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Circular Coloring, Circular Flow, and Homomorphism of Signed Graphs

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Résumé

Un graphe signé est un graphe G accompagné d'une fonction $\sigma : E(G) \to \{+,-\}$. La coloration et l'homomorphisme des graphes sont deux des problèmes au cœur de la théorie des graphes, et ces notions et problèmes peuvent être naturellement étendus aux graphes signés. Dans cette thèse, nous introduisons deux notions duales: la coloration circulaire des graphes signés et le flot circulaire dans les graphes signés mono-orientés. Nous explorons les paramètres correspondants: le nombre chromatique circulaire et l'indice de flot circulaire sur certaines classes de graphes signés. L'étude s'inscrit dans le cadre du problème de $\frac{2k+1}{k}$-flot circulaire de Jaeger et de sa conjecture duale de Jaeger-Zhang sur la cartographie des graphes planaires de grande maille aux cycles impairs. De plus, notre travail comble l'écart de parité en introduisant les analogues bipartis de ces conjectures.

Dans la premèire partie de la thèse, nous étendons la notion de coloration circulaire des graphes aux graphes signés, comme un raffinement de la $2k$-coloration des graphes signés de Zaslavsky. Nous développons des outils, par exemple, des cliques circulaires signées, des cliques circulaires bipartis signées et des indicateurs signés, pour déterminer le nombre chromatique circulaire de graphes signés. Nous fournissons des bornes sur le nombre chromatique circulaire de plusieurs classes importantes de graphes signés: les graphes simples d-dégénérés signés, les graphes simples planaires bipartis signés, les graphes simples planaires signés et les graphes simples signés dont le graphe sous-jacent est k-colorable avec grande maille arbitraire.

Dans la deuxième partie, en tant que notion duale de la coloration circulaire des graphes signés, nous introduisons la notion de flot circulaire dans les graphes signés mono-orientés. Ceci est différent du concept largement étudié des flots dans les graphes signés biorientés. Nous adaptons et étendons les idées de la théorie des flots sur les graphes aux graphes signés. Nous fournissons une série de résultats d'indices de flux de graphes signés avec des conditions d'arête-connectivité données. En particulier, la condition de $(6p-2)$-arête-connectivité s'avère suffisante pour qu'un graphe eulérien signé admette un $\frac{4p}{2p-1}$-flow circulaire. Lorsqu'il est limité aux graphes planaires, nous prouvons un résultat plus fort: tout graphe planaire biparti signé de maille négative au moins $6p-2$ est circulaire $\frac{4p}{2p-1}$-colorable. C'est jusqu'à présent la meilleure borne de maille négative de l'analogue biparti de la conjecture de Jaeger-Zhang.

La troisième partie se concentre sur le problème d'homomorphisme des graphes planaires signés (bipartis). Nous fournissons d'abord une borne supérieure améliorée et exacte sur le nombre chromatique circulaire des graphes simples planaires bipartis signés en utilisant le nombre de sommets comme un paramètre. Deuxièmement, motivés par une reformulation du théorème des 4 couleurs, avec la méthode de potentiel nouvellement développée, nous prouvons une borne inférieure de la densité d'arêtes des C_{-4}-graphes signés critiques, et par conséquent, nous prouvons que tout graphe planaire biparti signé de maille négative supérieure au égale à 8 admet un homomorphisme à C_{-4} et que cette limite de maille négative est la meilleure possible. Troisièmement, en utilisant un résultat de coloration des arêtes dont la preuve est basée sur le théorème des 4 couleurs, nous montrons que tout graphe planaire biparti signé de maille négative supérieure au égale à 6 admet un homomorphisme à $(K_{3,3}, M)$, qui pourrait être considéré comme un analogue du théorème de

Grötzsch. Enfin, nous confirmons que la condition $mad(G) < \frac{14}{5}$ est suffisante pour qu'un graphe signé admette un homomorphisme à (K_6, M).

