Mapping sparse signed graphs to (K_{2k}, M)

Zhouningxin Wang

IRIF, Université de Paris

(A joint work with Reza Naserasr, Riste Skrekovski, Rongxing Xu)

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1. Introduction
   - Homomorphism of signed graphs
   - Homomorphism of signed bipartite graphs
2. Our work
   - Mapping to (K_6, M) and (K_{2k}, M)
   - Tightness
3. Conclusion and Discussion

Signed graphs

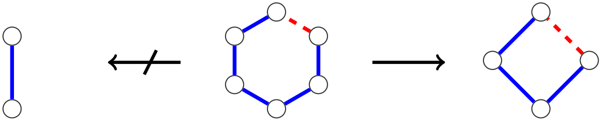
- A signed graph is a graph $G = (V, E)$ together with an assignment $\sigma : E(G) \to \{+, -\}$, denoted by (G, σ).
- A switching at vertex v is to switch the signs of all the edges incident to this vertex.
- We say (G, σ') is switching equivalent to (G, σ) if it is obtained from (G, σ) by switching at some vertices (allowing repetition).
- The sign of a closed walk is the product of signs of all the edges of this walk.

Theorem [T. Zaslavsky 1982]

Signed graphs (G, σ) and (G, σ') are switching equivalent if and only if they have a same set of negative cycles.

Homomorphism of signed graphs

- A homomorphism of signed graph (G, σ) to a signed graph (H, π) is a mapping φ from $V(G)$ and $E(G)$ to $V(H)$ and $E(H)$ (respectively) such that the adjacency, the incidence and the signs of closed walks are preserved.
- If there exists a homomorphism of (G, σ) to (H, π), we write $(G, \sigma) \to (H, \pi)$.



Edge-sign preserving homomorphism

- An edge-sign preserving homomorphism of signed graph (G, σ) to (H, π) is a mapping φ from $V(G)$ and $E(G)$ to $V(H)$ and $E(H)$ (respectively) such that for $uv \in E(G)$, $\varphi(u)\varphi(v) \in E(H)$ and $\sigma(uv) = \pi(\varphi(u)\varphi(v))$.
- If there exists an edge-sign preserving homomorphism of (G, σ) to (H, π), we write $(G, \sigma) \xrightarrow{s.p.} (H, \pi)$.

Proposition

Given signed graphs (G, σ) and (H, π),

$$(G, \sigma) \to (H, \pi) \Leftrightarrow \exists \sigma' \equiv \sigma, (G, \sigma') \xrightarrow{s.p.} (H, \pi).$$

Girth conditions

Considering the parity of the length of a closed walk and the sign of it, there are four possible types of closed walks:

- type 00 is a closed walk which is positive and of even length,
- type 01 is a closed walk which is positive and of odd length,
- type 10 is a closed walk which is negative and of even length,
- type 11 is a closed walk which is negative and of odd length.

The length of a shortest nontrivial closed walk in (G, σ) of type ij, ($ij \in \mathbb{Z}_2^2$), is denoted by $g_{ij}(G, \sigma)$.

No-homomorphism Lemma [R. Naserasr, E. Rollová and E. Sopena 2015]

If $(G, \sigma) \to (H, \pi)$, then $g_{ij}(G, \sigma) \geq g_{ij}(H, \pi)$ for $ij \in \mathbb{Z}_2^2$.

Necessary and Sufficient conditions

As No-homomorphism Lemma gives us a necessary condition for mapping (G, σ) to (H, π), is it also sufficient?

- For example, let H be a triangle and G be a Mycielski graph M_k for $k > 3$. The graph G is triangle-free but it has chromatic number $k > 3$.
- What kind of conditions can make it also sufficient? One possible constraint: maximum average degree.

Maximum average degree

Given a graph G, the maximum average degree, denoted $\mathrm{mad}(G)$, is the largest average degree taken over all the subgraphs of G.

Theorem [C. Charpentier, R. Naserasr and E. Sopena 2020]

Given a signed graph (H, π), there exists an $\epsilon > 0$ such that every signed graph (G, σ), satisfying $g_{ij}(G, \sigma) \geq g_{ij}(H, \pi)$ and $\mathrm{mad}(G) < 2 + \epsilon$, admits a homomorphism to (H, π).

A main question then is to find the best value of ϵ for a given signed graph (H, π).

