Circular chromatic number of signed graphs

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CanaDAM 2021

May 28, 2021
1 Introduction

2 Results on some classes of signed graphs
   - Signed $k$-chromatic graph
   - Signed bipartite planar graphs
   - Signed $d$-degenerate graphs
   - Signed planar graphs

3 Discussion
Homomorphism of signed graphs

- A **homomorphism** of signed graph \((G, \sigma)\) to a signed graph \((H, \pi)\) is a mapping \(\varphi : V(G) \rightarrow V(H)\) such that the adjacency and the signs of the closed walks are preserved. If there exists one, we write \((G, \sigma) \rightarrow (H, \pi)\).

- A homomorphism of \((G, \sigma)\) to \((H, \pi)\) is **edge-sign preserving** if it, furthermore, preserves the signs of edges. If there exists one, we write \((G, \sigma) \xrightarrow{s.p.} (H, \pi)\).

- \((G, \sigma) \rightarrow (H, \pi) \iff \exists \sigma' \equiv \sigma, (G, \sigma') \xrightarrow{s.p.} (H, \pi)\).
Examples: homomorphism of signed graphs

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Examples: homomorphism of signed graphs

\[ \text{switching equivalent} \]

\[ \text{S.P.} \]
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Circular coloring of signed graphs

Given a signed graph \((G, \sigma)\) (with no positive loop) and a real number \(r\), a circular \(r\)-coloring of \((G, \sigma)\) is a mapping \(f : V(G) \to C^r\) such that for each positive edge \(uv\) of \((G, \sigma)\),

\[
d_{C^r}(f(u), f(v)) \geq 1,
\]

and for each negative edge \(uv\) of \((G, \sigma)\),

\[
d_{C^r}(f(u), \overline{f(v)}) \geq 1.
\]

The circular chromatic number of \((G, \sigma)\) is defined as

\[
\chi_c(G, \sigma) = \inf\{r \geq 1 : (G, \sigma) \text{ admits a circular } r\text{-coloring}\}.
\]
Signed circular clique

For integers $p \geq 2q > 0$ such that $p$ is even, the signed circular clique $K_{p; q}^s$ has vertex set $[p] = \{0, 1, \ldots, p-1\}$, in which

- $ij$ is a positive edge if $q \leq |i - j| \leq p - q$;
- $ij$ is a negative edge if $|i - j| \leq \frac{p}{2} - q$ or $|i - j| \geq \frac{p}{2} + q$.

Let $\hat{K}_{p; q}^s$ be the signed subgraph of $K_{p; q}^s$ induced by vertices $\{0, 1, \ldots, \frac{p}{2} - 1\}$.

Figure: $K_{8; 3}^s$  
Figure: $\hat{K}_{8; 3}^s$
Signed circular clique

The circular chromatic number of \((G, \sigma)\) is

\[
\chi_c(G, \sigma) = \inf\{\frac{p}{q} : p \text{ is even and } (G, \sigma) \xrightarrow{s.p.} K_{p,q}^s\}
= \inf\{\frac{p}{q} : p \text{ is even and } (G, \sigma) \rightarrow \hat{K}_{p,q}^s\}
\]

For a non-zero integer \(\ell\), we denote by \(C_\ell\) the cycle of length \(|\ell|\) whose sign agrees with the sign of \(\ell\). We have

\[
\chi_c(C_{-2k}) = \frac{4k}{2k - 1} \quad \text{and} \quad \chi_c(C_{2k+1}) = \frac{2k + 1}{k}.
\]
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Discussion
Given a class $C$ of signed graphs,

$$\chi_c(C) = \sup\{\chi_c(G, \sigma) \mid (G, \sigma) \in C\}.$$  

- the class of signed $k$-chromatic graphs, denoted $SK_k$
- the class of signed bipartite planar graphs, denoted $SBP$
- the class of signed $d$-degenerate simple graphs, denoted $SD_d$
- the class of signed planar simple graphs, denoted $SP$.  

Signed $k$-chromatic graph

Proposition [R. Naserasr, W. and X. Zhu 2021]
For any positive integer $k \geq 2$, $\chi_c(SK_k) = 2k$.

