Density of C_{-4}-critical signed graphs

Zhouningxin Wang

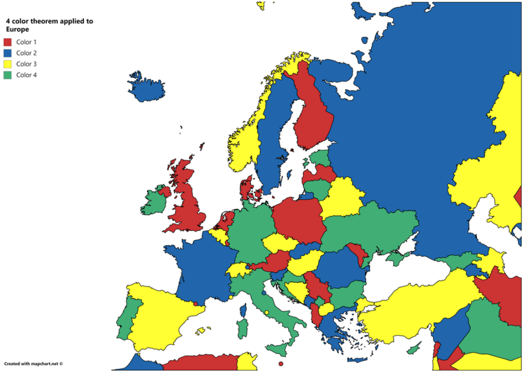
IRIF, Université de Paris

(A joint work with Reza Naserasr, Lan-Anh Pham)

2nd Feb. 2021

1. Introduction
   - Start from Four-Color Theorem
   - Homomorphism of signed graphs
   - (H, π)-critical signed graphs
   - Jaeger-Zhang conjecture and its bipartite analog
2. Density of C_{-4}-critical signed graphs
   - C_{-4}-critical signed graphs
   - Application to the planarity
3. Conclusion

Coloring the map with 4 colors

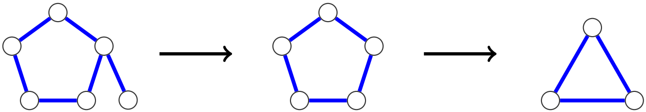


Four-Color Theorem

Every planar graph is 4-colorable.

H-coloring

- A homomorphism of a graph G to another graph H is a mapping from $V(G)$ to $V(H)$ such that the adjacency is preserved.
- If G admits a homomorphism to H, we also say G is H-colorable.



Four-Color Theorem restated

Every planar graph admits a K_4-coloring.

$(2k+1)$-coloring problem vs C_{2k+1}-coloring problem

Given a graph G, we define $T_k(G)$ to be the graph obtained from G by replacing each edge uv with a path of length k.

Indicator Construction Lemma [P. Hell, J. Nešetřil 1990]

A graph G is $(2k+1)$-colorable if and only if $T_{2k-1}(G)$ is C_{2k+1}-colorable.

- The C_{2k+1}-coloring problem captures the $(2k+1)$-coloring problem.
- The C_{2k+1}-coloring problem is NP-complete. (H.A. Maurer, J.H. Sudborough, E. Welzl 1981)

Can we make use of even cycles to capture $2k$-coloring problem?

Signed graphs

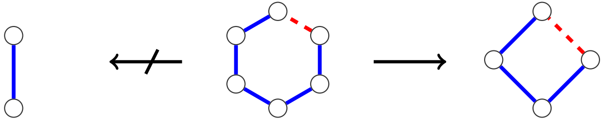
- A signed graph is a graph $G = (V, E)$ together with an assignment $\{+, -\}$ on its edges, denoted by (G, σ).
- A switching at vertex v is to switch the signs of all the edges incident to this vertex.
- We say (G, σ') is switching equivalent to (G, σ) if it is obtained from (G, σ) by switching at some vertices (allowing repetition).
- The sign of a closed walk is the product of signs of all the edges of this walk.

Theorem [T. Zaslavsky 1982]

Signed graphs (G, σ) and (G, σ') are switching equivalent if and only if they have a same set of negative cycles.

Homomorphism of signed graphs

- A homomorphism of a signed graph (G, σ) to (H, π) is a mapping φ from $V(G)$ and $E(G)$ to $V(H)$ and $E(H)$ (respectively) such that the adjacency, the incidence and the signs of closed walks are preserved.
- An edge-sign preserving homomorphism of a signed graph (G, σ) to (H, π) is a mapping φ from $V(G)$ and $E(G)$ to $V(H)$ and $E(H)$ (respectively) such that for $uv \in E(G)$, $\varphi(u)\varphi(v) \in E(H)$ and $\sigma(uv) = \pi(\varphi(u)\varphi(v))$.
- $(G, \sigma) \to (H, \pi) \Leftrightarrow \exists \sigma' \equiv \sigma, (G, \sigma') \xrightarrow{s.p.} (H, \pi)$.



