Circular Coloring of Signed Graphs

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Homomorphism of signed graphs

- A signed graph is a graph $G = (V, E)$ together with an assignment $\{+, -\}$ on its edges, denoted by (G, σ).
- A switching at vertex v is to switch the signs of all the edges incident to this vertex.
- The sign of a closed walk is the product of signs of all the edges of this walk.
- A homomorphism of signed graph (G, σ) to a signed graph (H, π) is a mapping φ from $V(G)$ and $E(G)$ correspondingly to $V(H)$ and $E(H)$ such that the adjacency, the incidence and the signs of the closed walks are preserved.
- If there exists one, we write $(G, \sigma) \to (H, \pi)$.

Homomorphism of signed graphs

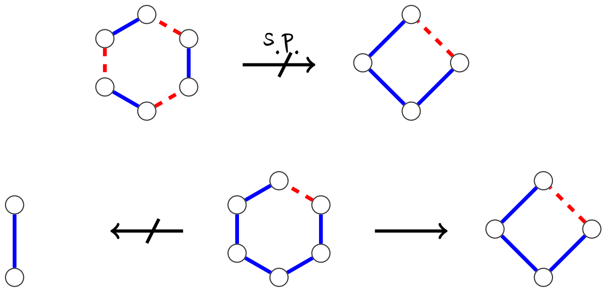
- An edge-sign preserving homomorphism of a signed graph (G, σ) to (H, π) is a mapping $f : V(G) \to V(H)$ such that for every positive (respectively, negative) edge uv of (G, σ), $f(u)f(v)$ is a positive (respectively, negative) edge of (H, π).
- If there exists one, we write $(G, \sigma) \xrightarrow{s.p.} (H, \pi)$.

Proposition

Given two signed graphs (G, σ) and (H, π),

$$(G, \sigma) \to (H, \pi) \Leftrightarrow \exists \sigma' \equiv \sigma, (G, \sigma') \xrightarrow{s.p.} (H, \pi).$$

Examples: homomorphism of signed graphs



Circular chromatic number of signed graphs

Given a signed graph (G, σ) with no positive loop and a real number r, a circular r-coloring of (G, σ) is a mapping $f : V(G) \to C^r$ such that for each positive edge uv of (G, σ),

$$d_{(\mathrm{mod}\ r)}(f(u), f(v)) \geq 1,$$

and for each negative edge uv of (G, σ),

$$d_{(\mathrm{mod}\ r)}(f(u), \overline{f(v)}) \geq 1.$$

The circular chromatic number of (G, σ) is defined as

$$\chi_c(G, \sigma) = \inf\{r \geq 1 : (G, \sigma) \text{ admits a circular } r\text{-coloring}\}.$$

Refinement of 0-free $2k$-coloring of signed graphs

Definition [T. Zaslavsky 1982]

Given a signed graph (G, σ) and a positive integer k, a 0-free $2k$-coloring of (G, σ) is a mapping $f : V(G) \to \{\pm 1, \pm 2, \ldots, \pm k\}$ such that for any edge uv of (G, σ), $f(u) \neq \sigma(uv)f(v)$.

Proposition

Assume (G, σ) is a signed graph and k is a positive integer. Then (G, σ) is 0-free $2k$-colorable if and only if (G, σ) is circular $2k$-colorable.

Equivalent definition

A circular r-coloring of a signed graph (G, σ) is a mapping $f : V(G) \to [0, r)$ such that for each positive edge uv,

$$1 \leq |f(u) - f(v)| \leq r - 1$$

and for each negative edge uv,

$$\text{either } |f(u) - f(v)| \leq \frac{r}{2} - 1 \text{ or } |f(u) - f(v)| \geq \frac{r}{2} + 1.$$

Equivalent definition: (p, q)-coloring

For $i, j, x \in \{0, 1, \dots, p-1\}$,

$$d_{(\text{mod } p)}(i, j) = \min\{|i-j|, p-|i-j|\} \text{ and } \bar{x} = x + \frac{p}{2} (\text{mod } p).$$

