, i.e. whenever the excess ℓ of the component is large. We give there methods on how to investigate the number of creations of ℓ-components as well as their sizes.

2. Connected components and transitions

2.1. Expected number of transitions

In this paragraph, we give a general formal expression of the expectations of the number of the two types of transitions. To this end, let $\alpha(\ell; k)$ be the expected number of times a new edge is added by means of the first type of transition $p \to \ell$ in order to create an ℓ-component with k edges (or with $k \times (b-1) - \ell$ vertices). Note again that in this case, the number of multicyclic components of the $\{\mathbb{H}(n, t)\}_{0\le t\le 1}$ process remains the same after the addition of this edge. Similarly, let $\beta(\ell; k)$ be the expected number of times an edge is added joining at least two multicyclic components in order to form a new ℓ-component with a total of k edges. In other terms, $\beta(\ell; k)$ is the expected number of times at least two multicyclic components and some hypertrees merge to form an ℓ-component.

We consider labeled structures first because labeled graphs and hypergraphs are much easier to enumerate than the corresponding unlabeled problems. For example, there is exactly one (resp. 3) unlabeled (resp. labeled) graph(s) with 3 vertices and 2 edges. Note that the enumeration of unlabeled (hyper)graphs requires a considerable amount of combinatorial theory including Pólya's theorem (cf. [13]). Note that two labeled (hyper)graphs G_1 and G_2 are considered the same if and only if there is a $1 - -1$ map from the vertices of G_1 onto the vertices of G_2 which preserves not only the adjacency but also the labeling.

We have the following lemma which computes the expected number of transitions $\alpha(\ell; k)$:

Lemma 2.1. *Let $a = k(b-1) - \ell$. Denote by $\rho(a, k)$ the number of ways to label an ℓ-component with a vertices such that one edge – whose deletion will not increase the number of multicyclic components but will suppress the newly created ℓ-component*

– is distinguished among the others. Then,

$$\alpha(\ell; k) = \binom{n}{a}\rho(a, k)\int_0^1 t^{k-1}(1-t)^{\binom{n}{b}-\binom{n-a}{b}-k}\,\mathrm{d}t. \tag{2}$$

Proof. There are $\binom{n}{a}$ choices of the $a = k(b-1) - \ell$ vertices of the newly created ℓ-component. By the definition of $\rho(a; k)$, there are $\binom{n}{a} \times \rho(a; k)$ possible ℓ-components. The probability that the previous component (the one before obtaining the current ℓ-component) belongs to $\{\mathbb{H}(n, t)\}_{0\le t\le 1}$ is given by

$$t^{k-1}(1-t)^{\sum_{i=1}^{b-1}\binom{n-a}{i}\binom{a}{b-i}+\binom{a}{b}-k+1} \tag{3}$$

where the summation in the exponent represents the number of edges not present between the considered component and the rest of the hypergraph. The conditional probability that the last edge is added during the time interval $(t, t+\mathrm{d}t)$ and not earlier is $\mathrm{d}t/(1-t)$. Using the identity

$$\sum_{i=1}^{b-1}\binom{n-a}{i}\binom{a}{b-i} = \binom{n}{b} - \binom{n-a}{b} - \binom{a}{b} \tag{4}$$

and integrating over all times after some algebra, we obtain (2). □

Similarly, if we let $\tau(a; k)$ to be the number of ways to label an ℓ-component with $a = (k-1) - \ell$ vertices and k edges such that one edge – whose suppression augments the number of multicyclic connected components – is distinguished among the others. Then, $\beta(\ell; k)$ can be computed as for $\alpha(\ell; k)$ using exactly $\tau(a; k)$ instead of $\rho(a; k)$.

Next, the following lemma gives some asymptotic values needed when using formula (2).

Lemma 2.2. *Let $b > 1$ be fixed and $a = (b-1)k - \ell$. We have*

$$\binom{n}{a}\int_0^1 t^{k-1}(1-t)^{\binom{n}{b}-\binom{n-a}{b}-k}\,\mathrm{d}t = \frac{1}{\sqrt{(b-1)}\,n^{\ell}}\,\frac{k^{(k-1)}\,[(b-1)!]^k}{\left(k(b-1)-\ell\right)^{kb-\ell}}$$
$$\times\ \exp\left(k(b-2) - \ell - \frac{(b-1)^4\,k^3}{24\,n^2}\right) \times \left(1 + O\left(\frac{k}{n} + \frac{k^2}{n^2} + \frac{k^4}{n^3} + \frac{k}{n^{b-1}} + \frac{1}{k}\right)\right). \tag{5}$$

Proof. First, using Stirling formula for factorial we get

$$\binom{n}{a} = \frac{1}{\sqrt{2\pi a}}\,\frac{n^a\,\mathrm{e}^a}{a^a}\,\exp\left(-\frac{a^2}{2n} - \frac{a^3}{6n^2} + O\left(\frac{k^4}{n^3} + \frac{1}{k}\right)\right). \tag{6}$$

For $(x, y) \in \mathbb{N}^2$, we have

$$\int_0^1 t^x(1-t)^y\mathrm{d}t = \frac{x!}{y!}(x+y+1)! = \frac{1}{(x+y+1)\binom{x+y}{x}}. \tag{7}$$

Setting $N = \binom{n}{b} - \binom{n-a}{b}$, using standard calculus we then obtain

$$N = \frac{n^{(b-1)}a}{(b-1)!}\left(1 - \frac{a(b-1)}{2n} + \frac{a^2(b-1)(b-2)}{6n^2} + O\left(\frac{1}{n}\right) + O\left(\frac{k}{n^2}\right)\right). \tag{8}$$

Now, using the above formulas we find that the integral equals

$$\begin{aligned}
\frac{1}{N\binom{N}{k-1}} &= \frac{\sqrt{2\pi k}}{N^k}\,\frac{(k-1)^{(k-1)}}{\mathrm{e}^{k-1}}\left(1 + O\left(\frac{k^2}{N} + \frac{1}{k}\right)\right)\\
&= \sqrt{\frac{2\pi}{k}}\,\frac{k^k}{N^k\mathrm{e}^k}\left(1 + O\left(\frac{k^2}{N} + \frac{1}{k}\right)\right)\\
&= \sqrt{\frac{2\pi}{k}}\,\frac{k^k}{\mathrm{e}^k}\,\frac{[(b-1)!]^k}{n^{k(b-1)}a^k}\left(1 + O\left(\frac{k}{n^{b-1}b} + \frac{1}{k}\right)\right)\\
&\quad\times\ \exp\left(-k\log\left(1 - \frac{a(b-1)}{2n} + \frac{a^2(b-1)(b-2)}{6n^2} + O\left(\frac{1}{n}\right) + O\left(\frac{k}{n^2}\right)\right)\right).
\end{aligned} \tag{9}$$