No-homomorphism Lemma

There are four possible types of closed walks in signed graphs:

- type 00 is a closed walk which is positive and of even length,
- type 01 is a closed walk which is positive and of odd length,
- type 10 is a closed walk which is negative and of even length,
- type 11 is a closed walk which is negative and of odd length.

The length of a shortest nontrivial closed walk in (G, σ) of type ij, $(ij \in \mathbb{Z}_2^2)$, is denoted by $g_{ij}(G, \sigma)$.

No-homomorphism Lemma

If $(G, \sigma) \to (H, \pi)$, then $g_{ij}(G, \sigma) \geq g_{ij}(H, \pi)$ for $ij \in \mathbb{Z}_2^2$.

k-coloring problem vs C_{-k}-coloring problem

Given a signed graph (G, σ), we define $T_k(G, \sigma)$ to be a signed graph obtained from (G, σ) by replacing each edge uv with a signed path of length k with sign $-\sigma(uv)$.

Lemma

A graph G is k-colorable if and only if $T_{k-2}(G, +)$ is C_{-k}-colorable.

In particular, $2k$-coloring problem of graphs is captured by C_{-2k}-coloring problem of signed (bipartite) graphs.

Special case when $k = 4$

A graph G is 4-colorable if and only if $T_2(G, +)$ is C_{-4}-colorable.

Proof of $G \to K_4 \Leftrightarrow T_2(G,+) \to C_{-4}$

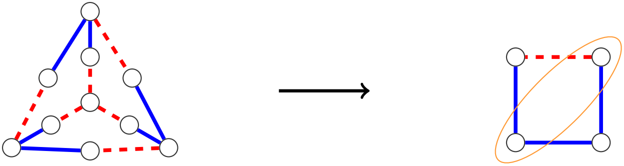


Figure: $G \to K_4 \Rightarrow T_2(G,+) \to C_{-4}$

- $\Rightarrow$: It suffices to show that $T_2(K_4) \to C_{-4}$.
- $\Leftarrow$: Let $\varphi : T_2(G,+) \to C_{-4}$. This mapping preserves the bipartition.

Edge-sign preserving homomorphism to C_{-4}

Lemma [C. Charpentier, R. Naserasr, and E. Sopena 2020]

A signed bipartite graph (G, σ) admits an edge-sign preserving homomorphism to C_{-4} if and only if (P_3, π) does not admit an edge-sign preserving homomorphism to (G, σ) where (P_3, π) is the signed path of length 3 given below.

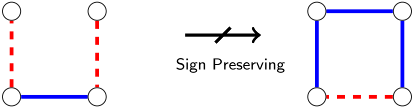


Figure: C_{-4} and its edge-sign preserving dual

NP-completeness of C_{-4}-coloring problem

- In order to map a signed bipartite graph (G, σ) to C_{-4}, it is necessary and sufficient to find an equivalent signature σ' of σ where no positive edge is incident with a negative edge at each of its end.
- Deciding whether there exists an edge-sign preserving homomorphism to C_{-4} is in polynomial time but finding such an equivalent signature is hard.
- The C_{-4}-coloring problem is NP-complete. (R. C. Brewster, F. Foucaud, P. Hell and R. Naserasr 2017)

k-critical and H-critical

- A graph is k-critical if it is k-chromatic but any proper graph of it is $(k-1)$-colorable.
- A graph is H-critical if it is not H-colorable but any proper graph of it is H-colorable. (P. A. Catlin 1988)
- k-critical $\Leftrightarrow$ K_{k-1}-critical

The popular question of H-critical graphs on n vertices is to bound below the number of edges as a function of n.

- Any C_3-critical (4-critical) graph on n vertices has at least $\frac{5n-2}{3}$ edges; (A. Kostochka and M. Yancey 2014)
- Any C_5-critical graph on n vertices has at least $\frac{5n-2}{4}$ edges; (Z. Dvorak and L. Postle 2017)
- Any C_7-critical graph on n vertices has at least $\frac{17n-2}{15}$ edges. (L. Postle and E. Smith-Roberge 2019)

(H, π)-critical signed graph

A signed graph (G, σ) is (H, π)-critical if the followings hold:

- $g_{ij}(G, \sigma) \geq g_{ij}(H, \pi)$;
- $(G, \sigma) \not\to (H, \pi)$;
- $(G', \sigma) \to (H, \pi)$ for any proper subgraph $(G', \sigma) \subset (G, \sigma)$.