Mots clés: graphe signé, homomorphisme, coloration circulaire, flot circulaire, théorème des 4 couleurs.

Abstract

A signed graph is a graph G together with an assignment $\sigma : E(G) \to \{+, -\}$. Graph coloring and homomorphism are two of the central problems in graph theory, and those notions and problems can be naturally extended to signed graphs. In this thesis, we introduce two dual notions: circular coloring of signed graphs and circular flow in mono-directed signed graphs, and explore the corresponding parameters: circular chromatic number and circular flow index on some classes of signed graphs. The study fits well into the framework of Jaeger's circular $\frac{2k+1}{k}$-flow problem and its dual Jaeger-Zhang conjecture on mapping planar graphs of large girth to odd cycles, and moreover, fills the parity gap by introducing the bipartite analogues of those conjectures.

In the first part of the thesis, we extend the notion of the circular coloring of graphs to signed graphs, as a refinement of Zaslavsky's $2k$-coloring of signed graphs. We develop some tools, for example, signed circular cliques, signed bipartite circular cliques and signed indicators, to determine the circular chromatic number of signed graphs. We provide bounds on the circular chromatic number of several important classes of signed graphs: signed d-degenerate simple graphs, signed bipartite planar simple graphs, signed planar simple graphs, and signed simple graphs whose underlying graph is k-colorable with arbitrary large girth.

In the second part, as a dual notion of the circular coloring of signed graphs, we introduce the notion of circular flow in mono-directed signed graphs. This is different from the widely-studied concept of the flows on bidirected signed graphs. We adapt and extend the ideas from the theory of flows on graphs to signed graphs. We provide a series of flow index results of signed graphs with given edge-connectivity conditions. Especially, the condition of $(6p-2)$-edge-connectivity is proved to be sufficient for a signed Eulerian graph to admit a circular $\frac{4p}{2p-1}$-flow. When restricted to planar graphs, we prove a stronger result that every signed bipartite planar graph of negative-girth at least $6p-2$ is circular $\frac{4p}{2p-1}$-colorable. This is so far the best negative-girth bound of the bipartite analogue of the Jaeger-Zhang conjecture.

The focus of the third part is on the homomorphism problem of signed (bipartite) planar graphs. We first provide an improved and also tight upper bound on the circular chromatic number of signed bipartite planar simple graphs using the number of vertices as a parameter. Secondly, motivated by a reformulation of the 4-color theorem, with the newly-developed potential method, we bound from below the edge-density of C_{-4}-critical signed graphs and consequently, we prove that every signed bipartite planar graph of negative-girth at least 8 maps to C_{-4} and that this girth bound is the best possible. Thirdly, using an edge-coloring result whose proof is based on the 4-color theorem, we show that every signed bipartite planar graph of negative-girth at least 6 admits a homomorphism to $(K_{3,3}, M)$, which could be regarded as an analogue of the Grötzsch Theorem. Last, we confirm that the condition $mad(G) < \frac{14}{5}$ is sufficient for a signed graph to admit a homomorphism to (K_6, M).

Keywords: signed graph, homomorphism, circular coloring, circular flow, the 4-color theorem.

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Introduction en français

Un des résultats les plus célèbres de la théorie des graphes est le théorème des quatre couleurs. En 1852, le mathématicien Francis Guthrie a essayé de colorier une carte des cantons d'Angleterre de sorte que deux région partageant une frontière commune reçoivent des couleurs différentes. Il a alors conjecturé que quatre couleurs suffisaient toujours pour colorier de cette façon une telle carte. Son frère Francis Frederick a transmis la question à son professeur Augustus de Morvan. Celui-ci a contribué ensuite à populariser ce problème, qui est devenu le problème bien connu des quatre couleurs. De nombreuses tentatives et preuves erronées ont eu lieu pendant le siècle suivant jusqu'en 1976 et la preuve de deux mathématiciens, Kenneth Appel et Wolfgang Haken. Ce fut une des premières preuves assistées par ordinateur, car elle nécessitait la vérification d'un grand nombre de configurations. La preuve par technique de déchargement de 1996 par Neil Robertson, Daniel P. Sanders, Paul Seymour et Robin Thomas, bien qu'ayant réduit le nombre de configurations, n'est toujours pas vérifiable par un être humain.