Theorem [R. Naserasr, W. and X. Zhu 2021]
For any integers $k, g \geq 2$ and any $\epsilon > 0$, there is a graph $G$ of girth at least $g$ and a signature $\sigma$ satisfying that

$$\chi(G) = k \text{ and } \chi_c(G, \sigma) > 2k - \epsilon.$$ 

We will prove that for any integer $p$, there is a signed graph $(G, \sigma)$ for which the followings hold:

- $G$ is of girth at least $g$ and is $k$-colorable.
- $(G, \sigma)$ is not circular $\frac{2kp}{p+1}$-colorable.
Lemma [N. Alon, A. Kostochka, B. Reiniger, D. West and X. Zhu 2016]

For any positive integers $k, q, g \geq 2$, there exists a $(k, q, g)$-graph.
Signed $k$-chromatic graph

Idea of the construction

- $H$: $(2kp, k, 2kg)$-graph.
- $\phi$: a standard $2kp$-labeling of the edges of $T$.
- $\ell(v)$: the level of $v$, i.e., the distance from $v$ to the root vertex in $T$. Let $\theta(v) = \ell(v) \pmod{k}$.

For each leaf $v$ of $T$,

- let $u_{v,1}, u_{v,2}, \ldots, u_{v,k}$ be the vertices on $P_v$ that are connected to $v$ by augmenting edges.
- let $u'_{v,i} \in P_v$ be the closest descendant of $u_{v,i}$ with $\theta(u'_{v,i}) = i$.
- let $e_{v,i}$ be the edge connecting $u'_{v,i}$ to its child on $P_v$.
- let $A_{v,i} = \{\phi(e_{v,i}), \phi(e_{v,i}) + 1, \ldots, \phi(e_{v,i}) + p\}$,
  $B_{v,i} = \{a + kp : a \in A_{v,i}\}$ and $C_{v,i} = A_{v,i} \cup B_{v,i}$.
Construction of the signed graph \((G, \sigma)\)

Note that \(B_{v,i}\) is a \(kp\)-shift of \(A_{v,i}\). Two possibilities:

- \(A_{v,i} \cap A_{v,j} \neq \emptyset, \ d_{(\text{mod } 2kp)}(\phi(e_{v,i}), \phi(e_{v,j})) \leq p.\)
- \(A_{v,i} \cap B_{v,j} \neq \emptyset, \ d_{(\text{mod } 2kp)}(\phi(e_{v,i}), \phi(e_{v,j})) \leq p.\)
Signed bipartite planar graphs

Proposition [R. Naserasr, W. and X. Zhu 2021]

\[ \chi_c(SBP) = 4. \]

Let \( \Gamma_1 \) be a positive 2-path connecting \( u_1 \) and \( v_1 \). For \( i \geq 2 \),

\[ \chi_c(\Gamma_n) = \frac{4n}{n+1} \]
Signed bipartite planar graphs

Results on signed bipartite planar graphs of girth $\geq 6$

- $\chi_c(SBP_6) \leq 3$. [R. Naserasr and W. 2021+]
  It’s a corollary of the result that every signed bipartite planar graph of negative girth 6 admits a homomorphism to $(K_{3,3}, M)$.

- $\chi_c(SBP_8) \leq \frac{8}{3}$. [R. Naserasr, L-A. Pham and W. 2020+]
  It’s a corollary of the result that $C_{-4}$-critical signed graph has density $|E(G)| \geq \frac{3|V(G)|-2}{4}$.
Signed d-degenerate graphs

Proposition [R. Naserasr, W. and X. Zhu 2021]

For any positive integer $d$, $\chi_c(SD_d) = 2\lfloor \frac{d}{2} \rfloor + 2$.

Sketch of the proof:

- First we show that every $(G, \sigma) \in SD_d$ admits a circular $(2\lfloor \frac{d}{2} \rfloor + 2)$-coloring.

For the tightness,

- For odd integer $d$, we have $\chi_c(K_{d+1}, +) = d + 1$.
- For $d = 2$, we consider the signed graph $\Gamma_n$ built before.
- For even integer $d \geq 4$, we construct a signed $d$-degenerate graph $\Omega_d$ such that $\chi_c(\Omega_d) = d + 2$. 
Signed \( d \)-degenerate graphs

**Proof for even \( d \geq 4 \)**

- Define a signed graph \( \Omega_d \) as follows.