We observe that:

- A graph G is k-critical if the signed graph $(G, +)$ is $(K_{k-1}, +)$-critical.
- By No-homomorphism Lemma, our first condition eliminates trivial cases.

C_{-4}-critical signed graph

We say a signed graph (G, σ) is C_{-4}-critical if the followings hold:

- (G, σ) is bipartite and its negative-girth is at least 4;
- $(G, \sigma) \not\to C_{-4}$;
- $(G', \sigma) \to C_{-4}$ for any proper subgraph $(G', \sigma) \subset (G, \sigma)$.

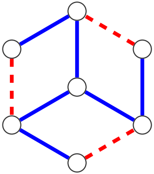


Figure: $\hat{W}$

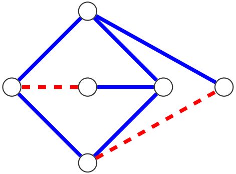


Figure: Γ

Jaeger-Zhang Conjecture

Jaeger-Zhang Conjecture [C.-Q. Zhang 2002]

Every planar graph of odd-girth at least $4k+1$ admits a homomorphism to C_{2k+1}.

- $k = 1$: Grötzsch's theorem;
- $k = 2$: true for odd-girth 11 (Z. Dvořák and L. Postle 2017);
- $k \geq 3$:
  - true for odd-girth $8k-3$ (X. Zhu 2001);
  - true for odd-girth $\frac{20k-2}{3}$ (O.V. Borodin, S.-J. Kim, A.V. Kostochka and D.B. West 2002);
  - true for odd-girth $6k+1$ (L. M. Lovász, C. Thomassen, Y. Wu and C. Q. Zhang 2013).

Signed bipartite analog of Jaeger-Zhang Conjecture

Signed bipartite analog of Jaeger-Zhang Conjecture [R. Naserasr, E. Rollová, É. Sopena 2015]

Every signed bipartite planar graph of negative-girth at least $4k-2$ admits a homomorphism to C_{-2k}.

- For mapping to C_{-4}, 8 is the best negative-girth condition;
- For any $k \geq 3$, true for negative-girth $8k-2$ (C. Charpentier, R. Naserasr, and E. Sopena 2020).

1 Introduction
- Start from Four-Color Theorem
- Homomorphism of signed graphs
- (H, π)-critical signed graphs
- Jaeger-Zhang conjecture and its bipartite analog

2 Density of C_{-4}-critical signed graphs
- C_{-4}-critical signed graphs
- Application to the planarity

3 Conclusion

Density of C_{-4}-critical signed graphs

Theorem

If $\hat{G}$ is a C_{-4}-critical signed graph which is not isomorphic to $\hat{W}$, then

$$|E(\hat{G})| \geq \frac{4|V(\hat{G})|}{3}.$$

This theorem is tight due to a construction on a series of C_{-4}-critical signed graphs with edge-density $\frac{4}{3}$.

Corollary

Every signed bipartite planar graph with negative-girth at least 8 admits a homomorphism to C_{-4}.

Technique of the proof

Assume to the contrary that a minimum counterexample $\hat{G} = (G, \sigma)$ exists.

- The minimum counterexample $\hat{G}$ is 2-connected.
- There must exist a 2-vertex in $\hat{G}$.
- There is no 3-thread in $\hat{G}$.

We denote $P_2(\hat{H})$ to be a graph obtained from $\hat{H}$ by adding a vertex v and joining it with two vertices in $\hat{H}$ (with any signature).

Technique of the proof

- The potential of a signed graph is defined to be

$$p(\hat{G}) = 4|V(\hat{G})| - 3|E(\hat{G})|.$$

  We will estimate the potentials of some subgraphs of $\hat{G}$ and find some forbidden configuration in $\hat{G}$.
- The minimum counterexample $\hat{G}$ is a C_{-4}-critical signed graph which is not isomorphic to $\hat{W}$, it satisfies $p(\hat{G}) \geq 1$, and that for any signed graph $\hat{H}$, $\hat{H} \neq \hat{W}$, with $|V(\hat{H})| < |V(\hat{G})|$ satisfying $p(\hat{H}) \geq 1$, $\hat{H}$ admits a homomorphism to C_{-4}.
- We will find more forbidden configurations and apply discharging technique.

