Circular Flow in Mono-directed Eulerian Signed Graphs

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Introduction

- Start from Jaeger’s flow conjecture
- Circular coloring of signed graphs
- Circular flow in mono-directed signed graphs
- Bipartite analog of Jaeger-Zhang conjecture

Circular flow in mono-directed Eulerian signed graphs

- Preliminaries
- Flows in Eulerian signed graphs
- Coloring of signed bipartite planar graphs

Conclusion

- Results
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Jaeger’s circular flow conjecture

Tutte’s 3-flow conjecture
Every 4-edge-connected graph admits a nowhere-zero 3-flow.

Jaeger’s circular flow conjecture
Every 4k-edge-connected graph admits a circular \( \frac{2k+1}{k} \)-flow.

- It has been disproved for \( k \geq 3 \) [M. Han, J. Li, Y. Wu, and C.Q. Zhang 2018];
- It has been verified for the 6k-edge-connectivity [L. M. Lovász, C. Thomassen, Y. Wu, and C.Q. Zhang 2013].
Duality: circular flow and circular coloring

Let $p$ and $q$ be two positive integers satisfying $p \geq 2q$.

A circular $\frac{p}{q}$-flow in a graph $G$ is a pair $(D, f)$ where $D$ is an orientation on $G$ and $f : E(G) \to \mathbb{Z}$ satisfying that $q \leq |f(e)| \leq p - q$ and for each vertex $v$, \[
\sum_{(v, w) \in D} f(vw) - \sum_{(u, v) \in D} f(uv) = 0.
\]

A circular $\frac{p}{q}$-coloring of a graph $G$ is a mapping $\varphi : V(G) \to \{1, 2, \ldots, p\}$ such that $q \leq |f(u) - f(v)| \leq p - q$ for each edge $uv \in E(G)$.


A plane graph $G$ admits a circular $\frac{p}{q}$-coloring if and only if its dual graph $G^*$ admits a circular $\frac{p}{q}$-flow.
Jaeger-Zhang Conjecture [C.-Q. Zhang 2002]

Every planar graph of odd-girth at least $4k + 1$ admits a circular $\frac{2k+1}{k}$-coloring.

- $k = 1$: Grötzsch’s theorem;
- $k = 2$: verified for odd-girth 11 [Z. Dvořák and L. Postle 2017; D. Cranston and J. Li 2020];
- $k = 3$: verified for odd-girth 17 [D. Cranston and J. Li 2020; L. Postle and E. Smith-Roberge 2022];
- $k \geq 4$:
  - verified for odd-girth $8k - 3$ [X. Zhu 2001];
  - verified for odd-girth $\frac{20k-2}{3}$ [O.V. Borodin, S.-J. Kim, A.V. Kostochka and D.B. West 2002];
  - verified for odd-girth $6k + 1$ [L. M. Lovász, C. Thomassen, Y. Wu and C. Q. Zhang 2013].
Signed graphs

- A signed graph \((G, \sigma)\) is a graph \(G\) together with an assignment \(\sigma : E(G) \to \{+, -\}\).
- The sign of a closed walk (especially, a cycle) is the product of signs of all the edges in it.
- A switching at vertex \(v\) is to switch the signs of all the edges incident to this vertex.

**Theorem [T. Zaslavsky 1982]**

Signed graphs \((G, \sigma)\) and \((G, \sigma')\) are switching equivalent if and only if they have the same set of negative cycles.

- The negative-girth of a signed graph is the length of a shortest negative cycle.
Homomorphism of signed graphs

- A **homomorphism** of \((G, \sigma)\) to \((H, \pi)\) is a mapping \(\varphi\) from \(V(G)\) and \(E(G)\) to \(V(H)\) and \(E(H)\) respectively, such that the adjacency, the incidence and the signs of closed walks are preserved. If there exists one, we write \((G, \sigma) \rightarrow (H, \pi)\).

