Mapping sparse signed graphs to (K_{2k}, M)

Zhouningxin Wang

IRIF, Université de Paris

(Joint work with Reza Naserasr, Riste Skrekovski, Rongxing Xu)

3rd May 2021

1 Introduction
- Homomorphism of signed graphs
- Homomorphism of signed bipartite graphs

2 Our work
- Mapping signed graphs to (K_6, M)
- Tightness

3 Conclusion and Discussion

Homomorphism of signed graphs

- A homomorphism of a signed graph (G, σ) to a signed graph (H, π) is a mapping φ from $V(G)$ and $E(G)$ to $V(H)$ and $E(H)$ (respectively) such that the adjacencies, the incidences and the signs of the closed walks are preserved. If there exists one, we write $(G, \sigma) \to (H, \pi)$.
- A homomorphism of (G, σ) to (H, π) is said to be edge-sign preserving if it, furthermore, preserves the signs of edges. If there exists one, we write $(G, \sigma) \xrightarrow{s.p.} (H, \pi)$.
- $(G, \sigma) \to (H, \pi) \Leftrightarrow \exists \sigma' \equiv \sigma, (G, \sigma') \xrightarrow{s.p.} (H, \pi)$.

Homomorphism of signed graphs

Double Switching Graphs $\mathrm{DSG}(G, \sigma)$

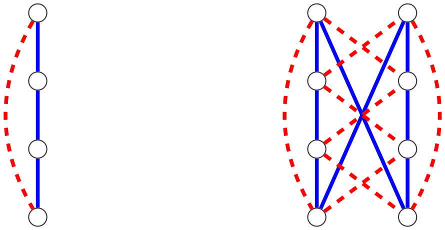


Figure: Signed graphs (C_4, e) and $\mathrm{DSG}(C_4, e)$

Theorem [R.C. Brewster and T. Graves 2009]

Given signed graphs (G, σ) and (H, π),

$$(G, \sigma) \to (H, \pi) \Leftrightarrow (G, \sigma) \xrightarrow{s.p.} \mathrm{DSG}(H, \pi).$$

Girth conditions

Given a signed graph (G, σ) and an element $ij \in \mathbb{Z}_2^2$, we define $g_{ij}(G, \sigma)$ to be the length of a shortest closed walk W whose number of negative edges modulo 2 is i and whose length modulo 2 is j.

No-homomorphism Lemma [R. Naserasr, E. Rollová and E. Sopena 2015]

If $(G, \sigma) \to (H, \pi)$, then $g_{ij}(G, \sigma) \geq g_{ij}(H, \pi)$ for $ij \in \mathbb{Z}_2^2$.

Necessary and Sufficient conditions

As No-homomorphism Lemma gives us a necessary condition for mapping (G, σ) to (H, π), is it also sufficient?

- For example, let H be a triangle and let G be a Mycielski graph M_k for $k > 3$. The graph G is triangle-free but it has chromatic number $k > 3$.
- What kind of conditions can make it also sufficient?
  One possible constraint: maximum average degree.

Maximum average degree

Given a graph G, the maximum average degree, denoted $\mathrm{mad}(G)$, is the largest average degree taken over all the subgraphs of G.

Theorem [C. Charpentier, R. Naserasr and E. Sopena 2020]

Given a signed graph (H, π), there exists an $\epsilon > 0$ such that every signed graph (G, σ), satisfying $g_{ij}(G, \sigma) \geq g_{ij}(H, \pi)$ and $\mathrm{mad}(G) < 2 + \epsilon$, admits a homomorphism to (H, π).

A main question then is to find the best value of ϵ for a given signed graph (H, π).

- For (K_4, e), the best value of ϵ was proved to be $\frac{4}{7}$.
- For (K_6, M), we prove that the best value of ϵ is $\frac{4}{5}$.
- For (K_{2k}, M), $k \geq 4$, we prove that the best value of ϵ is 1.

Indicator construction $S(G)$

Given a graph G, a signed graph $S(G)$ is built as follows:

- Take the vertex set $V(G)$;
- For each edge uv of G, we add two more vertices x_{uv} and y_{uv}, and connect them with both of u and v (noting that uv is not an edge of $S(G)$);
- For each 4-cycle $ux_{uv}vy_{uv}$, we assign a negative sign to one of the edges.

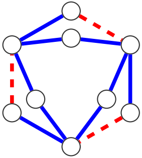


Figure: $S(K_3)$

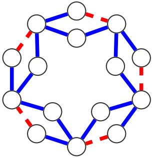


Figure: $S(C_5)$

A strengthening of Four-Color Theorem

Theorem [R. Naserasr, E. Rollová and E. Sopena 2015]

- A graph G is bipartite if and only if $S(G) \to (K_{2,2}, e)$.
- A graph G is k-colorable for $k \geq 3$ if and only if $S(G) \to (K_{k,k}, M)$.

Four-Color Theorem restated

For every planar simple graph G, $S(G) \to (K_{4,4}, M)$.

