Circular chromatic number of signed graphs

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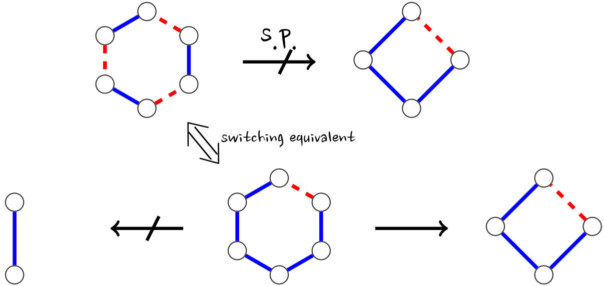
Homomorphism of signed graphs

- A homomorphism of signed graph (G, σ) to a signed graph (H, π) is a mapping $\varphi : V(G) \to V(H)$ such that the adjacency and the signs of the closed walks are preserved. If there exists one, we write $(G, \sigma) \to (H, \pi)$.
- A homomorphism of (G, σ) to (H, π) is edge-sign preserving if it, furthermore, preserves the signs of edges. If there exists one, we write $(G, \sigma) \xrightarrow{s.p.} (H, \pi)$.
- $(G, \sigma) \to (H, \pi) \Leftrightarrow \exists \sigma' \equiv \sigma, (G, \sigma') \xrightarrow{s.p.} (H, \pi)$.

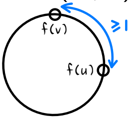
Examples: homomorphism of signed graphs



Circular coloring of signed graphs

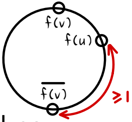
Given a signed graph (G, σ) (with no positive loop) and a real number r, a circular r-coloring of (G, σ) is a mapping $f : V(G) \to C^r$ such that for each positive edge uv of (G, σ),

$$d_{C^r}(f(u), f(v)) \geq 1,$$



and for each negative edge uv of (G, σ),

$$d_{C^r}(f(u), \overline{f(v)}) \geq 1.$$



The circular chromatic number of (G, σ) is defined as

$$\chi_c(G, \sigma) = \inf\{r \geq 1 : (G, \sigma) \text{ admits a circular } r\text{-coloring}\}.$$

Signed circular clique

For integers $p \geq 2q > 0$ such that p is even, the signed circular clique $K^s_{p;q}$ has vertex set $[p] = \{0, 1, \ldots, p-1\}$, in which

- ij is a positive edge if $q \leq |i-j| \leq p-q$;
- ij is a negative edge if $|i-j| \leq \frac{p}{2} - q$ or $|i-j| \geq \frac{p}{2} + q$.

Let $\hat{K}^s_{p;q}$ be the signed subgraph of $K^s_{p;q}$ induced by vertices $\{0, 1, \ldots, \frac{p}{2} - 1\}$.

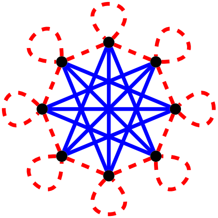


Figure: $K^s_{8;3}$

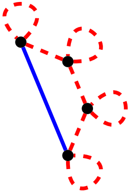


Figure: $\hat{K}^s_{8;3}$

Signed circular clique

The circular chromatic number of (G, σ) is

$$\chi_c(G,\sigma) = \inf\{\frac{p}{q} : p \text{ is even and } (G,\sigma) \xrightarrow{s.p.} K^s_{p;q}\}$$
$$= \inf\{\frac{p}{q} : p \text{ is even and } (G,\sigma) \to \hat{K}^s_{p;q}\}$$

For a non-zero integer ℓ, we denote by C_ℓ the cycle of length $|\ell|$ whose sign agrees with the sign of ℓ. We have

$$\chi_c(C_{-2k}) = \frac{4k}{2k-1} \text{ and } \chi_c(C_{2k+1}) = \frac{2k+1}{k}.$$

Given a class $\mathcal{C}$ of signed graphs,

$$\chi_c(\mathcal{C}) = sup\{\chi_c(G,\sigma) \mid (G,\sigma) \in \mathcal{C}\}.$$

- the class of signed k-chromatic graphs, denoted $\mathcal{SK}_k$
- the class of signed bipartite planar graphs, denoted $\mathcal{SBP}$,
- the class of signed d-degenerate simple graphs, denoted $\mathcal{SD}_d$,
- the class of signed planar simple graphs, denoted $\mathcal{SP}$.