- A homomorphism of \((G, \sigma)\) to \((H, \pi)\) is said to be **edge-sign preserving** if furthermore, it preserves the signs of the edges. If there exists one, we write \((G, \sigma) \xrightarrow{s.p.} (H, \pi)\).

\[(G, \sigma) \rightarrow (H, \pi) \iff \exists \sigma' \equiv \sigma, (G, \sigma') \xrightarrow{s.p.} (H, \pi).\]
Circular coloring of signed graphs

Let $C^r$ be a circle of circumference $r$.

**Definition [R. Naserasr, Z. Wang and X. Zhu 2021]**

Given a signed graph $(G, \sigma)$ with no positive loop and a real number $r$, a circular $r$-coloring of $(G, \sigma)$ is a mapping $\varphi : V(G) \to C^r$ such that for each positive edge $uv$ of $(G, \sigma)$,

$$d_{C^r}(\varphi(u), \varphi(v)) \geq 1,$$

and for each negative edge $uv$ of $(G, \sigma)$,

$$d_{C^r}(\varphi(u), \overline{\varphi(v)}) \geq 1.$$

The **circular chromatic number** of $(G, \sigma)$ is defined as

$$\chi_c(G, \sigma) = \inf\{r \geq 1 : (G, \sigma) \text{ admits a circular } r\text{-coloring}\}.$$
Circular $\frac{p}{q}$-coloring of signed graphs

For $x \in \{0, 1, \ldots, p - 1\}$, $\bar{x} = x + \frac{p}{2} \pmod{p}$.

Given a positive even integer $p$ and a positive integer $q$ satisfying $q \leq \frac{p}{2}$, a circular $\frac{p}{q}$-coloring of a signed graph $(G, \sigma)$ is a mapping $\varphi : V(G) \to \{0, 1, \ldots, p - 1\}$ such that for any positive edge $uv$,

$$q \leq |\varphi(u) - \varphi(v)| \leq p - q,$$

and for any negative edge $uv$,

$$|\varphi(u) - \varphi(v)| \leq \frac{p}{2} - q \quad \text{or} \quad |\varphi(u) - \varphi(v)| \geq \frac{p}{2} + q.$$
Orientation on signed graphs

**Figure:** A bi-directed signed $K_3$  

**Figure:** A mono-directed signed $K_3$
Circular flow in mono-directed Eulerian signed graphs

Circular $\frac{p}{q}$-flow in mono-directed signed graphs

Definition [J. Li, R. Naserasr, Z. Wang and X. Zhu 2022+]

Given a positive even integer $p$ and a positive integer $q$ where $q \leq \frac{p}{2}$, a circular $\frac{p}{q}$-flow in $(G, \sigma)$ is a pair $(D, f)$ where $D$ is an orientation on $G$ and $f : E(G) \rightarrow \mathbb{Z}$ satisfies the following conditions.

- For each positive edge $e$, $|f(e)| \in \{q, \ldots, p - q\}$.
- For each negative edge $e$, $|f(e)| \in \{0, \ldots, \frac{p}{2} - q\} \cup \{\frac{p}{2} + q, \ldots, p - 1\}$.
- For each vertex $v$ of $(G, \sigma)$, $\sum_{(v, w) \in D} f(vw) = \sum_{(u, v) \in D} f(uv)$.

The circular flow index of $(G, \sigma)$ is defined to be

$$\Phi_c(G, \sigma) = \min\{\frac{p}{q} \mid (G, \sigma) \text{ admits a circular } \frac{p}{q}\text{-flow}\}.$$
Duality: circular coloring and circular flow

Lemma [J. Li, R. Naserasr, Z. Wang and X. Zhu 2022+]

Let \((G, \sigma)\) be a signed plane graph and \((G^*, \sigma^*)\) be its dual signed graph. Then

\[ \chi_c(G, \sigma) = \Phi_c(G^*, \sigma^*). \]

Figure: Circular $\frac{8}{3}$-coloring of $C_{-4}$

Figure: Circular $\frac{8}{3}$-flow in $C^*_{-4}$
Circular flow in mono-directed signed graphs

Circular $\frac{2\ell}{\ell-1}$-flow and circular $\frac{2\ell}{\ell-1}$-coloring

Let $k$ be a positive integer.