Given an even integer p and a positive integer q satisfying $q \leq \frac{p}{2}$, a (p, q)-coloring of a signed graph (G, σ) is a mapping $f : V(G) \to \{0, 1, \dots, p-1\}$ such that for any positive edge uv,

$$d_{(\text{mod } p)}(f(u), f(v)) \geq q,$$

and for any negative edge uv,

$$d_{(\text{mod } p)}(f(u), \overline{f(v)}) \geq q.$$

The circular chromatic number of (G, σ) is

$$\chi_c(G, \sigma) = \inf\{\frac{p}{q} : (G, \sigma) \text{ has a } (p, q)\text{-coloring}\}.$$

Signed circular clique

Circular chromatic number of signed graphs could also be defined through graph homomorphism.

For integers $p \geq 2q > 0$ such that p is even, the signed circular clique $K^s_{p;q}$ has vertex set $[p] = \{0, 1, \ldots, p-1\}$, in which

- ij is a positive edge if $q \leq |i-j| \leq p-q$;
- ij is a negative edge if $|i-j| \leq \frac{p}{2} - q$ or $|i-j| \geq \frac{p}{2} + q$.

Lemma

Given even positive integers p, p', if $\frac{p}{q} \leq \frac{p'}{q'}$, then $K^s_{p;q} \xrightarrow{s.p.} K^s_{p';q'}$.

Let $\hat{K}^s_{p;q}$ be the signed subgraph of $K^s_{p;q}$ induced by vertices $\{0, 1, \ldots, \frac{p}{2} - 1\}$.

An example of signed circular clique

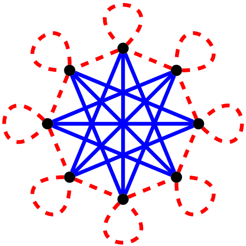


Figure: $K^s_{8;3}$

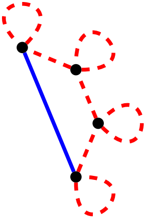


Figure: $\hat{K}^s_{8;3}$

Equivalent definition through homomorphism

Lemma

Given a signed graph (G, σ) and a positive even integer p, a positive integer q with $p \geq 2q$, the followings are equivalent:

- (G, σ) has a (p, q)-coloring;
- $(G, \sigma) \xrightarrow{s.p.} K^s_{p;q}$;
- $(G, \sigma) \to \hat{K}^s_{p;q}$.

The circular chromatic number of (G, σ) is

$$\chi_c(G, \sigma) = \inf\{\frac{p}{q} : p \text{ is even and } (G, \sigma) \xrightarrow{s.p.} K^s_{p;q}\}$$
$$= \inf\{\frac{p}{q} : p \text{ is even and } (G, \sigma) \to \hat{K}^s_{p;q}\}$$

Tool: signed indicator

Let G be a graph and let Ω be a signed graph.

- A signed indicator $\mathcal{I}$ is a triple $\mathcal{I} = (\Gamma, u, v)$ such that Γ is a signed graph and u, v are two distinct vertices of Γ.
- Given a signed indicator $\mathcal{I}$, we denote by $G(\mathcal{I})$ the signed graph obtained from G by replacing each edge with a copy of $\mathcal{I}$.
- Given two signed indicators $\mathcal{I}_+$ and $\mathcal{I}_-$, we denote by $\Omega(\mathcal{I}_+, \mathcal{I}_-)$ the signed graph obtained from Ω by replacing each positive edge with a copy of $\mathcal{I}_+$ and replacing each negative edge with a copy of $\mathcal{I}_-$.

Signed indicator

Assume $\mathcal{I} = (\Gamma, u, v)$ is a signed indicator and $r \geq 2$ is a real number.