Signed k-chromatic graph

Proposition [R. Naserasr, W. and X. Zhu 2021]

For any positive integer $k \geq 2$, $\chi_c(\mathcal{SK}_k) = 2k$.

Theorem [R. Naserasr, W. and X. Zhu 2021]

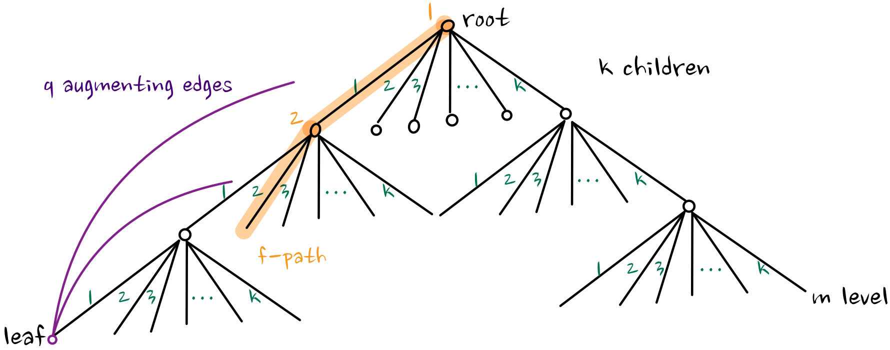
For any integers $k, g \geq 2$ and any $\epsilon > 0$, there is a graph G of girth at least g and a signature σ satisfying that

$$\chi(G) = k \text{ and } \chi_c(G, \sigma) > 2k - \epsilon.$$

We will prove that for any integer p, there is a signed graph (G, σ) for which the followings hold:

- G is of girth at least g and is k-colorable.
- (G, σ) is not circular $\frac{2kp}{p+1}$-colorable.

Tool: q-augmented k-ary tree of girth at least g



Lemma [N. Alon, A. Kostochka, B. Reiniger, D. West and X. Zhu 2016]

For any positive integers $k, q, g \geq 2$, there exists a (k, q, g)-graph.

Idea of the construction

- H: $(2kp, k, 2kg)$-graph.
- ϕ: a standard $2kp$-labeling of the edges of T.
- $\ell(v)$: the level of v, i.e., the distance from v to the root vertex in T. Let $\theta(v) = \ell(v) (\mathrm{mod}\ k)$.

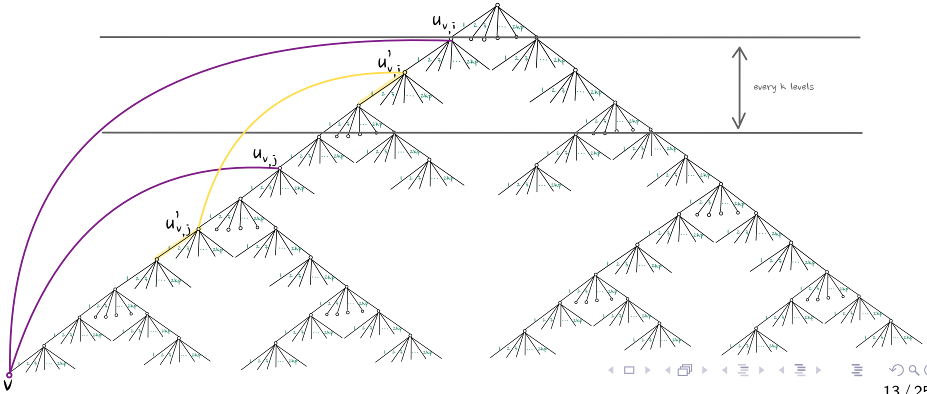
For each leaf v of T,

- let $u_{v,1}, u_{v,2}, \ldots, u_{v,k}$ be the vertices on P_v that are connected to v by augmenting edges.
- let $u'_{v,i} \in P_v$ be the closest descendant of $u_{v,i}$ with $\theta(u'_{v,i}) = i$.
- let $e_{v,i}$ be the edge connecting $u'_{v,i}$ to its child on P_v.
- let $A_{v,i} = \{\phi(e_{v,i}), \phi(e_{v,i}) + 1, \ldots, \phi(e_{v,i}) + p\}$, $B_{v,i} = \{a + kp : a \in A_{v,i}\}$ and $C_{v,i} = A_{v,i} \cup B_{v,i}$.

