Mapping sparse signed graphs to \((K_{2k}, M)\)

Zhouningxin Wang

IRIF, Université de Paris

(A joint work with Reza Naserasr, Riste Skrekovski, Rongxing Xu)

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1 Introduction

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2 Our work

- Mapping to \((K_6, M)\) and \((K_{2k}, M)\)
- Tightness

3 Conclusion and Discussion
Homomorphism of signed graphs

Signed graphs

- A **signed graph** is a graph $G = (V, E)$ together with an assignment $\sigma : E(G) \rightarrow \{+, -\}$, denoted by $(G, \sigma)$.
- A **switching** at vertex $v$ is to switch the signs of all the edges incident to this vertex.
- We say $(G, \sigma')$ is **switching equivalent** to $(G, \sigma)$ if it is obtained from $(G, \sigma)$ by switching at some vertices (allowing repetition).
- The **sign** of a closed walk is the product of signs of all the edges of this walk.

**Theorem [T. Zaslavsky 1982]**

Signed graphs $(G, \sigma)$ and $(G, \sigma')$ are switching equivalent if and only if they have a same set of negative cycles.
A homomorphism of signed graph \((G, \sigma)\) to a signed graph \((H, \pi)\) is a mapping \(\varphi\) from \(V(G)\) and \(E(G)\) to \(V(H)\) and \(E(H)\) (respectively) such that the adjacency, the incidence and the signs of closed walks are preserved.

If there exists a homomorphism of \((G, \sigma)\) to \((H, \pi)\), we write \((G, \sigma) \rightarrow (H, \pi)\).
Edge-sign preserving homomorphism

- An edge-sign preserving homomorphism of signed graph $(G, \sigma)$ to $(H, \pi)$ is a mapping $\varphi$ from $V(G)$ and $E(G)$ to $V(H)$ and $E(H)$ (respectively) such that for $uv \in E(G)$, $\varphi(u)\varphi(v) \in E(H)$ and $\sigma(uv) = \pi(\varphi(u)\varphi(v))$.
- If there exists an edge-sign preserving homomorphism of $(G, \sigma)$ to $(H, \pi)$, we write $(G, \sigma) \xrightarrow{s.p.} (H, \pi)$.

Proposition

Given signed graphs $(G, \sigma)$ and $(H, \pi)$,

$$(G, \sigma) \rightarrow (H, \pi) \Leftrightarrow \exists \sigma' \equiv \sigma, (G, \sigma') \xrightarrow{s.p.} (H, \pi).$$
Considering the parity of the length of a closed walk and the sign of it, there are four possible types of closed walks:

- **type 00** is a closed walk which is positive and of even length,
- **type 01** is a closed walk which is positive and of odd length,
- **type 10** is a closed walk which is negative and of even length,
- **type 11** is a closed walk which is negative and of odd length.

The length of a shortest nontrivial closed walk in \((G, \sigma)\) of type \(ij\), \((ij \in \mathbb{Z}_2^2)\), is denoted by \(g_{ij}(G, \sigma)\).

**No-homomorphism Lemma [R. Naserasr, E. Rollová and E. Sopena 2015]**

If \((G, \sigma) \rightarrow (H, \pi)\), then \(g_{ij}(G, \sigma) \geq g_{ij}(H, \pi)\) for \(ij \in \mathbb{Z}_2^2\).
As No-homomorphism Lemma gives us a necessary condition for mapping \((G, \sigma)\) to \((H, \pi)\), is it also sufficient?

- For example, let \(H\) be a triangle and \(G\) be a Mycielski graph \(M_k\) for \(k > 3\). The graph \(G\) is triangle-free but it has chromatic number \(k > 3\).

- What kind of conditions can make it also sufficient? One possible constraint: maximum average degree.
Maximum average degree

Given a graph $G$, the maximum average degree, denoted $\text{mad}(G)$, is the largest average degree taken over all the subgraphs of $G$.