Therefore by replacing a with $k(b-1)-\ell$ and using (6), it yields

$$\frac{\binom{n}{a}}{N\binom{N}{k-1}} = \frac{1}{\sqrt{(b-1)}\,n^{\ell}} \frac{k^{(k-1)}\,[(b-1)!]^k}{\Big(k(b-1)-\ell\Big)^{kb-\ell}} \exp\Big(k(b-2)-\ell\Big) \exp\left(-\frac{(b-1)^4\,k^3}{24\,n^2}\right)$$
$$\times \left(1+O\left(\frac{k}{n}+\frac{k^2}{n^2}+\frac{k^4}{n^3}+\frac{k}{n^{b-1}}+\frac{1}{k}\right)\right). \quad \square$$

Lemma 2.2 tells us that the expectations the random variables of interest rely on the asymptotic number of the considered connected components. In the rest of the paper, k is an integer in $\left[1, \frac{n}{b-1}\right]$. As k and n are both large, we need to quantify summations including the O-terms in Eq. (5). To this purpose, we have the following lemma:

Lemma 2.3. *Let $u > 0$. If $u \geq 0$ as $n \to \infty$, we have*

$$\sum_{k=1}^{\frac{n}{b-1}} \left[k^u \exp\left(-\frac{(b-1)^4 k^3}{24\,n^2}\right)\left(1+O\left(\frac{k}{n}+\frac{k^2}{n^2}+\frac{k^4}{n^3}+\frac{1}{k}\right)\right)\right]$$
$$= \frac{24^{\left(\frac{u+1}{3}\right)} n^{\frac{2(u+1)}{3}}}{(b-1)^{\frac{4(u+1)}{3}}}\,\Gamma\left(\frac{u+1}{3}\right) + O\left(n^{\frac{u+1}{2}}\right). \tag{10}$$

Proof. We start splitting the summation into three parts, $1 \leq k < n^{1/2}$, $n^{1/2} \leq k \leq n^{17/24}$ then $n^{17/24} < k \leq \frac{n}{b-1}$. If $1 \leq k < n^{1/2}$ the O-terms are $O(1)$ and $\mathrm{e}^{-\frac{(b-1)^4 k^3}{24 n^2}} = 1 - O(n^{-1/2})$. Therefore, since $\sum_{k=1}^{n^{1/2}} k^u \left(1 - O\left(\frac{1}{n^{1/2}}\right)\right) = O(n^{\frac{u+1}{2}})$ the first part of the summation is $O(n^{\frac{u+1}{2}})$. Next, for $n^{1/2} \leq k \leq n^{17/24}$ the O-terms are $O(n^{-1/6})$. By means of Euler–MacLaurin formula [8] and standard calculus, we get

$$\sum_{k=n^{1/2}}^{n^{17/24}} k^u \exp\left(-\frac{(b-1)^4 k^3}{24\,n^2}\right)\left(1+O\left(\frac{1}{n^{1/6}}\right)\right)$$
$$= \int_{n^{1/2}}^{n^{17/24}} x^u \exp\left(-\frac{(b-1)^4 x^3}{24\,n^2}\right) \mathrm{d}x + O(n^{u/2}) + \mathrm{e}^{-O(n^{1/8})}$$
$$= \frac{24^{\left(\frac{u+1}{3}\right)} n^{\frac{2(u+1)}{3}}}{(b-1)^{\frac{4(u+1)}{3}}}\,\Gamma\left(\frac{u+1}{3}\right) + O(n^{u/2}). \tag{11}$$

Finally, to complete the proof we observe that if $n^{17/24} < k \leq \frac{n}{b-1}$ the O-terms in (10) are $O(n)$ but every term of the summation is at most $n^u \exp\left(-O(n^{1/8})\right)$. $\square$

2.2. Enumerations of connected hypergraphs

As far as we know there are not so many results about the exact and asymptotic enumerations of connected uniform hypergraphs. In this paragraph, we recall some of the results established independently in [22,9,1] (the three papers actually use three different methods). In [1], the authors use the generating functions approach [14,19,12,29–31] to count exactly and asymptotically connected labeled b-uniform hypergraphs. Before giving the enumerative results, we need some definitions.

Definitions.
- If $A(z) = \sum_n a_n z^n$ and $B(z) = \sum_n b_n z^n$ are two formal power series, $A \preceq B$ means that $\forall n \in \mathbb{N}$, $a_n \leq b_n$.
- If $A(z) = \sum_n a_n z^n$, $[z^n]\,A(z)$ denotes a_n.
- Let $\Theta(z)$ be the following exponential generating function (EGF for short).

$$\theta(z) = 1 - \frac{T(z)^{(b-1)}}{(b-2)!} \tag{12}$$

where

$$T(z) = z \exp\left(\frac{T(z)^{(b-1)}}{(b-1)!}\right) \tag{13}$$

is the EGF of labeled rooted hypertrees which can be obtained using the symbolic method of generating functions [12].
- We denote by ϑ_z the combinatorial operator $z\frac{\partial}{\partial z}$.

We remark that when applied on the EGFs of hypergraphs, $\vartheta_z = z\frac{\partial}{\partial z}$ corresponds to distinguish a vertex, e.g. the root of a tree, amongst the others.

Among other results, the authors of [1] established the following:

Lemma 2.4. *Let $H_\ell(z)$ be the EGF of b-uniform connected hypergraphs with excess ℓ. Then,*

$$H_{-1}(z) = T(z) - \frac{(b-1)\,T(z)^b}{b!}$$
$$H_0(z) = -\frac{1}{2}\log\theta(z) + \frac{\theta(z)}{2} - \frac{1}{2}. \tag{14}$$

Moreover, for any $\ell \geq 1$ H_ℓ satisfies

$$\frac{\lambda_\ell(b-1)^{2\ell}}{3\,\ell\,T(z)^\ell\,\theta(z)^{3\ell}} - \frac{(\nu_\ell(b-2))(b-1)^{2\ell-1}}{(3\,\ell-1)\,T(z)^\ell\,\theta(z)^{3\ell-1}} \preceq H_\ell(z) \preceq \frac{\lambda_\ell(b-1)^{2\ell}}{3\,\ell\,T(z)^\ell\,\theta(z)^{3\ell}}, \tag{15}$$

where $\lambda_\ell = 3\left(\frac{3}{2}\right)^\ell \frac{\ell!}{2\pi}\left(1+O\left(\frac{1}{\ell}\right)\right)$ and $\nu_\ell = O(\ell\lambda_\ell)$. Furthermore, λ_ℓ is defined recursively by $\lambda_0 = \frac{1}{2}$ and

$$\lambda_\ell = \frac{1}{2}\lambda_{\ell-1}(3\ell-1) + \frac{1}{2}\sum_{t=0}^{\ell-1}\lambda_t\lambda_{\ell-1-t}, \quad (\ell \geq 1). \tag{16}$$