The following is a strengthening of the Four-Color Theorem (proof of which is based on an edge-coloring result of B. Guenin which in turn is based on the Four-Color Theorem).

Theorem [R. Naserasr, E. Rollová and E. Sopena 2013]

Every signed bipartite planar (simple) graph maps to $(K_{4,4}, M)$.

Mapping signed bipartite graphs to $(K_{4,4}, M)$

- The homomorphism problem of planar graphs to K_3, which is a non-trivial core subgraph of K_4, has been greatly studied.
- Grötzsch's theorem states that planar graph of girth at least 4 maps to K_3 and 3-coloring problem of planar graphs is proved to be NP-complete.
- It is natural to ask for each core subgraphs of $(K_{4,4}, M)$ which families of planar graphs map to. Two notable subgraphs:
  - $(K_{3,3}, M)$;
  - C_{-4}.

  The question of mapping signed bipartite planar graphs to $(K_{3,3}, M)$ captures 3-coloring problem of planar graphs.

Homomorphism to $(K_{k,k}, M)$ and (K_{2k}, M)

Theorem [R. Naserasr, R. Skrekovski, Z. Wang and R. Xu 2020+]

For a signed bipartite graph (G, σ),

$$(G, \sigma) \to (K_{k,k}, M) \Leftrightarrow (G, \sigma) \to (K_{2k}, M).$$

We prove:

- Every signed graph (G, σ) with $\mathrm{mad}(G) < \frac{14}{5}$ admits a homomorphism to (K_6, M).
- Every signed graph (G, σ) with $\mathrm{mad}(G) < 3$ admits a homomorphism to (K_{2k}, M) for $k \geq 4$.

1. Introduction
   - Homomorphism of signed graphs
   - Homomorphism of signed bipartite graphs
2. Our work
   - Mapping signed graphs to (K_6, M)
   - Tightness
3. Conclusion and Discussion

Homomorphism of signed graphs to (K_6, M)

Theorem [R. Naserasr, R. Skrekovski, Z. Wang and R. Xu 2020+]

Every signed graph with maximum average degree less than $\frac{14}{5}$ admits a homomorphism to (K_6, M). Moreover, the bound $\frac{14}{5}$ is the best possible.

Special case of Theorem 2.5 [O. V. Borodin, S.-J. Kim, A. V. Kostochka and D. B. West 2004]

If G is a graph of girth at least 7 and maximum average degree at most $\frac{28}{11}$, then $(G, \sigma) \to (K_6, M)$ for any signature σ.

(K_6, M) and DSG(K_6, M)

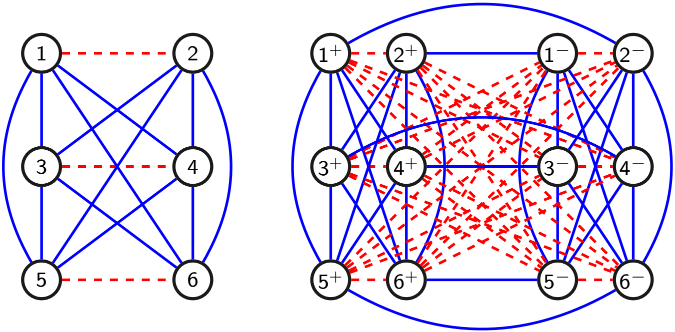


Figure: Signed graphs (K_6, M) and $\mathrm{DSG}(K_6, M)$

Sketch of the proof

- Assume to the contrary that a minimum counterexample (G, σ) exists.
- Let C be the vertex set of $\mathrm{DSG}(K_6, M)$ and let L be a list assignment of $V(G)$ where $L \subset C$. Study the properties of list $\mathrm{DSG}(K_6, M)$-coloring.
- By extending a partial list coloring of a subgraph to the entire signed graph (G, σ), we list all the forbidden configurations needed.
- Discharging technique.

Extending partial list-coloring: signed rooted tree

- A signed rooted tree (T, σ) is depicted in the figure.
- For a vertex x of (T, σ), we define the set of admissible colors, denoted $L^a(x)$, to be the set of the colors $c \in L(x)$ such that with the restriction of L onto T_x there exists an L-coloring ϕ of T_x where $\phi(x) = c$.

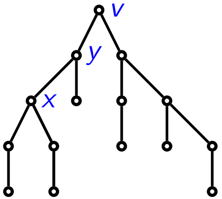


Figure: (T, σ) at root v and (T_x, σ) at root x

Extending partial list-coloring

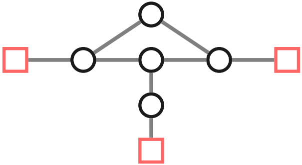
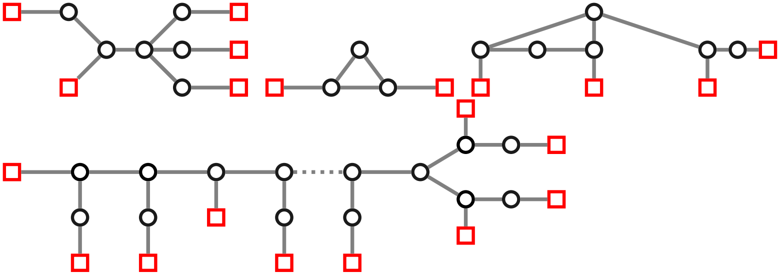


Figure: Extending partial list-coloring

- Pre-color the vertices of $G - H$ and modify the list of vertices of H corresponding to the coloring of $G - H$.
- Prove that this updated list assignment is extendable. Hence, H is a forbidden configuration of G.