- Let \( \varphi \) be a circular \( r \)-coloring of \( \Omega_d \) where \( r < d + 2 \). Without loss of generality, \( \varphi(x_1), \ldots, \varphi(x_d) \) are cyclically ordered on \( C^r \) and assume that \( d_{(\text{mod } r)}(\varphi(x_1), \varphi(x_2)) \) is maximized. We prove that there is no place for \( y_{1,1+\frac{d}{2}} \).
Signed planar graphs

Proposition [R. Naserasr, W. and X. Zhu 2021]

\[ 4 + \frac{2}{3} \leq \chi_c(\mathcal{SP}) \leq 6. \]
Signed planar graphs

Lemma [R. Naserasr, W. and X. Zhu 2021]

Let \( r = \frac{14}{3} - \epsilon \) with \( 0 < \epsilon \leq \frac{2}{3} \). For any circular \( r \)-coloring \( \phi \) of \( \tilde{W} \),
\[
d_{(\mod r)}(\phi(u), \phi(v)) \geq \frac{4}{9}.
\]

Let \( \Gamma \) be obtained from \( \tilde{W} \) by adding a negative edge \( uv \). Let \( I = (\Gamma, u, v) \).

Theorem [R. Naserasr, W. and X. Zhu 2021]

Let \( \Omega = K_4(I) \). Then \( \Omega \) is a signed planar simple graph with \( \chi_c(\Omega) = \frac{14}{3} \).
Results on signed planar graphs of large girth

- $\chi_c(\mathcal{SP}_4) = 4$. (By the 3-degeneracy of triangle-free planar graph)
- $\chi_c(\mathcal{SP}_7) \leq 3$. [R. Naserasr, R. Škrekovski, W. and R. Xu 2020+]

It’s a corollary of the result that every signed graph of $mad < \frac{14}{5}$ admits a homomorphism to $(K_6, M)$. 
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Let \( C_{\ell}^{o+} \) be signed cycle of length \( \ell \) where the number of positive edges is odd. Then \( \chi_c(C_{\ell}^{o+}) = \frac{2\ell}{\ell-1} \).

**Theorem [R. Naserasr, W. and X. Zhu 2021]**

Given a positive integer \( \ell \) and a signed graph \((G, \sigma)\) satisfying \( g_{ij}(G, \sigma) \geq g_{ij}(C_{\ell}^{o+}) \),

\[
\chi_c(G, \sigma) \leq \frac{2\ell}{\ell-1} \iff (G, \sigma) \rightarrow C_{\ell}^{o+}.
\]

**Question**

Given a positive integer \( \ell \), what is the smallest value \( f(\ell) \) (with \( f(\infty) = \infty \)) such that for every signed planar graph \((G, \sigma)\) satisfying \( g_{ij}(G, \sigma) \geq g_{ij}(C_{\ell}^{o+}) \) and \( g_{ij}(G, \sigma) \geq f(\ell) \) for all \( ij \in \mathbb{Z}_2^2 \), we have \( \chi_c(G, \sigma) \leq \frac{2\ell}{\ell-1} \).
Jaeger-Zhang conjecture

When $\ell = 2k + 1$,

**Jaeger-Zhang conjecture** [C.-Q. Zhang 2002]

Every planar graph of odd-girth $f(2k + 1) = 4k + 1$ admits a circular $\frac{2k+1}{k}$-coloring, i.e., $C_{2k+1}$-coloring.

- $f(3) = 5$ [Grötzsch’s theorem];
- $f(5) \leq 11$ [Z. Dvořák and L. Postle 2017][D. W. Cranston and J. Li 2020];
- $4k + 1 \leq f(2k + 1) \leq 6k + 1$ [C. Q. Zhang 2002; L. M. Lovász, C. Thomassen, Y. Wu and C. Q. Zhang 2013];
Bipartite analogue of Jaeger-Zhang conjecture

When $\ell = 2k$,

Bipartite analogue of Jaeger-Zhang conjecture [R. Naserasr, E. Rollová and É. Sopena 2015]

Every signed bipartite planar graph of negative-girth $f(2k)$ admits a circular $\frac{4k}{2k-1}$-coloring, i.e., $C_{-2k}$-coloring.

- $f(4) = 8$ [R. Naserasr, L. A. Pham and W. 2020+];
- $f(2k) \leq 8k - 2$ [C. Charpentier, R. Naserasr and E. Sopena 2020].
The end. Thank you!