3. Components of fixed complexities

As typical examples, let us work with unicyclic components. We will compute the expected number of transitions $-1 \to 0$. That is the number of times unicyclic connected components (i.e. 0-components) are created. We will also investigate the number of times unicyclic components merge with hypertrees growing in size but staying with the same complexity (excess 0). In these directions, we have the following result:

Theorem 3.1. *As $n \to \infty$, on the average a b-uniform random hypergraph has about $\frac{1}{3}\log n$ dynamic unicyclic components. The expected number of static 0-components is $\sim \frac{\sqrt{2}\pi^{3/2}24^{1/6}}{6\,\Gamma\left(\frac{5}{6}\right)}(b-1)^{-5/3}\,n^{1/3} \approx 1.974\ldots\,(b-1)^{-5/3}\,n^{1/3}$.*

Proof. The creation of unicyclic components can be obtained only by adding an edge joining 2 distinct vertices inside the same hypertree with $(b-2)$ other vertices from $(b-2)$ distinct hypertrees (to complete the edge). The number of such constructions is therefore given by the coefficients of the following EGF:

$$C'_0(z) = \frac{\left(\vartheta_z H_{-1}(z)\right)^{(b-2)}}{(b-2)!} \times \left(\frac{\vartheta_z^2 - \vartheta_z}{2}\left(H_{-1}(z)\right)\right), \tag{17}$$

where the combinatorial operator $\vartheta_z = z\frac{\partial}{\partial z}$ corresponds to marking a vertex of the hypergraph in order to distinguish it from the others. We refer the reader to Bergeron et al. [6] for the use of distinguishing/marking and pointing in combinatorial species. Recall that the EGFs are as described briefly in Lemma 2.4. Then, using $\vartheta_z H_{-1}(z) = T(z)$ and $\vartheta_z T(z) = \frac{T(z)}{\theta(z)}$ we find

$$C'_0(z) = \frac{T(z)^{b-2}}{2\,(b-2)!}\left(\frac{T(z)}{\theta(z)} - T(z)\right). \tag{18}$$

We also have (such expansions are similar to those in [24])

$$\frac{1}{\theta(z)} = \frac{1}{1 - \frac{T(z)^{(b-1)}}{(b-2)!}} = \sum_{k=0}^{\infty}\frac{k^k}{k!\,[(b-2)!]^k}\,z^{(b-1)k}. \tag{19}$$

(19) can be proved using Cauchy's integral formula as follows. Let $[z^n]\,1/\theta(z)$ be the coefficient of the nth term of the series $1/\theta(z)$. Substituting $u = T(z)$, we get successively

$$[z^n]\frac{1}{\theta(z)} = \frac{1}{2\pi i}\oint\frac{1}{\theta(z)}\frac{\mathrm{d}z}{z^{n+1}}$$
$$= \frac{1}{2\pi i}\oint\exp\left(n\frac{u^{(b-1)}}{(b-1)!}\right)\frac{\mathrm{d}u}{u^{n+1}}. \tag{20}$$

We obtain (19) by remarking that the RHS of (20) equals $[u^n]\exp\left(n\frac{u^{(b-1)}}{(b-1)!}\right)$.

Denoting by $\rho'((b-1)k, k)$ the number of ways to label a unicyclic component with $(b-1)k$ vertices and with a distinguished edge such that its deletion will leave a set of $(b-2)$ rooted hypertrees and another hypertree with two distinct and marked vertices, using (18) and (19)

$$\rho'\Big((b-1)k,k\Big) = ((b-1)k)!\left[z^{(b-1)k}\right] C'_0(z) \sim \frac{((b-1)k)!\,k^k}{2\,k!\,[(b-2)!]^k}. \tag{21}$$

Next, using Lemma 2.2 with the above equation, after standard calculations, we get

$$\begin{aligned}
&\sum_{k=1}^{\frac{n}{(b-1)}} \rho'\Big((b-1)k,k\Big)\binom{n}{(b-1)k}\int_0^1 t^{k-1}(1-t)^{\binom{n}{b}-\binom{n-(b-1)k}{b}-k}\,\mathrm{d}t \\
&= \sum_{k=1}^{\frac{n}{(b-1)}} \frac{(b-1)^{k(b-1)+1/2}\,k^{k(b-1)}\exp(-k(b-2))}{2\,[(b-2)!]^k} \\
&\quad\times \frac{k^{(k-1)}[(b-1)!]^k}{(k(b-1))^{(kb)}}\exp\left(k(b-2)-\frac{(b-1)^4k^3}{24n^2}\right)\times\left(1+O\left(\frac{k}{n}+\frac{k^2}{n^2}+\frac{k^4}{n^3}+\frac{1}{k}\right)\right) \\
&= \sum_{k=1}^{\frac{n}{(b-1)}} \frac{1}{2k}\times\exp\left(-\frac{(b-1)^4\,k^3}{24\,n^2}\right)\times\left(1+O\left(\frac{k}{n}+\frac{k^2}{n^2}+\frac{k^4}{n^3}+\frac{1}{k}\right)\right)
\end{aligned} \tag{22}$$

Splitting the summation again as we did for (10), we find that

$$(22) \sim \frac{1}{2}\int_1^{n/(b-1)} \frac{1}{x}\,\mathrm{e}^{-(b-1)^4x^3/24n^2}\,\mathrm{d}x. \tag{23}$$

To estimate the last integral of (22), we write

$$\begin{aligned}
\int_1^{n/(b-1)} \frac{1}{x}\,\mathrm{e}^{-(b-1)^4x^3/24n^2}\,\mathrm{d}x &= \int_1^{n^{2/3}/(b-1)^{4/3}} \frac{1}{x}\left(1+O\left(\frac{(b-1)^4x^2}{n^2}\right)\right)\mathrm{d}x \\
&\quad + O\left(\int_{n^{2/3}/(b-1)^{4/3}}^{n/(b-1)} \frac{1}{x}\mathrm{e}^{-(b-1)^4x^3/24n^2}\mathrm{d}x\right) \sim \log\left(n^{2/3}\right)+O(1).
\end{aligned} \tag{24}$$