**Theorem [C. Charpentier, R. Naserasr and E. Sopena 2020]**

Given a signed graph $(H, \pi)$, there exists an $\epsilon > 0$ such that every signed graph $(G, \sigma)$, satisfying $g_{ij}(G, \sigma) \geq g_{ij}(H, \pi)$ and $\text{mad}(G) < 2 + \epsilon$, admits a homomorphism to $(H, \pi)$.

A main question then is to find the best value of $\epsilon$ for a given signed graph $(H, \pi)$.

- For $(K_4, e)$, the best value of $\epsilon$ was proved to be $\frac{4}{7}$.
- For $(K_6, M)$, we prove that the best value of $\epsilon$ is $\frac{4}{5}$.
- For $(K_{2k}, M)$, $k \geq 4$, we prove that the best value of $\epsilon$ is 1.
Double Switching Graphs

Given a signed graph \((G, \sigma)\) on the vertex set \(V = \{x_1, \ldots, x_n\}\), the Double Switching Graph of \((G, \sigma)\), denoted \(DSG(G, \sigma)\), is a signed graph built as follows:

- We have two disjoint copies of \(V\), \(V^+ = \{x_1^+, x_2^+, \ldots, x_n^+\}\) and \(V^- = \{x_1^-, x_2^-, \ldots, x_n^-\}\) in \(DSG(G, \sigma)\).
- Each set of vertices \(V^+, V^-\) then induces a copy of \((G, \sigma)\).
- Furthermore, a vertex \(x_i^-\) connects to vertices in \(V^+\) as it is obtained from a switching on \(x_i\). More precisely, if \(x_ix_j\) is a positive (negative) edge in \((G, \sigma)\), then \(x_i^+x_j^+, x_i^-x_j^-\) are positive (negative) edges in \(DSG(G, \sigma)\), and \(x_i^+x_j^-, x_i^-x_j^+\) are negative (positive) edges in \(DSG(G, \sigma)\).
Homomorphism of signed graphs

Double Switching Graphs

\[ x_1 \quad x_2 \quad x_3 \quad x_4 \]
\[ x_1^+ \quad x_2^+ \quad x_3^+ \quad x_4^+ \]
\[ x_1^- \quad x_2^- \quad x_3^- \quad x_4^- \]

Figure: Signed graphs \((C_4, e)\) and DSG\((C_4, e)\)

**Theorem [R.C. Brewster and T. Graves 2009]**

Given signed graphs \((G, \sigma)\) and \((H, \pi)\),

\[ (G, \sigma) \rightarrow (H, \pi) \iff (G, \sigma) \xrightarrow{s.p.} \text{DSG}(H, \pi). \]
Indicator construction $S(G)$

Given a graph $G$, a signed graph $S(G)$ is built as follows:

- Take the vertex set $V(G)$;
- For each edge $uv$ of $G$, we add two more vertices $x_{uv}$ and $y_{uv}$, and connect them with both of $u$ and $v$ (noting that $uv$ is not an edge of $S(G)$);
- For each 4-cycle $ux_{uv}v_{uv}y_{uv}$, we assign a negative sign to one of the edges.

Figure: $S(K_3)$  
Figure: $S(C_5)$
A strengthening of Four-Color Theorem

**Theorem [R. Naserasr, E. Rollová and E. Sopena 2015]**

- A graph $G$ is bipartite if and only if $S(G) \rightarrow (K_{2,2}, e)$.
- A graph $G$ is $k$-colorable for $k \geq 3$ if and only if $S(G) \rightarrow (K_{k,k}, M)$.

**Four-Color Theorem restated**

For every planar simple graph $G$, $S(G) \rightarrow (K_{4,4}, M)$.

The following is a strengthening of the Four-Color Theorem (proof of which is based on an edge-coloring result of B. Guenin which in turn is based on the Four-Color Theorem).