- For (K_4, e), the best value of ϵ was proved to be $\frac{4}{7}$.
- For (K_6, M), we prove that the best value of ϵ is $\frac{4}{5}$.
- For (K_{2k}, M), $k \geq 4$, we prove that the best value of ϵ is 1.

Double Switching Graphs

Given a signed graph (G, σ) on the vertex set $V = \{x_1, \ldots, x_n\}$, the Double Switching Graph of (G, σ), denoted $\mathrm{DSG}(G, \sigma)$, is a signed graph built as follows:

- We have two disjoint copies of V, $V^+ = \{x_1^+, x_2^+, \ldots, x_n^+\}$ and $V^- = \{x_1^-, x_2^-, \ldots, x_n^-\}$ in $\mathrm{DSG}(G, \sigma)$.
- Each set of vertices V^+, V^- then induces a copy of (G, σ).
- Furthermore, a vertex x_i^- connects to vertices in V^+ as it is obtained from a switching on x_i. More precisely, if $x_i x_j$ is a positive (negative) edge in (G, σ), then $x_i^+ x_j^+, x_i^- x_j^-$ are positive (negative) edges in $\mathrm{DSG}(G, \sigma)$, and $x_i^+ x_j^-, x_i^- x_j^+$ are negative (positive) edges in $\mathrm{DSG}(G, \sigma)$.

Double Switching Graphs

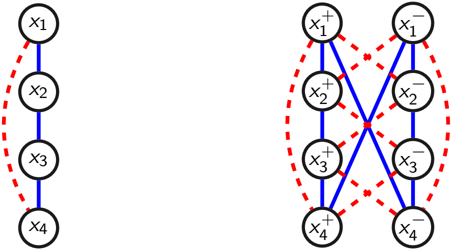


Figure: Signed graphs (C_4, e) and $\mathrm{DSG}(C_4, e)$

Theorem [R.C. Brewster and T. Graves 2009]

Given signed graphs (G, σ) and (H, π),

$$(G, \sigma) \to (H, \pi) \Leftrightarrow (G, \sigma) \xrightarrow{s.p.} \mathrm{DSG}(H, \pi).$$

Indicator construction $S(G)$

Given a graph G, a signed graph $S(G)$ is built as follows:

- Take the vertex set $V(G)$;
- For each edge uv of G, we add two more vertices x_{uv} and y_{uv}, and connect them with both of u and v (noting that uv is not an edge of $S(G)$);
- For each 4-cycle $ux_{uv}vy_{uv}$, we assign a negative sign to one of the edges.

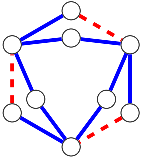


Figure: $S(K_3)$

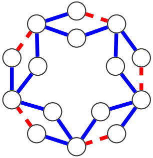


Figure: $S(C_5)$

A strengthening of Four-Color Theorem

Theorem [R. Naserasr, E. Rollová and E. Sopena 2015]

- A graph G is bipartite if and only if $S(G) \to (K_{2,2}, e)$.
- A graph G is k-colorable for $k \geq 3$ if and only if $S(G) \to (K_{k,k}, M)$.

Four-Color Theorem restated

For every planar simple graph G, $S(G) \to (K_{4,4}, M)$.

The following is a strengthening of the Four-Color Theorem (proof of which is based on an edge-coloring result of B. Guenin which in turn is based on the Four-Color Theorem).

Theorem [R. Naserasr, E. Rollová and E. Sopena 2013]

Every signed bipartite planar (simple) graph maps to $(K_{4,4}, M)$.

Mapping signed bipartite graphs to $(K_{4,4}, M)$

- For planar graphs, the homomorphism problem of planar graphs to K_3, which is a non-trivial core subgraph of K_4, has been greatly studied.
- Grötzsch's theorem states that planar graph of girth at least 4 maps to K_3 and 3-coloring problem of planar graphs is proved to be NP-complete.
- It is natural to ask for each core subgraphs of $(K_{4,4}, M)$ which families of planar graphs map to. Two notable subgraphs:
  - the negative 4-cycle;
  - $(K_{3,3}, M)$.

  The question of mapping signed bipartite planar graphs to $(K_{3,3}, M)$ captures 3-coloring problem of planar graphs.