Construction of the signed graph (G, σ)

Note that $B_{v,i}$ is a kp-shift of $A_{v,i}$. Two possibilities:

- $A_{v,i} \cap A_{v,j} \neq \emptyset$, $d_{(\text{mod } 2kp)}(\phi(e_{v,i}), \phi(e_{v,j})) \leq p$.
- $A_{v,i} \cap B_{v,j} \neq \emptyset$, $d_{(\text{mod } 2kp)}(\phi(e_{v,i}), \overline{\phi(e_{v,j})}) \leq p$.



Signed bipartite planar graphs

Proposition [R. Naserasr, W. and X. Zhu 2021]

$\chi_c(\mathcal{SBP}) = 4$.

Let Γ_1 be a positive 2-path connecting u_1 and v_1. For $i \geq 2$,

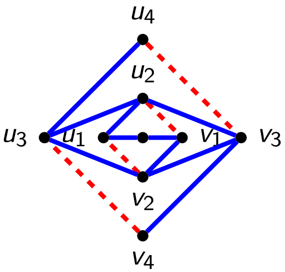


Figure: Γ_4

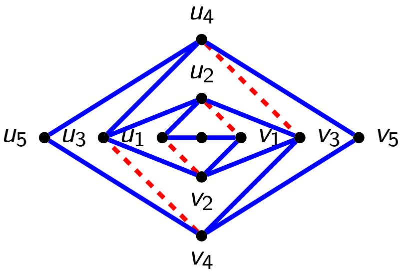


Figure: Γ_5

$\chi_c(\Gamma_n) = \frac{4n}{n+1}$

Results on signed bipartite planar graphs of girth ≥ 6

- $\chi_c(\mathcal{SBP}_6) \leq 3$. [R. Naserasr and W. 2021+]
  It's a corollary of the result that every signed bipartite planar graph of negative girth 6 admits a homomorphism to $(K_{3,3}, M)$.
- $\chi_c(\mathcal{SBP}_8) \leq \frac{8}{3}$. [R. Naserasr, L-A. Pham and W. 2020+]
  It's a corollary of the result that C_{-4}-critical signed graph has density $|E(G)| \geq \frac{3|V(G)|-2}{4}$.

Signed d-degenerate graphs

Proposition [R. Naserasr, W. and X. Zhu 2021]

For any positive integer d, $\chi_c(\mathcal{SD}_d) = 2\lfloor \frac{d}{2} \rfloor + 2$.

Sketch of the proof:

- First we show that every $(G, \sigma) \in \mathcal{SD}_d$ admits a circular $(2\lfloor \frac{d}{2} \rfloor + 2)$-coloring.

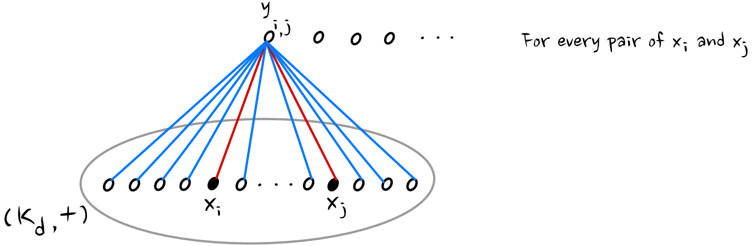
For the tightness,

- For odd integer d, we have $\chi_c(K_{d+1}, +) = d + 1$.
- For $d = 2$, we consider the signed graph Γ_n built before.
- For even integer $d \geq 4$, we construct a signed d-degenerate graph Ω_d such that $\chi_c(\Omega_d) = d + 2$.

Signed d-degenerate graphs

Proof for even $d \geq 4$

- Define a signed graph Ω_d as follows.



- Let φ be a circular r-coloring of Ω_d where $r < d + 2$. Without loss of generality, $\varphi(x_1), \ldots, \varphi(x_d)$ are cyclically ordered on C^r and assume that $d_{(\mathrm{mod}\ r)}(\varphi(x_1), \varphi(x_2))$ is maximized. We prove that there is no place for $y_{1,1+\frac{d}{2}}$.