Thus, the expected number of creations of unicyclic components is $\sim \frac{1}{3}\log n$. which completes the proof of the first part of the theorem. To prove the second part, we have to investigate the number of static 0-components, that is the number of times 0-components merge with hypertrees by the transition $0 \to 0$. The EGF of unicyclic components with a distinguished edge such that its suppression will leave a vertex-rooted unicyclic component and a set of $(b-1)$ rooted hypertrees is given by

$$C''_0(z) = \frac{T(z)^{b-1}}{(b-1)!}\,\vartheta_z\Big(H_0(z)\Big). \tag{25}$$

Let $\theta = 1 - T(z)^{b-1}/(b-2)!$. Since the EGF H_0 is given by (14), we have

$$C''_0(z) = \frac{(1-\theta)}{2}\left(1-\frac{1}{\theta}\right)^2. \tag{26}$$

We used $z\mathrm{d}\theta/\mathrm{d}z = -(b-1)/(b-2)!T(z)^{(b-1)}/\theta = (b-1)(1-1/\theta)$. Using tools from singularity analysis of generating functions [12], we find that T can be expanded as

$$T(z) = ((b-2)!)^{1/(b-1)} - \left(\frac{2}{b-1}\right)^{1/2}((b-2)!)^{1/(b-1)}\left(1-\frac{z}{z_0}\right)^{1/2}+\cdots \tag{27}$$

where $z_0 = ((b-2)!)^{1/(b-1)}\exp\left(-\frac{1}{b-1}\right)$. Denote by $\rho''((b-1)k,k)$ the number of ways to label a unicyclic component with $(b-1)k$ vertices and with a distinguished edge such that its deletion will leave a 0-component with a set of $(b-1)$ rooted hypertrees. Using singularity analysis of generating functions [12], we easily find that for any constant ℓ, the nth coefficient of the series $\frac{1}{\theta^\ell}$ has the same order as the one of $[2(b-1)]^{-\ell/2}(1-z/z_0)^{-\ell/2}$. More precisely

$$\left[z^n\right]\frac{1}{\theta^\ell} \sim [2(b-1)]^{-\ell/2}\,\frac{n^{\ell/2-1}}{\Gamma(\ell/2)}\,z_0^{-n}. \tag{28}$$

Hence, the EGF C''_0 described above behaves[1] like $1/2\theta^2$. Therefore, we have

$$\rho''\Big((b-1)k,k\Big) = ((b-1)k)!\left[z^{(b-1)k}\right] C''_0(z) \sim \sqrt{\frac{\pi}{8\,(b-1)}}\,\frac{k^{k(b-1)+1/2}}{\mathrm{e}^{k(b-2)}}\left(\frac{(b-1)^{k(b-1)}}{[(b-2)!]^k}\right). \tag{29}$$

[1] For two EGFs $A(z)$ and $B(z)$, we say that $A(z)$ behaves as $B(z)$ if $[z^n]A(z) \sim [z^n]B(z)$ as n is large.

Now, using Lemma 2.2 and summing over k after some cancellations, the computed expectation is

$$\sum_{k=1}^{n/(b-1)} \binom{n}{(b-1)k} \rho''((b-1)k,k) \int_0^1 t^{k-1}(1-t)^{\binom{n}{b}-\binom{n-(b-1)k}{b}-k}\,\mathrm{d}t$$
$$= \sum_{k=1}^{n/(b-1)} \frac{k^{(k-1)}[(b-1)!]^k}{\sqrt{(b-1)}(k(b-1))^{(kb)}} \exp(k(b-2))$$
$$\times \exp\left(-\frac{(b-1)^4k^3}{24n^2}\right)\sqrt{\frac{\pi}{8(b-1)}}\frac{k^{k(b-1)+1/2}}{\mathrm{e}^{k(b-2)}}\left(\frac{(b-1)^{k(b-1)}}{[(b-2)!]^k}\right)\left(1+O\left(\frac{k}{n}+\frac{k^2}{n^2}+\frac{k^4}{n^3}+\frac{1}{k}\right)\right)$$
$$= \sum_{k=1}^{n/(b-1)} \sqrt{\frac{\pi}{8(b-1)^2}}\frac{1}{k^{1/2}}\mathrm{e}^{-(b-1)^4k^3/24n^2}\left(1+O\left(\frac{k}{n}+\frac{k^2}{n^2}+\frac{k^4}{n^3}+\frac{1}{k}\right)\right). \tag{30}$$

We can now argue as for Lemma 2.3 to get rid of the O-terms and we find that this expectation is about $1,974748319\cdots(b-1)^{-\frac{5}{3}}n^{\frac{1}{3}}$. □

Note here that the result stated in Theorem 3.1 (humbly) generalizes the ones of Janson in [17] since by setting $b=2$, we retrieve his results concerning unicyclic (graph) components.

Next, we can investigate the number of vertices that ever belong to 0-components.

Theorem 3.2. *Let V_0 be the number of vertices that at some stage of the random graph process belong to unicyclic components. We have*

$$\mathbb{E}V_0 \sim \frac{1}{3}\frac{24^{1/3}\,\Gamma(1/3)}{(b-1)^{1/3}}\,n^{2/3}. \tag{31}$$

Proof. According to the above computations, the expected number of vertices added to $\mathcal{V}_0$ for the creation of such unicyclic components (transition $-1\to 0$) is about

$$\sum_{k=1}^{\frac{n}{(b-1)}} k(b-1)\times\rho'\Big((b-1)k,k\Big)\binom{n}{(b-1)k}\int_0^1 t^{k-1}(1-t)^{\binom{n}{b}-\binom{n-(b-1)k}{b}-k}\,\mathrm{d}t$$
$$\sim \sum_{k=1}^{n/(b-1)}\frac{(b-1)}{2}\exp\left(-\frac{(b-1)^4k^3}{24n^2}\right)\sim\frac{1}{6}\frac{24^{1/3}\,\Gamma(1/3)}{(b-1)^{1/3}}\,n^{2/3}. \tag{32}$$

Next, our main trick to compute the expected number of vertices added to already existing unicyclic components (by adding hypertrees) is the use of generating functions (this differs from the techniques in [17]). In the considered constructions, on the one hand we have a rooted unicyclic component and on the other hand a set of $(b-1)$ rooted hypertrees. The added vertices to the constructions come from the hypertrees. Using the operator $\vartheta_z = z\partial/\partial z$ upon the EGF $T(z)^{b-1}/(b-1)!$, we retrieve the number of vertices added to the already existing 0-components encoded in the generating function. Therefore, in formula (22) we have to replace $\rho'\Big((b-1)k,k\Big)$ by

$$((b-1)k)!\left[z^{(b-1)k}\right](\vartheta_z H_0(z))\left(\vartheta_z\frac{T(z)^{(b-1)}}{(b-1)!}\right)$$
$$=((b-1)k)!\left[z^{(b-1)k}\right]\frac{(b-1)}{2}\left(1-\frac{1}{\theta}\right)^2\frac{1-\theta}{\theta}\sim\frac{(b-1)k^{k+1}((b-1)k)!}{2\,k!\,((b-2)!)^k} \tag{33}$$

in order to compute the desired expectation (we used singularity analysis [12]). In the same vein as (22), we then obtain the expectation by summing. Incidentally, the result of the latter summation turns out to be asymptotically the same as (32). □