C'est pourquoi, 170 and après sa formulation, trouver une preuve alternative vérifiable par l'homme (voire élégante) au théorème dès quatre couleurs est toujours une question importante en théorie dès graphes. Il faut souligner de plus que cet énoncé simple et élégant « tout graphe plantaire est 4-colorable » est à l'origine d'une grande partie de la théorie des graphes du XXe siècle puisque la coloration de graphes est sans doute le sujet le plus largement étudié du domaine. Cette simple déclaration a conduit à la notion de nombre chromatique qui est le paramètre de graphe le plus largement étudié. Suite au problème des quatre couleurs, la coloration des graphes a été beaucoup développée avec de nombreuses branches et généralisations. La théorie des homomorphismes de graphes, importante dans cette thèse, est une généralisation naturelle de la notion de coloration.

Cette thèse est motivée par le théorème des quatre couleurs et ses reformulations et généralisations différentes. Tout d'abord, la notion de planarité a été caractérisée par Kuratowksi en 1930 : un graphe est planaire si et seulement si il ne contient pas K_5 ou $K_{3,3}$ comme mineur. Citons la conjecture de Hadwiger, qui est une généralisation du théorème des 4 couleurs basée sur la notion de mineur (et dont on peut considérer qu'elle à l'origine de la célèbre théorie des mineurs de Neil Robertson et Paul Seymour). Ensuite, un résultat de Tait sur la 3-colorabilité des arêtes des graphes planaires cubiques sans pont a inspiré une autre conjecture plus générale sur l'arête coloration des graphes planaires proposée par Paul Seymour. Enfin la théorie des flots développée par François Jaeger l'a conduit a formuler une autre généralisation sous forme de flot circulaires. Toutes ces conjectures et problèmes restent largement ouverts.

Dans cette thèse, nous considérons des généralisations de ces questions aux graphes signés. Cela servirait à une meilleure compréhension de la structure et des propriétés des graphes. Un graphe signé (G, σ) est un graphe G avec une affectation σ de signes $\{+, -\}$ à ses arêtes. Un graphe peut être simplement vu alors comme un graphe signé dont toutes les arêtes sont positives. Étant donnée une coupe (X, X^c) des sommets d'un graphe signe (G, σ), une coupe-inversion de la coupe (X, X^c) est l'operation qui inverse les signes de toutes les arêtes dans la coupe. La notion de graphes signés donne de nouvelles perspectives aux conjectures évoquées ci-dessus : On peut ainsi donner un renforcement de la conjecture de Hadwiger, connu comme la conjecture de Hadwiger impaire, en utilisant la notion de mineurs d'un graphe signé. Dans la théorie de la coloration usuelle ainsi que dans la théorie des homomorphismes de graphes, on observe que les cycles de longueur impaire jouent un rôle beaucoup plus important que ceux de longueur paire. La théorie des graphes signés a cette particularité d'exploiter pleinement les cycles impairs et les cycles pairs. Ainsi, nous pouvons capturer certains problèmes de coloration de graphes à travers des problèmes de coloration de graphes signés même restreints à la classe des graphes bipartis signés. Ces motivations et extensions sont présentées plus en détail dans ce qui suit.