**Theorem [R. Naserasr, E. Rollová and E. Sopena 2013]**

Every signed bipartite planar (simple) graph maps to $(K_{4,4}, M)$. 

Homomorphism of signed bipartite graphs
Introduction

Homomorphism of signed bipartite graphs

Mapping signed bipartite graphs to \((K_{4,4}, M)\)

- For planar graphs, the homomorphism problem of planar graphs to \(K_3\), which is a non-trivial core subgraph of \(K_4\), has been greatly studied.

- Grötzsch’s theorem states that planar graph of girth at least 4 maps to \(K_3\) and 3-coloring problem of planar graphs is proved to be NP-complete.

- It is natural to ask for each core subgraphs of \((K_{4,4}, M)\) which families of planar graphs map to. Two notable subgraphs:
  - the negative 4-cycle;
  - \((K_{3,3}, M)\).

The question of mapping signed bipartite planar graphs to \((K_{3,3}, M)\) captures 3-coloring problem of planar graphs.
Homomorphism to \((K_{k,k}, M)\) and \((K_{2k}, M)\)

Theorem

For a signed bipartite graph \((G, \sigma)\),

\[(G, \sigma) \rightarrow (K_{k,k}, M) \iff (G, \sigma) \rightarrow (K_{2k}, M).\]

We prove:

- Every signed graph \((G, \sigma)\) with \(\text{mad}(G) < \frac{14}{5}\) and satisfying \(g_{ij}(G, \sigma) \geq g_{ij}(K_6, M)\) admits a homomorphism to \((K_6, M)\).
- Every signed graph \((G, \sigma)\) with \(\text{mad}(G) < 3\) and satisfying \(g_{ij}(G, \sigma) \geq g_{ij}(K_{2k}, M)\) admits a homomorphism to \((K_{2k}, M)\) for \(k \geq 4\).
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3 Conclusion and Discussion
Mapping to \((K_6, M)\)

**Theorem**

Every signed graph with maximum average degree less than \(\frac{14}{5}\) admits a homomorphism to \((K_6, M)\). Moreover, the bound \(\frac{14}{5}\) is the best possible.

**Special case of Theorem 2.5 [O. V. Borodin, S.-J. Kim, A. V. Kostochka and D. B. West 2004]**

If \(G\) is a graph of girth at least 7 and maximum average degree at most \(\frac{28}{11}\), then \((G, \sigma) \rightarrow (K_6, M)\) for any signature \(\sigma\).
Mapping to \((K_6, M)\) and \((K_{2k}, M)\)

**Mapping to \((K_6, M)\)**

![Signed graphs \((K_6, M)\) and DSG\((K_6, M)\)](image)

*Figure: Signed graphs \((K_6, M)\) and DSG\((K_6, M)\)*
Sketch of the proof

- Assume to the contrary that a minimum counterexample $(G, \sigma)$ exists.
- Let $C$ be the vertex set of $\text{DSG}(K_6, M)$ and let $L$ be a list assignment of $V(G)$ where $L \subset C$. Study the properties of list $\text{DSG}(K_6, M)$-coloring.
- By extending a partial list coloring of a subgraph to the entire signed graph $(G, \sigma)$, we list all the forbidden configurations needed.
- Discharging technique.
Extending partial list-coloring: signed rooted tree

- A signed rooted tree \((T, \sigma)\) is depicted in the figure.
- For a vertex \(x\) of \((T, \sigma)\), we define the set of admissible colors, denoted \(L^a(x)\), to be the set of the colors \(c \in L(x)\) such that with the restriction of \(L\) onto \(T_x\) there exists an \(L\)-coloring \(\phi\) of \(T_x\) where \(\phi(x) = c\).