As an immediate corollary (see also [17, Corollaries 3 and 4]), we obtain

Corollary 3.3. *For any $C>2.3^{-2/3}\Gamma(1/3)=\frac{1}{3}24^{1/3}\,\Gamma(1/3)$ and large enough n,*

$$\mathbb{E}V_0^{max}\le C\frac{n^{2/3}}{(b-1)^{1/3}}. \tag{34}$$

Whenever the excess ℓ is fixed, that is $\ell=O(1)$, the methods developed here for unicyclic components can be generalized, using analytical tools such as those in [12]. In fact, using approach similar to that of Wright [29] in order to obtain the exact EGFs of ℓ-components (see [1]) we can prove that H_ℓ behaves like $\frac{w_\ell(b-1)^{2\ell}}{T(z)^\ell\theta^{3\ell}}$ where the w_ℓ are Wright's constants (see [18]), i.e. $w_1=5/24$, $w_2=5/16$ and

$$2\ell w_\ell = 3\ell(\ell-1)w_{\ell-1}+3\sum_{s=1}^{\ell-2}s(\ell-1-s)w_s w_{\ell-1-s},\quad \ell\ge 3. \tag{35}$$

Remark 3.4. Note that the sequences (w_ℓ) and (λ_ℓ) satisfy $w_\ell \sim \frac{\lambda_\ell}{3\ell} \sim \frac{3^\ell}{2^\ell}\frac{(\ell-1)!}{2\pi}$ as shown in [4,18,19].

Theorem 3.5.

$$\mathbb{E}V_1 \sim \frac{41\sqrt{\pi}\,\Gamma(5/6)}{3^{1/6}\,16}\frac{n^{2/3}}{(b-1)^{1/3}}. \tag{36}$$

For any fixed excess $\ell > 1$, we have

$$\mathbb{E}V_\ell \sim \sqrt{\pi}\,\ell w_\ell(2\ell+1)\frac{3^{\ell/2+4/3}\Gamma\left(\ell/2+1/3\right)}{2\Gamma\left(3\ell/2+3/2\right)}\frac{n^{2/3}}{(b-1)^{1/3}} \tag{37}$$

where the sequence (w_ℓ) is given by (35).

Proof. The expected number of vertices added to already existing ℓ-components can be computed using the same ideas as for unicyclic components. This time, the constructions are built with a rooted ℓ-component and $(b-1)$ unordered rooted hypertrees. Using the same trick as for (33), by means of generating functions we find the desired expectation as follows

$$\begin{aligned}&((b-1)k-\ell)!\left[z^{(b-1)k-\ell}\right](\vartheta_z H_\ell(z))\left(\vartheta_z\frac{T(z)^{(b-1)}}{(b-1)!}\right)\\&\quad\sim((b-1)k-\ell)!\left[z^{(b-1)k-\ell}\right]3\ell(b-1)^{2\ell+2}w_\ell\frac{1}{T(z)^\ell\theta^{3\ell+3}}.\end{aligned} \tag{38}$$

After some algebra mainly using (5), (10), (28) and (38) we obtain that the expected number of vertices added to already existing ℓ-components is about

$$\begin{aligned}&\sum_{k=0}^{(n-\ell)/(b-1)}\frac{(b-1)^{2\ell+1}}{n^\ell}\frac{3\sqrt{\pi}\,\ell w_\ell}{2^{3\ell/2+1}\Gamma(3\ell/2+3/2)}\exp\left(-\frac{(b-1)^4k^3}{24n^2}\right)k^{3\ell/2}\\&\quad\sim\sqrt{\pi}\,\ell w_\ell\frac{3^{\ell/2+1/3}\Gamma\left(\ell/2+1/3\right)}{\Gamma\left(3\ell/2+3/2\right)}\frac{n^{2/3}}{(b-1)^{1/3}}.\end{aligned} \tag{39}$$

Next, new ℓ-components are created via two kinds of transition: $p\to\ell$ with $-1\le p<\ell$ as in case (i) or $\oplus_i p_i\to\ell$ with $-1\le p_i<\ell$ as in case (ii). For the transition $p\to\ell$, the last added edge surrounds s vertices from a p-component and $b-s$ vertices of $b-s$ distinct hypertrees with $s=\ell-p+1$. The corresponding EGF is given by

$$\left(\frac{z^s}{s!}\frac{\mathrm{d}^s}{\mathrm{d}z^s}H_p(z)\right)\times\left(\frac{T(z)^{b-s}}{(b-s)!}\right). \tag{40}$$

It is easily seen that $\frac{z^s}{s!}\frac{\mathrm{d}^s}{\mathrm{d}z^s}H_p(z)$ behaves as $(b-1)^{2p+s}(3p)(3p+2)\cdots(3p+2s-2)/s!\;w_p/\theta^{3p+2s}$. Thus, the main terms are $\frac{\vartheta_z^2-\vartheta_z}{2}H_{\ell-1}(z)\frac{T(z)^{b-2}}{(b-2)!}$. Hence, if $p=\ell-1>0$ and $s=2$ then (40) behaves as

$$\frac{(b-1)^{2\ell+1}}{T(z)^\ell\theta^{3\ell+1}}\left(3(\ell-1)(3\ell-1)\frac{w_{\ell-1}}{2}\right). \tag{41}$$

If $p=\ell-1=0$, $s=2$ then (40) behaves simply as

$$\frac{(b-1)^3}{T(z)\theta^4}. \tag{42}$$

For $\ell>1$, the expected number of vertices added to create new ℓ-components via the transition $p\to\ell$ is then related to the EGF

$$\frac{3}{2}(b-1)^{2\ell+2}(\ell-1)(3\ell-1)(3\ell+1)w_{\ell-1}\frac{1}{T(z)\theta^{3\ell+3}} \tag{43}$$

where the operator ϑ_z has been applied on the EGF given by (41) which carry the main contribution $(p=\ell-1)$ of this kind of transition. If $\ell=1$, instead of (43) we have

$$\frac{4(b-1)^4}{T(z)\theta^6}. \tag{44}$$

As for (38) and (39), we then find that the expected number of vertices involved in the creations of new ℓ-components is about

$$\sqrt{\pi}(\ell-1)(9\ell^2-1)w_{\ell-1}\frac{3^{\ell/2+1/3}\Gamma\left(\ell/2+1/3\right)}{2\Gamma\left(3\ell/2+3/2\right)}\frac{n^{2/3}}{(b-1)^{1/3}} \tag{45}$$

for $\ell > 1$ and

$$2\frac{\sqrt{\pi}}{3^{1/6}}\Gamma(5/6)\frac{n^{2/3}}{(b-1)^{1/3}}, \quad (\ell = 1). \tag{46}$$