Le problème de flot de Jaeger et sa dualité

Une tentative notable de généralisation du problème des quatre couleurs est la conjecture du 4-flot proposée par W.T. Tutte en 1966 [Tut66] :

La conjecture du 4-flot. *Chaque graphe sans pont sans mineur de Petersen admet un 4-flot.*

Développant davantage la théorie des flots, W. T. Tutte a proposé les deux conjectures suivantes :

La conjecture du 3-flot. *Chaque graphe 4-arête-connexe admet un 3-flot.*

La conjecture du 5-flot. *Chaque graphe sans pont admet un 5-flot.*

Vers la conjecture du 5-flot, P.D. Seymour a prouvé dans [Sey81] que tout graphe 2-arête-connexe admet un 6-flot. La meilleure borne d'arête-connectivité pour la conjecture du 3-flot a été donnée par L. M. Lovász, C. Thomassen, Y. Wu et C.Q. Zhang [LTWZ13]. Un cas particulier de leurs résultats implique que tout graphe 6-arête-connecté admet un 3-flot.

Généralisant la conjecture du 3-flot de Tutte, F. Jaeger a proposé ce qui suit, connu aujourd'hui sous le nom de conjecture de flot circulaire de Jaeger : Étant donné un entier positif k, tout graphe $4k$-arête-connexe admet un $\frac{2k+1}{k}$-flot circulaire. M. Han, J. Li, Y. Wu et C.Q. Zhang ont réfuté cette conjecture en 2018 [HLWZ18]. La $4k$-arête-connectivité n'est pas une condition suffisante pour qu'un graphe admette un $\frac{2k+1}{k}$-flot circulaire. D'autre part, à l'appui de la conjecture, il a été vérifié en 2013 [LTWZ13] que tout graphe $6k$-arête-connexe admet un $\frac{2k+1}{k}$-flot circulaire. C'est jusqu'à présent la condition suffisante la plus connue. Il est naturel de se demander quelle est la meilleure arête-connectivité suffisante pour qu'un graphe admette un $\frac{2k+1}{k}$-flot circulaire.

Question 1. *Étant donné un entier $k \geqslant 1$, quel est le plus petit entier $f(k)$ tel que tout multigraphe $f(k)$-arête-connexe admette un $\frac{2k+1}{k}$-flot circulaire ?*

La coloration circulaire et le flot circulaire sur les graphes sont des notions duales qui contribuent chacune au développement de l'autre. Le nombre chromatique circulaire est un concept largement étudié pour les graphes. La restriction de la conjecture de $\frac{2k+1}{k}$-flot circulaire de Jaeger aux graphes planaires, par dualité, est d'un intérêt particulier car il s'agit de donner la borne supérieure sur le nombre chromatique circulaire de graphes planaires ayant des grandes mailles.

Conjecture 1. *Tout graphe planaire de maille supérieure ou égale à $4k$ admet une $\frac{2k+1}{k}$-coloration circulaire.*

De plus, C.Q. Zhang a renforcé Conjecture 1 et a proposé ce qui suit, connu comme la conjecture de Jaeger-Zhang.

Conjecture 2. [La conjecture de Jaeger-Zhang][Zha02] *Tout graphe planaire de maille impaire supérieure ou égale à $4k+1$ admet une $\frac{2k+1}{k}$-coloration circulaire.*

A l'appui de Conjecture 1, en 2001, J. Nešetřil et X. Zhu [NZ96], et indépendamment, A. Galluccio, L.A. Goddyn, et P. Hell [GGH01] ont prouvé que la condition de maille supérieure ou égale à $10k-4$ est suffisante pour qu'un graphe planaire admette une $\frac{2k+1}{k}$-coloration circulaire. Dans [KZ00], W. Klostermeyer et C.Q. Zhang ont introduit le lemme de pliage, à l'aide duquel ils ont prouvé q'une maille impaire supérieure ou égale à $10k-3$ était suffisante pour qu'un graphe planaire admette une $\frac{2k+1}{k}$-coloration circulaire. Cette condition de maille impaire a été améliorée à $8k-3$ par X. Zhu [Zhu01a]. En 2004, elle a été encore amélioré par O.V. Borodine, S.-J. Kim, AV Kostochka et D.B. Ouest à $\frac{20k-2}{3}$ [BKKW04]. Le meilleur résultat pour la valeur générale de k, aujourd'hui, est obtenu en appliquant le résultat de [LTWZ13] aux graphes duals des graphes planaires. Cela implique que tout graphe planaire de maille impaire supérieure ou égale à $6k+1$ admet une $\frac{2k+1}{k}$-coloration circulaire.