Key Lemma

Lemma (Potential of subgraphs)

Let $\hat{G} = (G, \sigma)$ be a minimum counterexample and let $\hat{H}$ be a subgraph of $\hat{G}$. Then

1. $p(\hat{H}) \geq 1$ if $\hat{G} = \hat{H}$;
2. $p(\hat{H}) \geq 3$ if $\hat{G} = P_2(\hat{H})$;
3. $p(\hat{H}) \geq 4$ otherwise.

Sketch of the proof

- Suppose to the contrary that $\hat{G}$ contains a proper subgraph $\hat{H}$ which does not satisfy $\hat{G} = P_2(\hat{H})$, and satisfies $p(\hat{H}) \leq 3$. We take the maximum such $\hat{H}$.
- Notice that $\hat{H}$ is a proper induced subgraph of size at least 5. Let φ be a mapping of $\hat{H}$ to C_{-4}.
- Define $\hat{G}_1$ to be a signed (multi)graph obtained from $\hat{G}$ by first identifying vertices of $\hat{H}$ which are mapped to a same vertex of C_{-4} under φ. We conclude that $\hat{G}_1 \not\to C_{-4}$.
- Two possibilities: Either $\hat{G}_1$ contains a C_{-2}, or $\hat{G}_1$ contains a C_{-4}-critical subgraph $\hat{G}_2$.

Sketch of the proof

- Case 1: $\hat{G}_1$ contains a C_{-2}. Then by computing $p(\hat{H} + P_{-2})$ and using the maximality of $\hat{H}$, we can obtain the contradiction.
- Case 2: $\hat{G}_1$ contains a C_{-4}-critical subgraph $\hat{G}_2$. First of all by the minimum counterexample, we have $p(\hat{G}_2) \leq 1$. Then we define signed graph $\hat{G}_3$ by combing $\hat{G}_2$ and $\hat{H}$ with some modifications. By the relation of $\hat{H} \subsetneq \hat{G}_3 \subset \hat{G}$, it leads to a contradiction with $p(\hat{H}) \leq 3$.

Forbidden configurations

Lemma

Two 4-cycles in the minimum counterexample $\hat{G}$ cannot share one edge or two edges.

Lemma

Let vv_1u be a 2-thread in the minimum counterexample $\hat{G}$. Suppose that v is a 3-vertex and let v_2, v_3 be the other two neighbors of v. Then the path v_2vv_3 must be contained in a negative 4-cycle in $\hat{G}$.

Lemma

A vertex of degree 3 in the minimum counterexample $\hat{G}$ does not have two neighbors of degree 2.

Constructions of C_{-4}-critical signed graphs of density $\frac{4}{3}$

Given a graph G, let $\tilde{G}$ be a signed graph obtained by replacing each edge of G by C_{-2}.

Lemma

A graph G is $(k+1)$-critical if and only if $T_{2k-2}(\tilde{G})$ is C_{-2k}-critical.

As odd cycles are the only 3-critical graphs, $T_2(\tilde{C}_{2k+1})$, for each $k \geq 1$, is a C_{-4}-critical signed graph whose density is $\frac{4}{3} = \frac{8k+4}{6k+3}$.

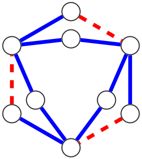


Figure: $T_2(\tilde{C}_3)$

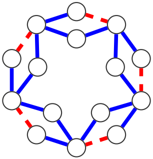


Figure: $T_2(\tilde{C}_5)$

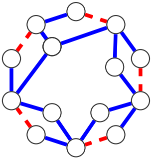


Figure: $\hat{G}'_5$

Constructions of sparse C_{-4}-critical signed graphs

We have a C_{-4}-critical signed graph on n vertices for each $n \geq 9$.
Let $\hat{G}_1$ and $\hat{G}_2$ be two C_{-4}-critical signed graphs.

- Assuming that there is a 2-vertex u in $\hat{G}_1$ with u_1, u_2 being its neighbors and a 2-vertex v in $\hat{G}_2$ with v_1, v_2 being its neighbors, we build a signed graph $\mathcal{F}(\hat{G}_1, \hat{G}_2)$ from disjoint union of $\hat{G}_1$ and $\hat{G}_2$ by deleting u and v, and adding a positive edge u_1v_1 and a negative edge u_2v_2.