For transitions $\oplus_i p_i \to \ell$, the main contributions correspond to a set of $(b-2)$ rooted hypertrees, a rooted $(\ell-1-s)$-component and a rooted s-component (with s varying from 0 to $\ell - 1$):

$$\frac{1}{2}\sum_{s=0}^{\ell-1}(\ell-1-s)\oplus s\oplus \underbrace{-1\oplus -1\oplus\cdots\oplus -1}_{(b-2)\text{ hypertrees}}. \tag{47}$$

The corresponding EGF is given by

$$\frac{1}{2}\sum_{s=0}^{\ell-1}(\vartheta_z H_{\ell-1-s})\,(\vartheta_z H_s)\left(\frac{T(z)^{b-2}}{(b-2)!}\right). \tag{48}$$

For $\ell > 1$ using (28), it can be shown that (48) behaves as

$$\frac{3(\ell-1)(b-1)^{2\ell+1}w_{\ell-1}}{2T(z)^{\ell}\theta^{3\ell+1}} + \frac{1}{2}\sum_{s=1}^{\ell-2}\frac{9(b-1)^{2\ell+1}s(\ell-1-s)w_s w_{\ell-1-s}}{T(z)^{\ell}\theta^{3\ell+1}}$$
$$= \Big(6\ell w_\ell - 3(3\ell-1)(\ell-1)w_{\ell-1}\Big)\frac{(b-1)^{2\ell+1}}{T(z)^{\ell}\theta^{3\ell+1}}, \quad (\ell > 1). \tag{49}$$

The expected number of vertices used to create new ℓ-components via transitions $\oplus_i p_i \to \ell$ is related to the EGF

$$\Big(6\ell w_\ell - 3(3\ell-1)(\ell-1)w_{\ell-1}\Big)(3\ell+1)\frac{(b-1)^{2\ell+2}}{T(z)^{\ell}\theta^{3\ell+3}}, \quad (\ell > 1). \tag{50}$$

Thus, we find the expected number of vertices involved in the creations of new ℓ-components via the transitions $\oplus_i p_i \to \ell$

$$\sqrt{\pi}\Big(2\ell w_\ell - (3\ell-1)(\ell-1)w_{\ell-1}\Big)(3\ell+1)\frac{3^{\ell/2+1/3}\Gamma(\ell/2+1/3)}{\Gamma(3\ell/2+3/2)}\frac{n^{2/3}}{(b-1)^{1/3}}. \tag{51}$$

If $\ell = 1$, instead of (49) we simply have $\frac{(b-1)^3}{8\theta^4}$. In the same vein as for (44) and (46) from these constructions, the expected number of vertices involved in the birth of 1-components is about

$$\frac{\sqrt{\pi}}{3^{1/6}4}\Gamma(5/6)\frac{n^{2/3}}{(b-1)^{1/3}}, \quad (\ell = 1). \tag{52}$$

Summing (39), (45) and (51) we obtain (37). Similarly if $\ell = 1$, summing (39) with $\ell = 1$, (46), (52) we get (36). □

4. Multicyclic components with extremal complexities

In this section, we turn on the birth and growth of components with higher complexities that is for excess tending to infinity with the number of the vertices. First, we will compute the expectations of the number of creations of ℓ-components for $\ell \geq 1$. To this purpose, we need several intermediate lemmas. Define $h_n(\xi, \beta)$ as follows

$$\frac{1}{T(z)^{\xi}\left(1-\frac{T(z)^{b-1}}{(b-2)!}\right)^{3\xi+\beta}} = \sum_{n\geq 0} h_n(\xi,\beta)\frac{z^n}{n!}. \tag{53}$$

The following lemma is an application of the saddle point method [8,12] which is well suited to cope with our analysis:

Lemma 4.1. *Let $\xi \equiv \xi(n)$ be such that $\xi(b-1) \to 0$ but $\xi(b-1)n \to \infty$ and let β be a fixed number. Then $h_n(\xi n, \beta)$ defined in (53) satisfies*

$$h_n(\xi n,\beta) = \frac{n!}{\sqrt{2\pi n\Big(b-1\Big)}\Big((b-1)!\Big)^{\frac{\xi n+n}{b-1}}}\Big(1-(b-1)u_0\Big)^{(1-\beta)}$$
$$\times \exp\left(n\Phi(u_0)\right)\left(1+O\left(\sqrt{\frac{\xi}{b-1}}\right)+O\left(\frac{1}{\xi(b-1)n}\right)\right), \tag{54}$$

where

$$\Phi(u) = u - \left(\frac{\xi+1}{b-1}\right)\ln u - 3\,\xi\,\ln(1-(b-1)u)$$

$$u_0 = \frac{3\xi b - 2\xi + 2 - \sqrt{\Delta}}{2(b-1)} \quad \textit{with } \Delta = 9\,\xi^2 b^2 - 12\,\xi^2 b + 12\,\xi b + 4\,\xi^2 - 12\,\xi. \tag{55}$$

Proof. One can start with Cauchy's integral formula. Note that the radius of convergence of the series $T(z)$ is given by $\sqrt[(b-1)]{(b-2)!}\,\exp(-1/(b-1))$. As for Lagrange inversion, we make the substitution $u = T(z)^{(b-1)}/(b-1)!$ and get successively

$$T(z) = \sqrt[(b-1)]{(b-1)!u}, \qquad z = \sqrt[(b-1)]{(b-1)!\,u}\,e^{-u} \quad \text{and}$$

$$dz = \left(\frac{1}{(b-1)\,u} - 1\right)\left((b-1)!\,u\right)^{\frac{1}{(b-1)}} e^{-u} du. \tag{56}$$