- A signed graph $(G, +)$ admits a circular $\frac{2k+1}{k}$-coloring if and only if $(G, +) \rightarrow C_{2k+1}$.

- A signed bipartite graph $(G, \sigma)$ admits a circular $\frac{4k}{2k-1}$-coloring if and only if $(G, \sigma) \rightarrow C_{-2k}$. [R. Naserasr and Z. Wang 2021]
Bipartite analog of Jaeger-Zhang conjecture

Signed bipartite analog of Jaeger-Zhang conjecture

Signed Eulerian analog of Jaeger’s circular flow conjecture

Every $g(k)$-edge-connected Eulerian signed graph admits a circular $\frac{4k}{2k-1}$-flow.

Signed bipartite analog of Jaeger-Zhang conjecture

Every signed bipartite planar graph of negative-girth at least $f(k)$ admits a homomorphism to $C_{-2k}$. 
Signed bipartite analog of Jaeger-Zhang conjecture

Every signed bipartite planar graph of negative-girth at least $f(k)$ admits a homomorphism to $C_{-2k}$.

- It was conjectured that $f(k) = 4k - 2$ [R. Naserasr, E. Rollová, and É. Sopena 2015];
- $k = 2$: verified for negative-girth 8 (best possible) [R. Naserasr, L-A. Pham, and Z. Wang 2022];
- $k = 3, 4$: verified for negative-girth 14 and 20 [J. Li, Y. Shi, Z. Wang, and C. Wei 2022+];
- $k \geq 5$:
  - verified for negative-girth $8k - 2$ [C. Charpentier, R. Naserasr, and E. Sopena 2020];
  - verified for negative-girth $6k - 2$ [J. Li, R. Naserasr, Z. Wang, and X. Zhu 2022+].
Main results

**Theorem [J. Li, R. Naserasr, Z. Wang and X. Zhu 2022+]**

Every \((6k - 2)\)-edge-connected Eulerian signed graph admits a circular \(\frac{4k}{2k-1}\)-flow.

**Theorem [J. Li, R. Naserasr, Z. Wang and X. Zhu 2022+]**

Every signed bipartite planar graph of negative-girth at least \(6k - 2\) admits a homomorphism to \(C_{-2k}\).
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(\mathbb{Z}_{2k}, \beta)-orientation on graphs

Definition [J. Li, Y. Wu and C.Q. Zhang 2020]

Given a graph $G$, a function $\beta : V(G) \rightarrow \{0, \pm 1, \ldots, \pm k\}$ is a \textit{parity-compliant $2k$-boundary} of $G$ if for every vertex $v \in V(G)$,

$$\beta(v) \equiv d(v) \pmod{2} \quad \text{and} \quad \sum_{v \in V(G)} \beta(v) \equiv 0 \pmod{2k}.$$

Given a parity-compliant $2k$-boundary $\beta$, an orientation $D$ on $G$ is called a \textit{(\mathbb{Z}_{2k}, \beta)-orientation} if for every vertex $v \in V(G)$,

$$\overrightarrow{d_D}(v) - \overleftarrow{d_D}(v) \equiv \beta(v) \pmod{2k}.$$
Preliminaries

$$(\mathbb{Z}_{2k}, \beta)$$-orientation on graphs

**Theorem [L.M. Lovasz, C. Thomassen, Y. Wu and C.Q. Zhang 2013; J. Li, Y. Wu and C.Q. Zhang 2020]**

Let $G$ be a graph with a parity-compliant $2k$-boundary $\beta$ for $k \geq 3$. Let $z_0$ be a vertex of $V(G)$ such that $d(z_0) \leq 2k - 2 + |\beta(z_0)|$. Assume that $D_{z_0}$ is an orientation on $E(z_0)$ which achieves the boundary $\beta(z_0)$. Let $V_0 = \{v \in V(G) \setminus \{z_0\} \mid \beta(v) = 0\}$. If $V_0 \neq \emptyset$, we let $v_0$ be a vertex of $V_0$ with the smallest degree. Assume that $d(A) \geq 2k - 2 + |\beta(A)|$ for any $A \subset V(G) \setminus \{z_0\}$ with $A \neq \{v_0\}$ and $|V(G) \setminus A| > 1$. Then the partial orientation $D_{z_0}$ can be extended to a $(\mathbb{Z}_{2k}, \beta)$-orientation on the entire graph $G$.