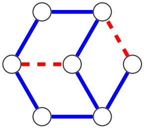


Figure: $\hat{W}$

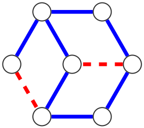


Figure: $\hat{W}$

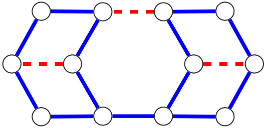


Figure: $\mathcal{F}(\hat{W}, \hat{W})$

Constructions of sparse C_{-4}-critical signed graphs

Analog of Hajo's construction

- Assuming that there is a positive edge x_1y_1 in $\hat{G}_1$ and a negative edge x_2y_2 in $\hat{G}_2$, we build a signed graph $\mathcal{H}(\hat{G}_1, \hat{G}_2)$ from disjoint union of $\hat{G}_1$ and $\hat{G}_2$ by deleting x_1y_1, x_2y_2 and identifying x_1 with x_2 and y_1 with y_2.

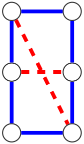


Figure: Γ

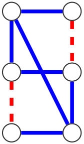


Figure: Γ

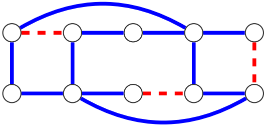


Figure: $\mathcal{H}(\Gamma, \Gamma)$

Mapping signed bipartite planar graphs to C_{-4}

A signed graph (G, σ) is 2k-colorable if there exists a mapping $c : V(G) \to \{\pm 1, \ldots, \pm k\}$ such that for each edge uv of G, $c(x) \neq \sigma(uv)c(y)$.

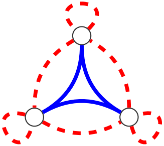


Figure: $\tilde{K}_3^+$

Conjecture [E. Máčajová, A. Raspaud, M. Škoviera 2016]

Every signed planar simple graph is 4-colorable.

Mapping signed bipartite planar graphs to C_{-4}

Theorem [F. Kardoš, J. Narboni 2020]

There exists a signed planar simple graph which is not 4-colorable.

Lemma

A signed graph (G, σ) is $2k$-colorable if and only if $T_{2k-2}(G, \sigma)$ is C_{-2k}-colorable.

When $k = 2$, (G, σ) is 4-colorable if and only if $T_2(G, \sigma)$ is C_{-4}-colorable. Therefore, there exists a signed graph $T_2(G, \sigma)$ which does not admit a homomorphism to C_{-4}.

Theorem

There exists a bipartite planar graph G of girth 6 with a signature σ such that $(G, \sigma) \not\to C_{-4}$.

Mapping signed bipartite planar graphs to C_{-4}

- By Folding Lemma, starting from a signed bipartite planar graph whose shortest negative cycles are of length at least 8, we get a homomorphic image $\hat{G}$ with a planar embedding where all faces are (negative) 8-cycles.
- Applying Euler's Formula on this graph, we have $|E(G)| \leq \frac{3(|V(G)|-2)}{4}$.

Theorem

Every signed bipartite planar graph with negative-girth at least 8 admits a homomorphism to C_{-4}. Moreover, the girth condition is the best possible.

1. Introduction
   - Start from Four-Color Theorem
   - Homomorphism of signed graphs
   - (H, π)-critical signed graphs
   - Jaeger-Zhang conjecture and its bipartite analog

2. Density of C_{-4}-critical signed graphs
   - C_{-4}-critical signed graphs
   - Application to the planarity

3. Conclusion

Relation with circular coloring of signed graphs

- In a joint work with Xuding Zhu, we have defined circular chromatic number of signed graphs. We prove that for any signed bipartite graph (G, σ),

$$X_c(G, \sigma) \leq \frac{8}{3} \Leftrightarrow (G, \sigma) \to C_{-4}.$$

- So our work can be restated as: Any $\frac{8}{3}$-critical signed bipartite graph has at least $\frac{4n}{3}$ edges except for $\hat{W}$.

Discussion

- We look for some strong sufficient conditions for signed bipartite planar graphs mapping to C_{-4}.

Conjecture

Let G be a bipartite planar graph of girth at least 6. Let σ be a signature on G such that in (G, σ) all 6-cycles are of a same sign. Then $(G, \sigma) \to C_{-4}$.

- It contains Four-Color Theorem by T_2 construction on a planar simple graph.

Discussion

- We determined that the best girth condition for mapping signed bipartite planar graphs to C_{-4} is 8 rather than 6.

Question

What is the girth condition for signed bipartite planar graphs mapping to C_{-2k}?

The end. Thank you!