From the Cauchy integral formula, we then obtain

$$h_n(\xi\,n,\beta) = \frac{n!}{2\pi i\left((b-1)!\right)^{(\xi\,n+n)/(b-1)}} \oint \frac{(1-(b-1)u)^{1-\beta}}{(b-1)\,u}\,\exp(n\Phi(u))\,du, \tag{57}$$

where $\Phi(u) = u - \left(\frac{\xi+1}{b-1}\right)\ln u - 3\,\xi\,\ln(1-(b-1)u)$. The big power in the integrand, *viz.* $\exp(n\Phi(u))$, suggests us to use the saddle point method. Investigating the roots of $\Phi'(u) = 0$, we find two saddle points, $u_0 = \frac{3\xi b - 2\xi + 2 - \sqrt{\Delta}}{2(b-1)}$ and $u_1 = \frac{3\xi b - 2\xi + 2 + \sqrt{\Delta}}{2(b-1)}$ with $\Delta = 9\,\xi^2 b^2 - 12\,\xi^2 b + 12\,\xi\,b + 4\,\xi^2 - 12\,\xi$. Moreover, we have $\Phi''(u) = \frac{\xi+1}{(b-1)u^2} + 3\,\frac{\xi(-b+1)^2}{(1-(b-1)u)^2}$ so that for $u \notin \{0, 1/(b-1)\}$, $\Phi''(u) > 0$. The main point of the application of the saddle point method here is that $\Phi'(u_0) = 0$ and $\Phi''(u_0) > 0$, hence $n\Phi(u_0\exp(i\tau))$ is well approximated by $n\Phi(u_0) - n{u_0}^2\Phi''(u_0)\frac{\tau^2}{2}$ in the vicinity of $\tau = 0$. If we integrate (57) around a circle passing vertically through $u = u_0$ in the z-plane, we obtain

$$h_n(\xi n,\beta) = \frac{n!}{2\pi\left((b-1)!\right)^{(\xi n+n)/(b-1)}} \int_{-\pi}^{\pi} \frac{\left(1-(b-1)u_0 e^{i\tau}\right)^{1-\beta}}{(b-1)}\,\exp\left(n\Phi(u_0 e^{i\tau})\right) d\tau \tag{58}$$

where

$$\Phi(u_0 e^{i\tau}) = u_0\cos\tau + iu_0\sin\tau - \frac{\xi+1}{b-1}\ln u_0 - i\frac{\xi+1}{b-1}\tau - 3\xi\,\ln(1-(b-1)u_0 e^{i\tau}). \tag{59}$$

Denoting by $\mathfrak{Re}(z)$ the real part of z, if $f(\tau) = \mathfrak{Re}(\Phi(u_0 e^{i\tau}))$ we have

$$f(\tau) = u_0\cos\tau - \frac{\xi+1}{b-1}\ln u_0 - 3\xi\,\ln u_0 - 3\xi\,\ln(b-1) - \frac{3\xi}{2}\ln\left(1 + \frac{1}{(b-1)^2 u_0^2} - \frac{2\cos\tau}{(b-1)u_0}\right). \tag{60}$$

It comes

$$f'(\tau) = \frac{d}{d\tau}\mathfrak{Re}(h(u_0 e^{i\tau})) = -u_0\sin\tau - \frac{3\xi\,\sin\tau}{u_0(b-1) + \frac{1}{(b-1)u_0} - 2\cos\tau}. \tag{61}$$

Therefore, if $\tau = 0$ $f'(\tau) = 0$. Also, $f(\tau)$ is a symmetric function of τ and in $[-\pi, -\tau_0] \cup [\tau_0, \pi]$, for any given $\tau_0 \in (0,\pi)$, and $f(\tau)$ takes its maximum value for $\tau = \tau_0$. Since $|\exp(\Phi(u))| = \exp(\mathfrak{Re}(\Phi(u)))$, when splitting the integral in (58) into three parts, viz. "$\int_{-\pi}^{-\tau_0} + \int_{-\tau_0}^{\tau_0} + \int_{\tau_0}^{\pi}$", we know that it suffices to integrate from $-\tau_0$ to τ_0, for a convenient value of τ_0, because the others can be bounded by the magnitude of the integrand at τ_0. In fact, we have $\Phi(u_0 e^{i\theta}) = \Phi(u_0) + \sum_{p\geq 2}\phi_p(e^{i\theta}-1)^p$ with $\phi_p = \frac{{u_0}^p}{p!}\Phi^{(p)}(u_0)$. We easily compute $\Phi^{(p)}(u_0) = (-1)^p(p-1)!\left(\frac{\xi+1}{(b-1)u_0{}^p} + \frac{3\xi(1-b)^p}{(1-(b-1)u_0)^p}\right)$, for $p \geq 2$. Whenever $\xi b \to 0$, we have

$$(b-1)u_0 = 1 - \sqrt{3\,(b-1)\,\xi} + (3/2\,b-1)\,\xi + O\left(b^{3/2}\xi^{3/2}\right). \tag{62}$$

Therefore, we obtain after a bit of algebra

$$|\phi_p| \leq O\left(\frac{2^p}{\xi^{\frac{p}{2}-1}(b-1)^{\frac{p}{2}}}\right), \quad \text{as } \xi(b-1) \to 0. \tag{63}$$

On the other hand,

$$|e^{i\tau} - 1| = \sqrt{2(1-\cos\tau)} < \tau, \quad \tau > 0. \tag{64}$$

Thus, the summation can be bounded for values of τ and ξ such that $\tau \to 0$, $\xi b \to 0$ ($\xi \to 0$) but $\frac{\tau}{\sqrt{\xi}} \to 0$ and we have

$$\Big| \sum_{p\geq 4} \phi_p (e^{i\tau}-1)^p \Big| \leq \sum_{p\geq 4} |\phi_p \tau^p| \leq \sum_{p\geq 4} O\Big(\frac{2^p\tau^p}{\xi^{\frac{p}{2}-1}(b-1)^{\frac{p}{2}}}\Big) = O\Big(\frac{\tau^4}{\xi(b-1)}\Big). \tag{65}$$

It follows that for $\tau \to 0$, $\xi(b-1) \to 0$ and $\frac{\tau}{\sqrt{\xi(b-1)}} \to 0$, $\Phi(u_0 e^{i\tau})$ can be rewritten as

$$\begin{aligned}\Phi(u_0e^{i\tau}) \;&= \Phi(u_0) - \frac{1}{(b-1)}\left(1 - \frac{\sqrt{\xi}}{\sqrt{3(b-1)}}\frac{3b-4}{2} + \frac{(9b^2-12b+4)}{12(b-1)}\xi\right)\tau^2 \\ &\quad - \frac{i}{(b-1)}\left(1 - \frac{(3b-4)\sqrt{\xi}}{2\sqrt{3(b-1)}} + \frac{(9b^2-12b+4)}{12(b-1)}\,\xi\right)\tau^3 + O\left(\frac{\tau^4}{\xi(b-1)}\right).\end{aligned} \tag{66}$$