**Theorem [J. Li, Y. Wu and C.Q. Zhang 2020]**

Let $G$ be a $(3k - 3)$-edge-connected graph, where $k \geq 3$. For any parity-compliant $2k$-boundary $\beta$ of $G$, $G$ admits a $(\mathbb{Z}_{2k}, \beta)$-orientation.
Tutte’s lemma [W.T. Tutte 1954]

If a graph admits a modulo $k$-flow $(D, f)$, then it admits an integer $k$-flow $(D, f')$ such that $f'(e) \equiv f(e) \pmod{k}$ for every edge $e$.

Lemma [J. Li, R. Naserasr, Z. Wang and X. Zhu 2022+]

Given an Eulerian signed graph $(G, \sigma)$, the following claims are equivalent:

- $(G, \sigma)$ admits a circular $\frac{4k}{2k-1}$-flow;
- $(G, \sigma)$ admits a modulo $4k$-flow $(D, f)$ such that for each positive edge $e$, $f(e) \in \{2k-1, 2k+1\}$, and for each negative edge $e$, $f(e) \in \{-1, 1\}$;
- $(G, \sigma)$ admits a $(\mathbb{Z}_{4k}, \beta)$-orientation with $\beta(v) \equiv 2k \cdot d^+(v) \pmod{4k}$ for each vertex $v \in V(G)$. 

Flows in Eulerian signed graphs

Sketch of the proof

• Assume that $D$ is a $(\mathbb{Z}_{4k}, \beta)$-orientation on $G$ with $\beta(v) \equiv 2k \cdot d^+(v) \pmod{4k}$. Let $D'$ be an arbitrary orientation on $G$.

• Define $f_1 : E(G) \to \mathbb{Z}_{4k}$ such that $f_1(e) = 1$ if $e$ is oriented in $D$ the same as in $D'$ and $f_1(e) = -1$ otherwise. We claim that such a pair $(D', f_1)$ is a modulo $4k$-flow in $G$ satisfying that $\partial_{D'} f_1(v) \equiv \beta(v) \pmod{4k}$ for each $v \in V(G)$.

• Define $g : E(G) \to \mathbb{Z}_{4k}$ such that $g(e) = 2k$ if $e$ is a positive edge and $g(e) = 0$ if $e$ is a negative edge. Thus $\partial_{D'} g(v) \equiv 2k \cdot d^+(v) \pmod{4k}$ for each $v \in V(G)$.

• Let $f = f_1 + g$. Then $f : E(\hat{G}) \to \mathbb{Z}_{4k}$ satisfies the following conditions:
  (1) For each positive edge $e$, $f(e) = f_1(e) + 2k \in \{2k - 1, 2k + 1\}$.
  (2) For each negative edge $e$, $f(e) = f_1(e) \in \{-1, 1\}$.
  (3) $\partial_{D'} f(v) = \partial_{D'} f_1(v) + \partial_{D'} g(v) = \beta(v) + 2k \cdot d^+(v) \equiv 0 \pmod{4k}$.

Such $(D', f)$ is a required modulo $4k$-flow in $(G, \sigma)$. 
Circular $\frac{4k}{2k-1}$-flow in Eulerian signed graphs

**Theorem [J. Li, Y. Wu and C.Q. Zhang 2020]**

Let $G$ be a $(3k - 3)$-edge-connected graph, where $k \geq 3$. For any parity-compliant $2k$-boundary $\beta$ of $G$, $G$ admits a $(\mathbb{Z}_{2k}, \beta)$-orientation.