![Figure: \((T, \sigma)\) at root \(v\) and \((T_x, \sigma)\) at root \(x\)]
Extending partial list-coloring

- Pre-color the vertices of $G - H$ and modify the list of vertices of $H$ corresponding to the coloring of $G - H$.
- Prove that this updated list assignment is extendable. Hence, $H$ is a forbidden configuration of $G$.
Some of forbidden configurations

- 2₁-vertex, 3₂-vertex, 4₄-vertex, 5₅-vertex;

- It’s worth mentioning that we have a series of infinite forbidden configurations with some patterns.
Mapping to $(K_6, M)$ and $(K_{2k}, M)$

**Theorem**

Every signed graph with maximum average degree less than 3 admits a homomorphism to $(K_8, M)$. Moreover, the bound 3 is the best possible.

**Theorem**

Every signed graph with maximum average degree less than 3 admits a homomorphism to $(K_{2k}, M)$ for $k \geq 4$. Moreover, the bound 3 is the best possible.
Proposition

There exists a signed graph \((G, \sigma)\) with \(\text{mad}(G) = \frac{14}{5}\) which does not admit a homomorphism to \((K_6, M)\).

Figure: A signed graph with \(\text{mad} = \frac{14}{5}\) does not map to \((K_6, M)\)
Tightness

Proposition

There exists a signed bipartite planar graph \((G, \sigma)\) satisfying 
\[ g_{ij}(G, \sigma) \geq g_{ij}(K_{3,3}, M) \] which does not admit a homomorphism to 
\((K_{3,3}, M)\).

Figure: A signed bipartite planar graph does not map to \((K_{3,3}, M)\)
There exists a series of signed graphs \((G_l, \sigma)\), built from a negative \(l\)-cycle by adding a positive triangle on each edge, which do not map to \((K_{2k}, M)\) for \(k \geq 4\).

Figure: A tight example \((G_l, \sigma)\)
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Application to planarity

Corollary
Given a planar graph $G$ of girth 7, for every signature $\sigma$, $(G, \sigma) \rightarrow (K_6, M)$.

- We do not know whether 7 is the best possible girth condition.

Grötzsch’s theorem restated
Given a triangle-free planar graph $G$, the signed bipartite (planar) graph $S(G)$ maps to $(K_6, M)$.

- Note that $S(G)$ has negative 4-cycles but has no 6-cycle. Moreover, if $G$ is of girth 5, then $S(G)$ has no 8-cycles.
Steinberg’s type questions for \((K_6, M)\)

- Steinberg’s conjecture: Planar graphs with no cycle of length 4, 5, 6 are 3-colorable.
- This conjecture is disproved recently (V. Cohen-Addad, M. Hebdige, D. Král’, Z. Li and E. Salgado 2017).
- Planar graphs with no cycle of length 4, 5, 6, 7 are 3-colorable (O. V. Borodin, A. N. Glebov, A. Raspaud and M. R. Salavatipour 2005).

It’s natural to ask:

Steinberg’s type questions

What is the smallest value of \(k\), \(k \geq 3\), such that every signed bipartite planar graph with no 4-cycles sharing an edge and no cycles of length 6, 8, \ldots, 2k, admits a homomorphism to \((K_6, M)\)?
Mapping signed bipartite planar graphs to signed even cycles

- If a signed bipartite planar graph has no cycle of length smaller than 6, then it maps to \((C_4, e)\). (R. Naserasr, L. A. Pham and Z. Wang 2020+)

- If a signed bipartite planar graph has no cycle of length smaller than 4, then it maps to \((K_{3,3}, M)\).

Question
What is a sufficient girth condition for a signed bipartite planar graph to map to \(C_{-2k}\)?
Steinberg’s type questions for negative even cycles

- If $k$ is a prime number, then there exists an integer $f(k)$ such that any planar graph with no cycle of length $1, 2, \ldots, 2k, 2k + 2, \ldots, f(k)$ admits a mapping to $C_{2k+1}$. (X. Hu and J. Li 2020+)

- We can ask similar questions for mapping signed bipartite planar graphs to negative even cycles.
The end. Thank you!