Il existe également des tentatives et des améliorations de la Conjecture 2 pour de petites valeurs spécifiques de k. Pour $k=1$, c'est une reformulation du célèbre théorème de Grötzsch. Pour des preuves de ce théorème, par exemple, voir [DKT11; DKT20; KY14b; Tho03]. Pour $k=2$, le résultat le plus connu découle de la densité d'arêtes des graphes C_5-critiques prouvé dans [DP17]. Il s'ensuit de ce résultat que tout graphe planaire de maille impaire supérieure ou égale à 11 admet une $\frac{5}{2}$-coloration circulaire. En étendant les techniques de [DP17], pour $k=3$, il est prouvé dans [PS22] que tout graphe planaire de maille impaire supérieure ou égale à 17 admet une $\frac{7}{3}$-coloration circulaire. Des preuves indépendantes de ces deux résultats (pour $k=2,3$) sont données dans [CL20].

Notons qu'un graphe admet une $\frac{2k+1}{k}$-coloration circulaire si et seulement s'il admet un homomorphisme à C_{2k+1}. Inspiré par la conjecture de Jaeger-Zhang et la curiosité de l'absence du rôle des valeurs paires dans le problème de l'homomorphisme aux cycles, dans cette thèse, nous étendons la notion de coloration circulaire des graphes aux graphes signés qui est un raffinement de $2k$-coloration sans zéro des graphes signés définis par T. Zaslavsky [Zas82a]. Sur la base de la notion de coloration circulaire des graphes signés, nous avons étudié une conjecture analogique qui comble l'écart de parité. Il a été conjecturé dans [NRS15] que tout graphe planaire biparti signé de maille négative supérieure ou égale à $4k-2$ admet une $\frac{4k}{2k-1}$-coloration circulaire. Un résultat à l'appui dans [CNS20] montre que la maille négative supérieure ou égale à $8k-2$ suffit pour qu'un graphe planaire biparti signé admette une $\frac{4k}{2k-1}$-coloration circulaire.

Une question plus générale étendant la Conjecture 2 aux graphes planaires signés est la suivante :

Question 2. *Étant donné un entier positif $\ell \geqslant 2$, quelle est le plus petit entier $g(\ell)$ tel que tout graphe planaire signé de maille au moins $g(\ell)$ admette une $\frac{2\ell}{\ell-1}$-coloration circulaire ?*

Notons que la Conjecture 1 et son analogue biparti sont des restrictions de Question 2 sur la classe des graphes planaires signés avec tous les arêtes positifs pour ℓ impair et la classe des graphes planaires bipartis signés pour ℓ pair, respectivement. De plus, dans la conjecture de Jaeger-Zhang et son analogue biparti, la condition de maille a été renforcée pour être respectivement la condition de maille impaire et la condition de maille négative.

Homomorphisme aux cycles négatifs

Outre les graphes complets, les graphes les plus étudiés dans la théorie des homomorphismes de graphes sont des cycles impairs. En 1990, P. Hell et J. Nešetřil [HN90] ont montré que le problème de $(2k+1)$-coloration des graphes peut être capturé par le problème d'homomorphisme de graphes aux C_{2k+1}, par une opération simple de subdivision de graphe. De plus, un graphe G admet un homomorphisme à C_{2k+1} si et seulement si le nombre chromatique circulaire de G est borné par $\frac{2k+1}{k}$. Ainsi la classe des cycles impairs joue un rôle important dans l'étude des problèmes d'homomorphisme et de coloration circulaire des graphes. Motivés par l'observation ci-dessus et la Conjecture 2, les homomorphismes aux cycles impairs ont été beaucoup étudiés, par exemple, voir [Cat88; BHI+08; DP17; PS22]. Mais les cycles pairs sont moins intéressants comme cibles d'homomorphisme car ils sont homomorphiquement équivalents à K_2.