**Theorem [J. Li, R. Naserasr, Z. Wang and X. Zhu 2022+]**

For any Eulerian signed graph $(G, \sigma)$, if the underlying graph $G$ is $(6k - 2)$-edge-connected, then $\Phi_c(G, \sigma) \leq \frac{4k}{2k-1}$.

**Corollary**

Every signed bipartite planar graph of girth at least $6k - 2$ admits a circular $\frac{4k}{2k-1}$-coloring, i.e., it admits a homomorphism to $C_{-2k}$. 
Bipartite folding lemma

Let \((G, \sigma)\) be a signed bipartite plane graph whose shortest negative cycle is of length \(2k\). Assume that \(C\) is a facial cycle that is not a negative \(2k\)-cycle. Then there are vertices \(v_{i-1}, v_i, \) and \(v_{i+1}\) consecutive in the cyclic order of the boundary of \(C\), such that identifying \(v_{i-1}\) and \(v_{i+1}\), after a possible switching at one of the two vertices, the resulting signed graph remains a signed bipartite plane graph whose shortest negative cycle is still of length \(2k\).
Lemma [J. Li, R. Naserasr, Z. Wang and X. Zhu 2022+]  

Given a positive integer $k$, a graph $G$ and a vertex $z$ of it, assume that the cut $(\{z\}, V(G) \setminus \{z\})$ is of size at most $6k - 2$, but every other cut $(X, X^c)$ is of size at least $6k - 2$. Then given any parity-compliant $4k$-boundary $\beta$ of $G$ and any orientation $D_z$ of the edges incident to $z$ satisfying that $\overleftarrow{d_{D_z}(z)} - \overrightarrow{d_{D_z}(z)} \equiv \beta(z) \pmod{4k}$, $D_z$ can be extended to a $(\mathbb{Z}_{4k}, \beta)$-orientation on $G$.

Given a parity-compliant $4k$-boundary $\beta$, let $D_z$ be the pre-orientation on the edges incident to $z$ achieving $\beta(z)$. Let $D'_z$ be a pre-orientation obtained from $D_z$ by changing one in-arc, say $(w, z)$, of $z$ to an out-arc and let $\beta'$ be defined as follows:

$$\beta'(v) = \begin{cases} 
\beta(v) + 2 & \text{if } v = z, \\
\beta(v) - 2, & \text{if } v = w, \\
\beta(v), & \text{otherwise.}
\end{cases}$$
Mapping signed bipartite planar graphs to $C_{-2k}$

**Theorem [J. Li, R. Naserasr, Z. Wang and X. Zhu 2022+]**

Every signed bipartite planar graph of negative-girth at least $6k - 2$ admits a homomorphism to $C_{-2k}$.

Assume that $(G, \sigma)$ is a minimum counterexample and $(G^*, \sigma^*)$ is its dual signed graph.

By the bipartite folding lemma, we may assume that $(G, \sigma)$ is a signed bipartite plane graph of negative-girth $6k - 2$ in which each facial cycle is a negative $(6k - 2)$-cycle and $(G, \sigma)$ admits no circular $\frac{4k}{2k-1}$-coloring.

Thus $(G^*, \sigma^*)$ is a $(6k - 2)$-regular signed Eulerian graph and admits no circular $\frac{4k}{2k-1}$-flow.
Coloring of signed bipartite planar graphs

Sketch of the proof

- Assume that \((X, X^c)\) is an edge-cut of size smaller than \(6k - 2\) of \(G^*\) and \(|X|\) is minimized. Let \(\hat{H}\) and \(\hat{H}^c\) denote the signed subgraphs of \(\hat{G}^*\) induced by \(X\) and \(X^c\).

- First, \(\hat{G}^*/\hat{H}\) admits a circular \(\frac{4k}{2k - 1}\)-flow by the minimality of \((G, \sigma)\). Let \(D\) be such a \((\mathbb{Z}_{4k}, \beta)\)-orientation on \(\hat{G}^*/\hat{H}\) with \(\beta(v) \equiv 2k \cdot d^+(v) \pmod{4k}\).