- For $a, b \in [0, r)$, we say the color pair (a, b) is feasible for $\mathcal{I}$ (with respect to r) if there is a circular r-coloring ϕ of Γ such that $\phi(u) = a$ and $\phi(v) = b$.
- Define

$$Z(\mathcal{I}, r) = \{b \in [0, \frac{r}{2}] : (0, b) \text{ is feasible for } \mathcal{I} \text{ with respect to } r\}.$$

Example

If Γ is a positive 2-path connecting u and v, and $\mathcal{I} = (\Gamma, u, v)$, then for any ϵ, $0 < \epsilon < 1$, and $r = 4 - 2\epsilon$,

$$Z(\mathcal{I}, r) = [0, \frac{r}{2} - \epsilon].$$

Signed indicator

Lemma

Assume that $\mathcal{I} = (\Gamma, u, v)$ is a signed indicator, $r \geq 2$ is a real number and $Z(\mathcal{I}, r) = [t, \frac{r}{2} - t]$ for some $0 < t < \frac{r}{4}$. Then for any graph G,

$$\chi_c(G(\mathcal{I})) = 2t\chi_c(G).$$

Lemma

Assume that $\mathcal{I}_+$ and $\mathcal{I}_-$ are indicators, $r \geq 2$ is a real number and

$$Z(\mathcal{I}_+, r) = [t, \frac{r}{2}],\ Z(\mathcal{I}_-, r) = [0, \frac{r}{2} - t]$$

for some $0 < t < \frac{r}{2}$. Then for any signed graph Ω,

$$\chi_c(\Omega(\mathcal{I}_+, \mathcal{I}_-)) = t\chi_c(\Omega).$$

Tight cycle argument

Assume (G, σ) is a signed graph and $\phi : V(G) \to [0, r)$ is a circular r-coloring of (G, σ). The partial orientation $D = D_\phi(G, \sigma)$ of G with respect to a circular r-coloring ϕ is defined as follows: (u, v) is an arc of D if and only if one of the following holds:

- uv is a positive edge and $(\phi(v) - \phi(u)) (\text{mod } r) = 1$.
- uv is a negative edge and $(\overline{\phi(v)} - \phi(u)) (\text{mod } r) = 1$.

Arcs in $D_\phi(G, \sigma)$ are called tight arcs of (G, σ) with respect to ϕ. A directed cycle in $D_\phi(G, \sigma)$ is called a tight cycle with respect to ϕ.

Tight cycle argument

Lemma

Let (G, σ) be a signed graph and let ϕ be a circular r-coloring of (G, σ). If $D_\phi(G, \sigma)$ is acyclic, then there exists an $r_0 \lneqq r$ such that (G, σ) admits an r_0-circular coloring.

Notice that assume $D_\phi(G, \sigma)$ is acyclic and among all such ϕ, $D_\phi(G, \sigma)$ has minimum number of arcs, then $D_\phi(G, \sigma)$ has no arc.

Lemma

Given a signed graph (G, σ), $\chi_c(G, \sigma) = r$ if and only if (G, σ) is circular r-colorable and every circular r-coloring ϕ of (G, σ) has a tight cycle.

Tight cycle argument

Proposition

Any signed graph (G, σ), which is not a forest, has a cycle with s positive edges and t negative edges such that

$$\chi_c(G, \sigma) = \frac{2(s+t)}{2a+t}$$

for some integer a.

Corollary

Given a signed graph (G, σ) on n vertices, $\chi_c(G, \sigma) = \frac{p}{q}$ for some $p \leq 2n$ and q.

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3 Discussion

Some classes of signed graphs

Given a class $\mathcal{C}$ of signed graphs,

$$\chi_c(\mathcal{C}) = sup\{\chi_c(G,\sigma) \mid (G,\sigma) \in \mathcal{C}\}.$$

- $\mathcal{SBP}$ the class of signed bipartite planar simple graphs,
- $\mathcal{SD}_d$ the class of signed d-degenerate simple graphs,
- $\mathcal{SP}$ the class of signed planar simple graphs.

Signed bipartite planar graphs

Proposition

$\chi_c(\mathcal{SBP}) = 4$.