Homomorphism to $(K_{k,k}, M)$ and (K_{2k}, M)

Theorem

For a signed bipartite graph (G, σ),

$$(G, \sigma) \to (K_{k,k}, M) \Leftrightarrow (G, \sigma) \to (K_{2k}, M).$$

We prove:

- Every signed graph (G, σ) with $\mathrm{mad}(G) < \frac{14}{5}$ and satisfying $g_{ij}(G, \sigma) \geq g_{ij}(K_6, M)$ admits a homomorphism to (K_6, M).
- Every signed graph (G, σ) with $\mathrm{mad}(G) < 3$ and satisfying $g_{ij}(G, \sigma) \geq g_{ij}(K_{2k}, M)$ admits a homomorphism to (K_{2k}, M) for $k \geq 4$.

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Mapping to (K_6, M)

Theorem

Every signed graph with maximum average degree less than $\frac{14}{5}$ admits a homomorphism to (K_6, M). Moreover, the bound $\frac{14}{5}$ is the best possible.

Special case of Theorem 2.5 [O. V. Borodin, S.-J. Kim, A. V. Kostochka and D. B. West 2004]

If G is a graph of girth at least 7 and maximum average degree at most $\frac{28}{11}$, then $(G, \sigma) \to (K_6, M)$ for any signature σ.

Mapping to (K_6, M)

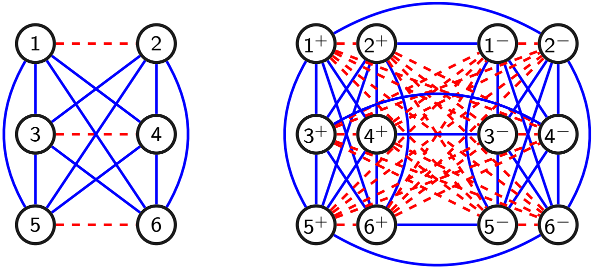


Figure: Signed graphs (K_6, M) and $\mathrm{DSG}(K_6, M)$

Sketch of the proof

- Assume to the contrary that a minimum counterexample (G, σ) exists.
- Let C be the vertex set of $\mathrm{DSG}(K_6, M)$ and let L be a list assignment of $V(G)$ where $L \subset C$. Study the properties of list $\mathrm{DSG}(K_6, M)$-coloring.
- By extending a partial list coloring of a subgraph to the entire signed graph (G, σ), we list all the forbidden configurations needed.
- Discharging technique.

Extending partial list-coloring: signed rooted tree

- A signed rooted tree (T, σ) is depicted in the figure.
- For a vertex x of (T, σ), we define the set of admissible colors, denoted $L^a(x)$, to be the set of the colors $c \in L(x)$ such that with the restriction of L onto T_x there exists an L-coloring ϕ of T_x where $\phi(x) = c$.

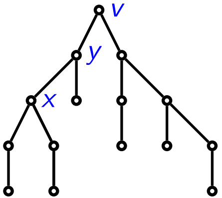


Figure: (T, σ) at root v and (T_x, σ) at root x

Extending partial list-coloring

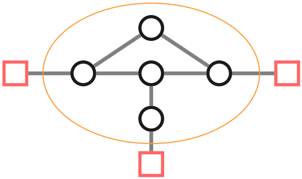
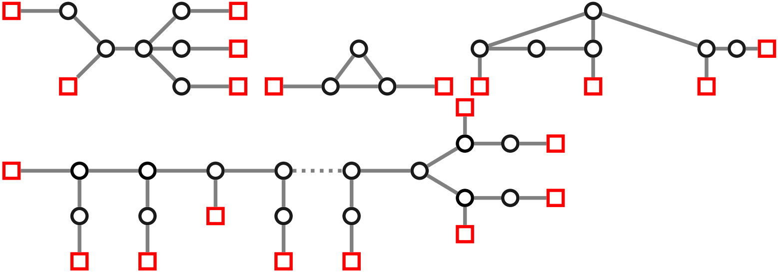


Figure: Extending partial list-coloring

- Pre-color the vertices of $G - H$ and modify the list of vertices of H corresponding to the coloring of $G - H$.
- Prove that this updated list assignment is extendable. Hence, H is a forbidden configuration of G.

Some of forbidden configurations

- 2_1-vertex, 3_2-vertex, 4_4-vertex, 5_5-vertex;



- It's worth mentioning that we have a series of infinite forbidden configurations with some patterns.