- Let \(G_1\) be the graph obtained from \(\hat{G}^*\) by identifying all the vertices of \(X^c\) and we denote by \(z_0\) the new vertex.

  - Let \(D_{z_0}\) denote the orientation of \(D\) restricted on \(E(z_0)\) and let \(\beta\) be a parity-compliant \(4k\)-boundary of \(G_1\) such that
    \[
    \beta(z_0) = \overleftarrow{d_{D_{z_0}}(z_0)} - \overrightarrow{d_{D_{z_0}}(z_0)}.
    \]

  - We conclude that \(D'_{z_0}\) can be extended to a \((\mathbb{Z}_{4k}, \beta)\)-orientation on \(G'_1\), thus also a \((\mathbb{Z}_{4k}, \beta)\)-orientation on \(G_1\).

So the \((\mathbb{Z}_{4k}, \beta)\)-orientation of \(\hat{G}^*/\hat{H}\) is extended to \(\hat{H}\) and thus \(\hat{G}^*\) admits a \((\mathbb{Z}_{4k}, \beta)\)-orientation with \(\beta(v) \equiv 2p \cdot d^+(v) \pmod{4k}\).
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## Results

### Recent results

#### Circular flow index of highly edge-connected signed graphs

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<td>$\Phi_c \leq 12$</td>
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<tr>
<td>3</td>
<td>*</td>
<td>$\Phi_c \leq 6$</td>
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<tr>
<td>4</td>
<td>*</td>
<td>$\Phi_c \leq 4$</td>
</tr>
<tr>
<td>5</td>
<td>$\Phi_c \leq 3$ [2]</td>
<td>*</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>$\Phi_c &lt; 4$ (tight)</td>
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<tr>
<td>7+planar</td>
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<td>$\Phi_c \leq \frac{12}{5}$ [LSWW22+]</td>
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<tr>
<td>$3k - 1$</td>
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<td>$\Phi_c \leq \frac{2k}{k-1}$</td>
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<tr>
<td>$3k$</td>
<td>*</td>
<td>$\Phi_c &lt; \frac{2k}{k-1}$</td>
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<td>$3k + 1$</td>
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<td>$\Phi_c \leq \frac{4k+2}{2k-1}$</td>
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<tr>
<td>$6k - 2 +$ Eulerian</td>
<td>*</td>
<td>$\Phi_c \leq \frac{4k}{2k-1}$</td>
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</tbody>
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Conjectures

**Lemma [J. Li, R. Naserasr, Z. Wang and X. Zhu 2022+]**

Given a graph $G$, we have $\Phi_c(T_2(G)) = 2\Phi_c(G)$.

Reformulate Tutte’s 5-flow conjecture:

**Conjecture [1]**

Every 2-edge-connected signed graph admits a circular 10-flow.

**Proposition [Z. Pan and X. Zhu 2003]**

For any rational number $r \in [2, 10]$, there exists a 2-edge-connected signed graph whose circular flow index is $r$. 
Conjectures

- Reduction of Tutte’s 5-flow conjecture to 3-edge-connected cubic graphs

**Question**

Does every 3-edge-connected signed graph admit a circular 5-flow?

- Stronger Tutte’s 3-flow conjecture

**Conjecture [2]**

Every 5-edge-connected signed graph admits a circular 3-flow.

- Tutte’s 4-flow conjecture restated

**Conjecture**

Every 2-edge-connected signed Petersen-minor-free graph admits a circular 8-flow.
Given an integer $k \geq 1$, what is the smallest integer $f_1(k)$ such that every $f_1(k)$-edge-connected signed graphs admits a circular $\frac{2k+1}{k}$-flow?

Given an integer $k \geq 1$, what is the smallest integer $f_2(k)$ such that every $f_2(k)$-edge-connected signed graphs admits a circular $\frac{4k}{2k-1}$-flow?

For Eulerian signed graphs:

Given an integer $k \geq 1$, what is the smallest integer $g(k)$ such that every (negative-) $g(k)$-edge-connected Eulerian signed graphs admits a circular $\frac{4k}{2k-1}$-flow?
Thanks for your attention!