Let Γ_1 be a positive 2-path connecting u_1 and v_1. For $i \geq 2$,

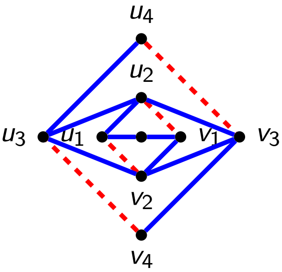


Figure: Γ_4

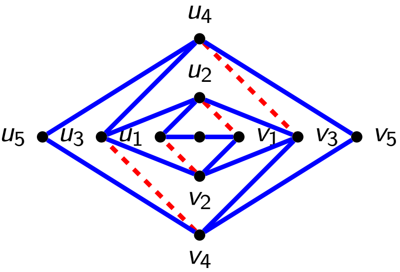


Figure: Γ_5

$$\chi_c(\Gamma_n) = \frac{4n}{n+1}$$

Results on signed bipartite planar graphs of girth ≥ 6

- $\chi_c(\mathcal{SBP}_6) \leq 3$. (Corollary of a result that every signed bipartite planar graph of negative girth 6 admits a homomorphism to $(K_{3,3}, M)$ [R. Naserasr and Z. Wang 2021+])
- $\chi_c(\mathcal{SBP}_8) \leq \frac{8}{3}$. (Corollary of a result that C_{-4}-critical signed graph has density $|E(G)| \geq \frac{3|V(G)|-2}{4}$ [R. Naserasr, L-A. Pham and Z. Wang 2020+])

Signed d-degenerate graphs

Proposition

For any positive integer d, $\chi_c(\mathcal{SD}_d) = 2\lfloor\frac{d}{2}\rfloor + 2$.

Sketch of the proof:

- First we show that every signed d-degenerate graph admits a circular $(2\lfloor\frac{d}{2}\rfloor + 2)$-coloring.

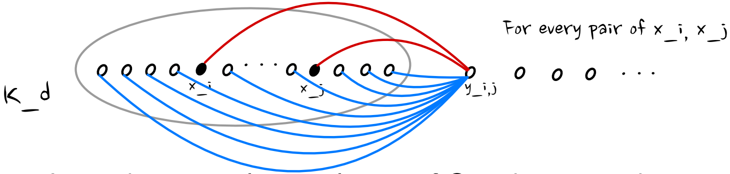
For the tightness,

- For odd integer d, we consider the signed complete graphs $(K_{d+1}, +)$.
- For $d = 2$, we consider the signed graph Γ_n built before.
- For even integer $d \geq 4$, we construct a signed d-degenerate graph (G, σ) satisfying that $\chi_c(G, \sigma) = d + 2$.

Signed d-degenerate graphs

Proof for even $d \geq 4$

- Define a signed graph Ω_d as follows.



- Let φ be a circular r-coloring of Ω_d where $r < d + 2$. Without loss of generality, $\varphi(x_1), \ldots, \varphi(x_d)$ are cyclically ordered on C^r and assume that $d_{(\mathrm{mod}\ r)}(\varphi(x_1), \varphi(x_2))$ is maximized. We prove that there is no place for $y_{1,1+\frac{d}{2}}$.

Signed planar graphs

Proposition

$$4 + \frac{2}{3} \leq \chi_c(\mathcal{SP}) \leq 6.$$

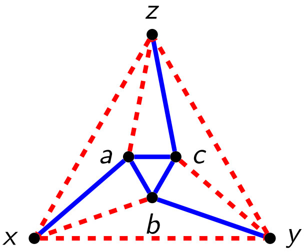


Figure: Mini-gadget (T, π)

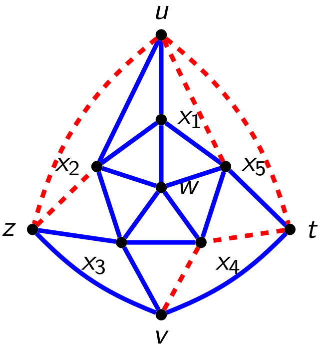


Figure: A signed Wenger Graph $\tilde{W}$

Signed planar graphs

Lemma

Let $r = \frac{14}{3} - \epsilon$ with $0 < \epsilon \leq \frac{2}{3}$. For any circular r-coloring ϕ of $\tilde{W}$, $d_{(\bmod\ r)}(\phi(u), \phi(v)) \geq \frac{4}{9}$.

Let Γ be obtained from $\tilde{W}$ by adding a negative edge uv. Let $\mathcal{I} = (\Gamma, u, v)$.