Some of forbidden configurations

- 2_1-vertex, 3_2-vertex, 4_4-vertex, 5_5-vertex;

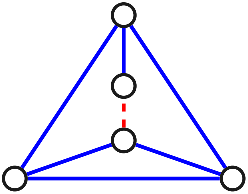


- It's worth mentioning that we have a series of infinite forbidden configurations with some patterns.

Tightness

Proposition [R. Naserasr, R. Skrekovski, Z. Wang and R. Xu 2020+]

There exists a signed graph (G, σ) with $\mathrm{mad}(G) = \frac{14}{5}$ which does not admit a homomorphism to (K_6, M).



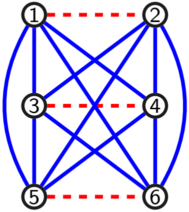


Figure: A signed graph with $\mathrm{mad} = \frac{14}{5}$ does not map to (K_6, M)

Tightness

Proposition [R. Naserasr, R. Skrekovski, Z. Wang and R. Xu 2020+]

There exists a signed bipartite planar graph (G, σ) satisfying $g_{ij}(G, \sigma) \geq g_{ij}(K_{3,3}, M)$ which does not admit a homomorphism to $(K_{3,3}, M)$.

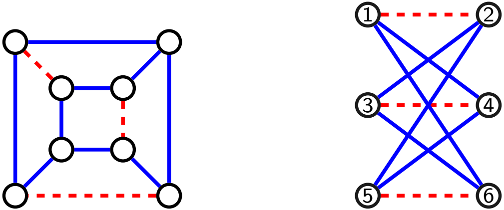


Figure: A signed bipartite planar graph does not map to $(K_{3,3}, M)$

1. Introduction
   - Homomorphism of signed graphs
   - Homomorphism of signed bipartite graphs
2. Our work
   - Mapping signed graphs to (K_6, M)
   - Tightness
3. Conclusion and Discussion

Application to planarity

Corollary [R. Naserasr, R. Skrekovski, Z. Wang and R. Xu 2020+]

Every signed planar graph of girth 7 admits a homomorphism to (K_6, M).

- We do not know whether 7 is the best possible girth condition.

Grötzsch's theorem restated

Given a triangle-free planar graph G, the signed bipartite planar graph $S(G)$ maps to (K_6, M).

- Note that $S(G)$ has negative 4-cycles but has no 6-cycle. Moreover, if G is of girth 5, then $S(G)$ has no 8-cycles.

Steinberg's type questions for (K_6, M)-coloring

Steinberg's conjecture

Planar graphs with no cycle of length 4 and 5 are 3-colorable.

- This conjecture is disproved recently [V. Cohen-Addad, M. Hebdige, D. Král', Z. Li and E. Salgado 2017].
- Planar graphs with no cycle of length $4, 5, 6, 7$ are 3-colorable [O. V. Borodin, A. N. Glebov, A. Raspaud and M. R. Salavatipour 2005].

Steinberg's type questions

What is the smallest value of k, $k \geq 3$, such that every signed bipartite planar graph with no 4-cycles sharing an edge and no cycles of length $6, 8, \dots, 2k$, admits a homomorphism to (K_6, M)?

Mapping signed bipartite planar graphs to signed even cycles

- If a signed bipartite planar graph has no cycle of length smaller than 6, then it maps to (C_4, e). [R. Naserasr, L. A. Pham and Z. Wang 2020+]
- If a signed bipartite planar graph has no cycle of length smaller than 4, then it maps to $(K_{3,3}, M)$. [R. Naserasr and Z. Wang 2021+]

Question

What is a sufficient girth condition for mapping a signed bipartite planar graph to C_{-2k}?

Steinberg's type questions for C_{-2k}-coloring

- If k is a prime number, then there exists an integer $f(k)$ such that any planar graph with no cycle of length $1, 2, \ldots, 2k$, $2k + 2, \ldots, f(k)$ admits a mapping to C_{2k+1}. [X. Hu and J. Li 2020+]
- We can ask similar questions for mapping signed bipartite planar graphs to negative even cycles.

Steinberg's type questions

What is the smallest value of k, $k \geq 3$, such that every signed bipartite planar graph with no cycle of length $2, 4, \ldots, f(k)$, admits a homomorphism to C_{-2k}?

The end. Thank you!