Signed planar graphs

Proposition [R. Naserasr, W. and X. Zhu 2021]

$$4 + \frac{2}{3} \le \chi_c(\mathcal{SP}) \le 6.$$

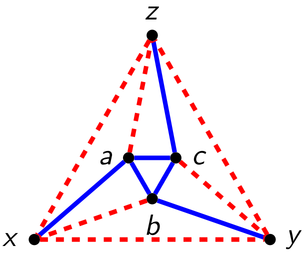


Figure: Mini-gadget (T, π)

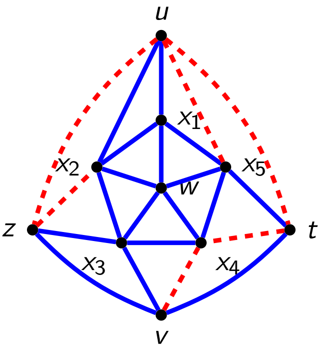


Figure: A signed Wenger Graph $\tilde{W}$

Signed planar graphs

Lemma [R. Naserasr, W. and X. Zhu 2021]

Let $r = \frac{14}{3} - \epsilon$ with $0 < \epsilon \leq \frac{2}{3}$. For any circular r-coloring ϕ of $\tilde{W}$, $d_{(\bmod r)}(\phi(u), \phi(v)) \geq \frac{4}{9}$.

Let Γ be obtained from $\tilde{W}$ by adding a negative edge uv. Let $\mathcal{I} = (\Gamma, u, v)$.

Theorem [R. Naserasr, W. and X. Zhu 2021]

Let $\Omega = K_4(\mathcal{I})$. Then Ω is a signed planar simple graph with $\chi_c(\Omega) = \frac{14}{3}$.

Results on signed planar graphs of large girth

- $\chi_c(\mathcal{SP}_4) = 4$. (By the 3-degeneracy of triangle-free planar graph)
- $\chi_c(\mathcal{SP}_7) \leq 3$. [R. Naserasr, R. Škrekovski, W. and R. Xu 2020+]
  It's a corollary of the result that every signed graph of $mad < \frac{14}{5}$ admits a homomorphism to (K_6, M).

1. Introduction
2. Results on some classes of signed graphs
   - Signed *k*-chromatic graph
   - Signed bipartite planar graphs
   - Signed d-degenerate graphs
   - Signed planar graphs
3. Discussion

Circular chromatic number of signed planar graphs

Let C_ℓ^{o+} be signed cycle of length ℓ where the number of positive edges is odd. Then $\chi_c(C_\ell^{o+}) = \frac{2\ell}{\ell-1}$.

Theorem [R. Naserasr, W. and X. Zhu 2021]

Given a positive integer ℓ and a signed graph (G, σ) satisfying $g_{ij}(G, \sigma) \geq g_{ij}(C_\ell^{o+})$,

$$\chi_c(G, \sigma) \leq \frac{2\ell}{\ell - 1} \Leftrightarrow (G, \sigma) \to C_\ell^{o+}.$$

Question

Given a positive integer ℓ, what is the smallest value $f(\ell)$ (with $f(\infty) = \infty$) such that for every signed planar graph (G, σ) satisfying $g_{ij}(G, \sigma) \geq g_{ij}(C_\ell^{o+})$ and $g_{ij}(G, \sigma) \geq f(\ell)$ for all $ij \in \mathbb{Z}_2^2$, we have $\chi_c(G, \sigma) \leq \frac{2\ell}{\ell-1}$.

Jaeger-Zhang conjecture

When $\ell = 2k+1$,

Jaeger-Zhang conjecture [C.-Q. Zhang 2002]

Every planar graph of odd-girth $f(2k+1) = 4k+1$ admits a circular $\frac{2k+1}{k}$-coloring, i.e., C_{2k+1}-coloring.

- $f(3) = 5$ [Grötzsch's theorem];
- $f(5) \leq 11$ [Z. Dvořák and L. Postle 2017][D. W. Cranston and J. Li 2020];
- $4k+1 \leq f(2k+1) \leq 6k+1$ [C. Q. Zhang 2002; L. M. Lovász, C. Thomassen, Y. Wu and C. Q. Zhang 2013];

Bipartite analogue of Jaeger-Zhang conjecture

When $\ell = 2k$,

Bipartite analogue of Jaeger-Zhang conjecture [R. Naserasr, E. Rollová and É. Sopena 2015]

Every signed bipartite planar graph of negative-girth $f(2k)$ admits a circular $\frac{4k}{2k-1}$-coloring, i.e., C_{-2k}-coloring.

- $f(4) = 8$ [R. Naserasr, L. A. Pham and W. 2020+];
- $f(2k) \leq 8k - 2$ [C. Charpentier, R. Naserasr and E. Sopena 2020].

The end. Thank you!