Theorem

Let $\Omega = K_4(\mathcal{I})$. Then Ω is a signed planar simple graph with $\chi_c(\Omega) = \frac{14}{3}$.

Sketch of the proof of the theorem

- First we show that Ω admits a circular $\frac{14}{3}$-coloring. We find a circular $\frac{14}{3}$-coloring ϕ of Γ such that $\phi(u) = \phi(v) = 0$ and then extend it to each of inner triangles.
- Let ϕ be a circular r-coloring of Ω for $r < \frac{14}{3}$. For any $1 \leq i < j \leq 4$, $\frac{4}{9} \leq d_{(\bmod r)}(\phi(v_i), \phi(v_j)) \leq \frac{r}{2} - 1$.
  Assume that $\phi(x_1), \phi(x_2), \phi(x_3), \phi(x_4)$ are on C^r in this cyclic order.
    - $\ell([\phi(v_1), \phi(v_4)]) = \ell([\phi(v_1), \phi(v_2)]) + \ell([\phi(v_2), \phi(v_3)]) + \ell([\phi(v_3), \phi(v_4)]) \geq 3 \times \frac{4}{9} = \frac{4}{3} > \frac{r}{2} - 1$,
    - $\ell([\phi(v_4), \phi(v_1)]) \geq r - (\ell([\phi(v_1), \phi(v_3)]) + \ell([\phi(v_2), \phi(v_4)])) \geq 2 > \frac{r}{2} - 1$.

  It's a contradiction.

Results on signed planar graphs of girth ≥ 4

- $\chi_c(\mathcal{SP}_4) = 4$. (By the 3-degeneracy of triangle-free planar graph)
- $\chi_c(\mathcal{SP}_7) \leq 3$. (Corollary of a result that every signed graph of $mad < \frac{14}{5}$ admits a homomorphism to (K_6, M) [R. Naserasr, R. Škrekovski, Z. Wang and R. Xu 2020+])

Signed circular chromatic number

For a graph G without loops, the signed circular chromatic number $\chi_c^s(G)$ of G is defined as

$$\chi_c^s(G) = \max\{\chi_c(G, \sigma) : \sigma \text{ is a signature of } G\}.$$

Proposition

For every graph G, $\chi_c^s(G) \leq 2\chi_c(G)$.

Signed chromatic number of k-chromatic graph

Theorem

For any integers $k, g \geq 2$ and any $\epsilon > 0$, there is a graph G of girth at least g satisfying that $\chi(G) = k$ and $\chi_c^s(G) > 2k - \epsilon$.

We will prove that for any integer p, there is a graph G for which the followings hold:

- G is of girth at least g and has chromatic number at most k.
- There is a signature σ such that (G, σ) is not circular $\frac{2kp}{p+1}$-colorable.

Augmented tree

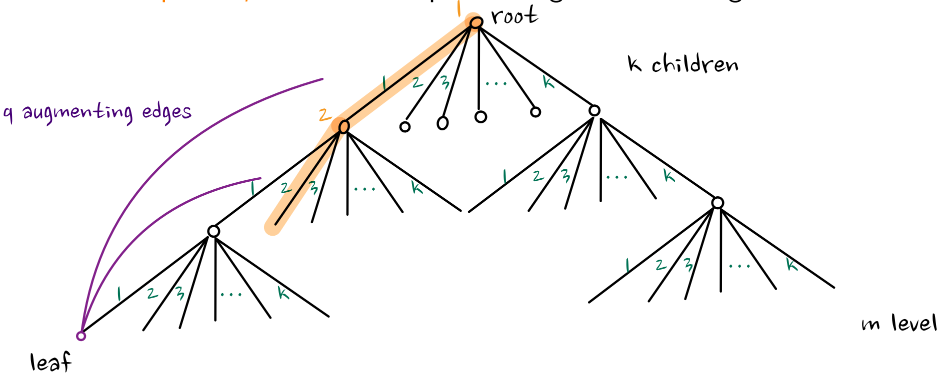
- A complete k-ary tree is a rooted tree in which each non-leaf vertex has k children and all the leaves are of the same level.
- A q-augmented k-ary tree is obtained from a complete k-ary tree by adding, for each leaf v, q edges connecting v to q of its ancestors. These q edges are called the augmenting edges from v.
- For positive integers k, q, g, a (k, q, g)-graph is a q-augmented k-ary tree which is bipartite and has girth at least g.