Therefore, if $\xi(b-1) \to 0$ but $\xi(b-1)n \to \infty$, if we let $\tau_0 = \frac{\ln n}{\sqrt{n u_0^2 \Phi''(u_0)}}$ (with $u_0^2\Phi''(u_0) = \frac{2}{b-1} + O(\sqrt{\xi(b-1)})$) we can remark (as already said) that it suffices to integrate (58) from $-\tau_0$ to τ_0, using the magnitude of the integrand at τ_0 to bound the resulting error. The rest of the proof is now standard application of the saddle point method (see for instance De Bruijn [8, Chapters 5 & 6]) leading to (54). □

Lemma 4.2. *Let $a = k(b-1) - \ell$. Denote by $c_\ell(a,k)$ the number of ways to label an ℓ-component with a vertices such that one edge – whose deletion will suppress the occurrence of the created ℓ-component – is distinguished among the others. As ℓ tends to ∞ with the number of vertices a such that $\ell = o\left(\sqrt[3]{\frac{a}{b}}\right)$ then*

$$c_\ell(a,k) \sim a!\left[z^a\right]\left(\frac{(b-1)^{2\ell+1}\lambda_\ell}{T(z)^\ell\theta^{3\ell+1}}\right), \tag{67}$$

where $\theta = 1 - T(z)^{b-1}/(b-2)!$ and the sequence (λ_ℓ) is defined with (16).

Proof. The main ideas are as follows. The inequalities given by Eq. (15) in Lemma 2.4 tell us that when ℓ is large, the main constructions that lead to the creations of new ℓ-components arise from transitions $(\ell-1) \to \ell$ and $s \oplus (\ell-1-s) \oplus -1 \oplus \cdots \oplus -1 \to \ell$ (with $b-2$ hypertrees and $s \in [0, \ell-1]$). Such constructions are respectively counted by $(\frac{\vartheta_z^2-\vartheta_z}{2}H_{\ell-1}(z)) \times \frac{T(z)^{b-2}}{(b-2)!}$ and $(\frac{1}{2}\sum_{s=0}^{\ell-1}\vartheta_z H_s(z)\ \vartheta_z H_{\ell-1-s}(z)) \times \frac{T(z)^{b-2}}{(b-2)!}$. Using (15) with (54), one can show that the coefficient of the sum of these EGFs has the same asymptotical behaviour as

$$\frac{(3\ell-1)(b-1)^{2\ell+1}\lambda_{\ell-1}}{2\,T(z)^\ell\theta^{3\ell+1}} \tag{68}$$

and as

$$\frac{(b-1)^{2\ell+1}}{2\,T(z)^\ell\,\theta^{3\ell+1}}\sum_{s=0}^{\ell-1}\lambda_s\,\lambda_{\ell-s-1}. \tag{69}$$

Summing (68) and (69) and using the definition of the sequence λ_ℓ given by (16) we obtain (67). □

We then have the following result giving the average number of dynamic ℓ-components as ℓ is large:

Theorem 4.3. *As $\ell \to \infty$ with n but such that $\ell = o\left(\sqrt[3]{n}\right)$, the expected number of creations of ℓ-component is ~ 1.*

Proof. Using Lemmas 2.2, 2.3, 4.1 and 4.2, we compute that the average number of dynamic ℓ-components during a random hypergraph process is about

$$\frac{3\ell^{1/2}}{2\pi^{1/2}}\frac{(b-1)^{2\ell}}{n^\ell}\left(\frac{e}{12\ell}\right)^{\ell/2}\sum_{k=0}^{(n+\ell)/(b-1)} k^{3\ell/2-1}\exp\left(-\frac{(b-1)^4k^3}{24\,n^2}\right) \sim 1. \quad \square \tag{70}$$

For the number of static ℓ-components, we get

Theorem 4.4. *As ℓ and n are both large and $\ell = o\left(\sqrt[3]{n}\right)$, the expected number of static ℓ-components is about*

$$\left(\frac{2n}{3(b-1)^5\ell}\right)^{1/3}. \tag{71}$$

Proof. As for (25), the EGF corresponding to ℓ-components swallowing $(b-1)$ unordered hypertrees is $\frac{T(z)^{b-1}}{(b-1)!}\,\vartheta_z H_\ell(z)$ which behaves as $(b-1)(1-\theta)\times\frac{(b-1)^{2\ell}\lambda_\ell}{T(z)^\ell\theta^{3\ell+2}}$. Therefore, the same ideas as for Theorem 4.3 apply. □

Theorem 4.5. *Let V_ℓ be the number of vertices that at some stage of the random graph process belong to an ℓ-component. As $\ell, n\to\infty$ but $\ell = o\left(\sqrt[3]{n}\right)$,*

$$\mathbb{E}V_\ell \sim \frac{12^{1/3}\ell^{1/3}n^{2/3}}{(b-1)^{1/3}}. \tag{72}$$

Proof. As for (38), the expected number of vertices added to already existing ℓ-components can be found by means of EGF, viz.

$$\vartheta_z H_\ell(z)\left(\vartheta_z\frac{T(z)^{b-1}}{(b-1)!}\right)\sim\frac{(b-1)^{2\ell+1}\lambda_\ell}{T(z)^\ell\theta^{3\ell+3}}. \tag{73}$$

Apart a factor "$1/\theta^2$", (73) is similar to (67). Thus, the expected number of vertices added to ℓ-components stated in the theorem follows. Again the expectation of the number of vertices added to ℓ-components can be computed via the derivative of the EGFs

$$\frac{T(z)^{b-2}}{(b-2)!}\left(\frac{\vartheta_z^2-\vartheta_z}{2}\,H_{\ell-1}(z)+\frac{1}{2}\sum_{s=0}^{\ell-1}\vartheta_z H_s(z)\vartheta_z H_{\ell-s-1}(z)\right)\sim\left(\frac{(b-1)^{2\ell+1}\lambda_\ell}{T(z)^\ell\theta^{3\ell+1}}\right) \tag{74}$$

which behaves as

$$\frac{3\ell(b-1)^{2\ell+2}\lambda_\ell}{T(z)^\ell\theta^{3\ell+3}}. \tag{75}$$

leading to the result. □

5. Conclusion

In this paper, we have studied the birth and growth in complexity of connected components in an evolving hypergraph. Using enumerative and analytic combinatorics with the methods initiated by Janson in [16,17], we have shown how to quantify asymptotic properties of random hypergraphs. Amongst other things, we study complex components that increase their complexity by receiving new edges and/or by merging/swallowing other components.

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