Mapping to (K_8, M)

Theorem

Every signed graph with maximum average degree less than 3 admits a homomorphism to (K_8, M). Moreover, the bound 3 is the best possible.

Theorem

Every signed graph with maximum average degree less than 3 admits a homomorphism to (K_{2k}, M) for $k \geq 4$. Moreover, the bound 3 is the best possible.

Tightness

Proposition

There exists a signed graph (G, σ) with $\mathrm{mad}(G) = \frac{14}{5}$ which does not admit a homomorphism to (K_6, M).

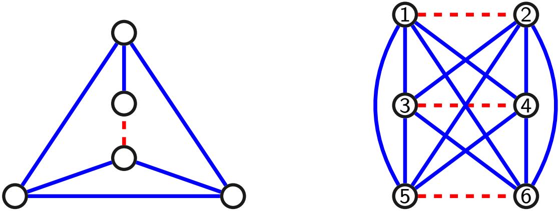
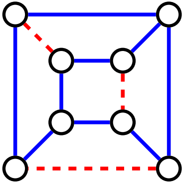


Figure: A signed graph with $\mathrm{mad} = \frac{14}{5}$ does not map to (K_6, M)

Tightness

Proposition

There exists a signed bipartite planar graph (G, σ) satisfying $g_{ij}(G, \sigma) \geq g_{ij}(K_{3,3}, M)$ which does not admit a homomorphism to $(K_{3,3}, M)$.



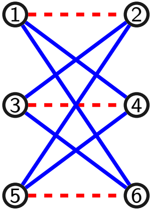


Figure: A signed bipartite planar graph does not map to $(K_{3,3}, M)$

Tightness

Proposition

There exists a series of signed graphs (G_l, σ), built from a negative l-cycle by adding a positive triangle on each edge, which do not map to (K_{2k}, M) for $k \geq 4$.

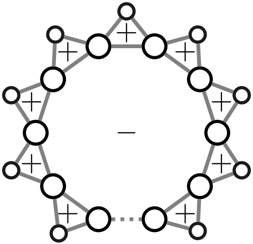


Figure: A tight example (G_l, σ)

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Application to planarity

Corollary

Given a planar graph G of girth 7, for every signature σ, $(G, \sigma) \to (K_6, M)$.

- We do not know whether 7 is the best possible girth condition.

Grötzsch's theorem restated

Given a triangle-free planar graph G, the signed bipartite (planar) graph $S(G)$ maps to (K_6, M).

- Note that $S(G)$ has negative 4-cycles but has no 6-cycle. Moreover, if G is of girth 5, then $S(G)$ has no 8-cycles.

Steinberg's type questions for (K_6, M)

- Steinberg's conjecture: Planar graphs with no cycle of length $4, 5, 6$ are 3-colorable.
- This conjecture is disproved recently (V. Cohen-Addad, M. Hebdige, D. Král', Z. Li and E. Salgado 2017).
- Planar graphs with no cycle of length $4, 5, 6, 7$ are 3-colorable (O. V. Borodin, A. N. Glebov, A. Raspaud and M. R. Salavatipour 2005).

It's natural to ask:

Steinberg's type questions

What is the smallest value of k, $k \geq 3$, such that every signed bipartite planar graph with no 4-cycles sharing an edge and no cycles of length $6, 8, \ldots, 2k$, admits a homomorphism to (K_6, M)?

Mapping signed bipartite planar graphs to signed even cycles

- If a signed bipartite planar graph has no cycle of length smaller than 6, then it maps to (C_4, e). (R. Naserasr, L. A. Pham and Z. Wang 2020+)
- If a signed bipartite planar graph has no cycle of length smaller than 4, then it maps to $(K_{3,3}, M)$.

Question

What is a sufficient girth condition for a signed bipartite planar graph to map to C_{-2k}?

Steinberg's type questions for negative even cycles

- If k is a prime number, then there exists an integer $f(k)$ such that any planar graph with no cycle of length $1, 2, \ldots, 2k$, $2k+2, \ldots, f(k)$ admits a mapping to C_{2k+1}. (X. Hu and J. Li 2020+)
- We can ask similar questions for mapping signed bipartite planar graphs to negative even cycles.

The end. Thank you!