Lemma [N. Alon, A. Kostochka, B. Reiniger, D. West and X. Zhu 2016]

For any positive integers $k, q, g \geq 2$, there exists a (k, q, g)-graph.

Augmented tree and standard labeling

- A standard labeling of a complete k-ary tree T;
- A f-path P_f of T with respect to a given k-coloring f.



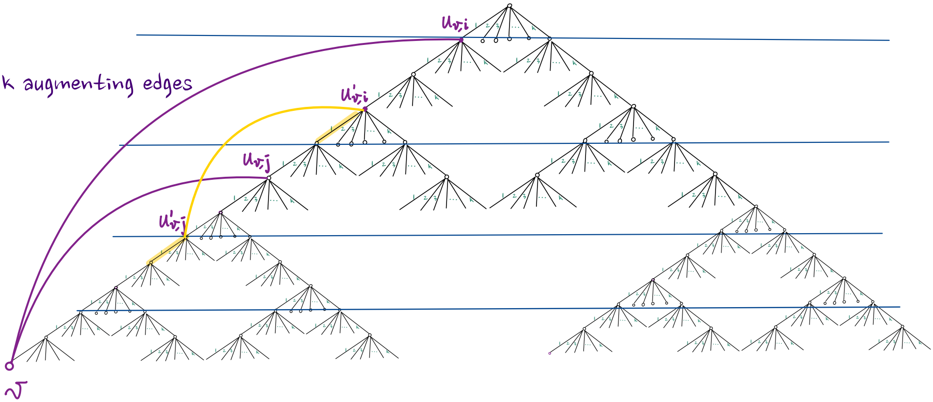
Construction of k-chromatic graph G

- H: $(2kp, k, 2kg)$-graph with underline tree T.
- ϕ: a standard $2kp$-labeling of the edges of T.
- $\ell(v)$: the level of v, i.e., the distance from v to the root vertex in T. Let $\theta(v) = \ell(v) (\mathrm{mod}\ k)$.

For each leaf v of T, let $u_{v,1}, u_{v,2}, \ldots, u_{v,k}$ be the vertices on P_v that are connected to v by augmenting edges. Let $u'_{v,i} \in P_v$ be the closest descendant of $u_{v,i}$ with $\theta(u'_{v,i}) = i$ and let $e_{v,i}$ be the edge connecting $u'_{v,i}$ to its child on P_v.
Let $s_{v,i} = \phi(e_{v,i})$ and let

- $A_{v,i} = \{s_{v,i}, s_{v,i} + 1, \ldots, s_{v,i} + p\}$,
- $B_{v,i} = \{a + kp : a \in A_{v,i}\}$,
- $C_{v,i} = A_{v,i} \cup B_{v,i}$.

k-augmented $2kp$-ary tree of girth $\geq 2kg$



Construction of the signature σ on G

Note that $B_{v,i}$ is a kp-shift of $A_{v,i}$. Two possibilities:

- $A_{v,i} \cap A_{v,j} \neq \emptyset$ (then $B_{v,i} \cap B_{v,j} \neq \emptyset$)

$$d_{(\bmod 2kp)}(\phi(e_{v,i}), \phi(e_{v,j})) \leq p.$$

- $A_{v,i} \cap B_{v,j} \neq \emptyset$ (then $B_{v,i} \cap A_{v,j} \neq \emptyset$)

$$d_{(\bmod 2kp)}(\phi(e_{v,i}), \overline{\phi(e_{v,j})}) \leq p.$$

Let L be the set of leaves of T. For each $v \in L$, we define one edge e_v on $V(T)$ as follows:

- If $d_{(\bmod 2kp)}(\phi(e_{v,i}), \phi(e_{v,j})) \leq p$, then let e_v be a positive edge connecting $u'_{v,i}$ and $u'_{v,j}$.
- If $d_{(\bmod 2kp)}(\phi(e_{v,i}), \overline{\phi(e_{v,j})}) \leq p$, then let e_v be a negative edge connecting $u'_{v,i}$ and $u'_{v,j}$.

Proof for "(G, σ) is not circular $\frac{2kp}{p+1}$-colorable"

Let (G, σ) be the signed graph with vertex set $V(T)$ and with edge set $\{e_v : v \in L\}$ where the signs of the edges are defined as above.

- Assume f is a $(2kp, p+1)$-coloring of (G, σ).
- As f is also a $2kp$-coloring of the vertices of T, there is a unique f-path P_v. Assume that $e_v = u'_{v,i}u'_{v,j}$. By definition,

$$f(u'_{v,i}) = \phi(e_{v,i}) \text{ and } f(u'_{v,j}) = \phi(e_{v,j}).$$

- If e_v is a positive edge, then $d_{(\bmod 2kp)}(\phi(e_{v,i}), \phi(e_{v,j})) \leq p$. If e_v is a negative edge, then $d_{(\bmod 2kp)}(\phi(e_{v,i}), \overline{\phi(e_{v,j})}) \leq p$. It is a contradiction.

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Mapping signed planar graphs to signed cycles

Let C_ℓ^{o+} be signed cycle of length ℓ where the number of positive edges is odd. Then $\chi_c(C_\ell^{o+}) = \frac{2\ell}{\ell-1}$.

Theorem

Given a positive integer ℓ and a signed graph (G, σ) satisfying $g_{ij}(G, \sigma) \geq g_{ij}(C_\ell^{o+})$ for $ij \in \mathbb{Z}_2^2$, we have $\chi_c(G, \sigma) \leq \frac{2\ell}{\ell-1}$ if and only if $(G, \sigma) \to C_\ell^{o+}$.

Question

Given a positive integer ℓ, what is the smallest value $f(\ell)$ (with $f(\infty) = \infty$) such that for every signed planar graph (G, σ) satisfying $g_{ij}(G, \sigma) \geq g_{ij}(C_\ell^{o+})$ and $g_{ij}(G, \sigma) \geq f(\ell)$ for all $ij \in \mathbb{Z}_2^2$, we have $\chi_c(G, \sigma) \leq \frac{2\ell}{\ell-1}$.

Jaeger-Zhang conjecture

When $\ell = 2k+1$,

Jaeger-Zhang conjecture [C.-Q. Zhang 2002]

Every planar graph of odd-girth $f(2k+1) = 4k+1$ admits a circular $\frac{2k+1}{k}$-coloring, i.e., C_{2k+1}-coloring.

- $f(3) = 5$ [Grötzsch's theorem];
- $f(5) \leq 11$ [Z. Dvořák and L. Postle 2017][D. W. Cranston and J. Li 2020];
- $4k+1 \leq f(2k+1) \leq 6k+1$ [C. Q. Zhang 2002; L. M. Lovász, C. Thomassen, Y. Wu and C. Q. Zhang 2013];

Bipartite analogue of Jaeger-Zhang conjecture

When $\ell = 2k$,

Bipartite analogue of Jaeger-Zhang conjecture

Every signed bipartite planar graph of negative-girth $f(2k)$ admits a circular $\frac{4k}{2k-1}$-coloring, i.e., C_{-2k}-coloring.

- $f(4) = 8$ [R. Naserasr, L. A. Pham and Z. Wang 2020+];
- $f(2k) \leq 8k - 2$ [C. Charpentier, R. Naserasr and E. Sopena 2020].

Odd-Hadwiger Conjecture

Theorem [P.A. Catlin 1979]

If $(G,-)$ has no $(K_4,-)$-minor, then $\chi_c(G,+) \leq 3$.

The Odd-Hadwiger conjecture was proposed independently by B. Gerard and P. Seymour.

Odd-Hadwiger conjecture

If a signed graph $(G,-)$ has no $(K_{k+1},-)$-minor, then $\chi_c(G,+) \leq k$.

Question

Assuming (G,σ) has no $(K_{k+1},-)$-minor, what is the best upper bound on $\chi_c(G,-\sigma)$?

The end. Thank you!