Nous essayons de combler l'écart de parité en utilisant des cycles signés. Étant donné un graphe G, nous définissons $T_{k-2}(G)$ comme étant le graphe signé obtenu à partir de G en remplaçant chaque arête de G par un chemin négatif de longueur $k-2$. Nous avons le résultat généralisé suivant.

Proposition 1. *Pour tout entier positif k, un graphe G est k-colorable si et seulement si $T_{k-2}(G)$ admet un homomorphisme à un cycle négatif C_{-k} de longueur k.*

En particulier, lorsque k=4, nous pourrions reformuler le théorème des quatre couleurs.

Théorème 1. [Le théorème des quatre couleurs reformulé] *Étant donné un graphe planaire G, le graphe biparti signé $T_2(G)$ admet un homomorphisme à C_{-4}.*

Pour un entier pair k, le graphe signé $T_{k-2}(G)$ est toujours biparti. Nous nous intéressons plus particulièrement au problème d'homomorphisme des graphes planaires bipartis signés à cycles pairs négatifs. Avec la notion de coloration circulaire des graphes signés, il existe un lien similaire entre la coloration circulaire des graphes bipartis signés et les homomorphismes aux cycles pairs négatifs : Un graphe biparti signé (G, σ) admet un homomorphisme à C_{-2k} si et seulement si (G, σ) est circulaire $\frac{4k}{2k-1}$-colorable.

Question 3. [Analogue biparti de la conjecture de Jaeger-Zhang] *Étant donné un entier positif $k \geqslant 2$, quelle est le plus petit entier $g(k)$ tel que tout graphe planaire biparti signé de maille négative au moins $g(k)$ admette un homomorphisme à C_{-2k} ?*

Contributions et organisations

Coloration circulaire des graphes signés

Dans le Chapitre 3, nous étendons la notion de coloration circulaire des graphes aux graphes signés. Le nombre chromatique circulaire de graphes signés, noté $\chi_c(G, \sigma)$, est un paramètre qui mérite d'être étudié. Nous montrons que le nombre chromatique circulaire est invariant par coupe-inversion et donnons plusieurs définitions équivalentes de notre coloration circulaire. En particulier, nous introduisons les cliques circulaires signées $K^s_{p;q}$ et montrons qu'un graphe signé est circulaire $\frac{p}{q}$-colorable si et seulement s'il admet une homomorphisme qui préserve le signe des arêtes à $K^s_{p;q}$. Ces notions étant relativement nouvelles, nous développons des outils, par exemple des indicateurs signés, pour calculer le nombre chromatique circulaire de graphes signés. En supplément, restreints à la classe des graphes biparties signés, nous introduisons des cliques circulaires bipartis signées.

Dans le Chapitre 4, nous déterminons des bornes (supérieures, inférieures ou les deux) pour le nombre chromatique circulaire de plusieurs classes de graphes signés : $\chi_c(\mathcal{SD}_d) = 2\lfloor \frac{d}{2} \rfloor + 2$, $\chi_c(\mathcal{SBP}) = 4$, $\frac{14}{3} \leqslant \chi_c(\mathcal{SP}) \leqslant 6$ et $\chi_c(\mathcal{SK}) = 2k$ où $\mathcal{SD}_d$ est la classe des graphes simples d-dégénérés signés, $\mathcal{SBP}$ est la classe des graphes simples planaires bipartis signés, $\mathcal{SP}$ est la classe des graphes simples planaires signés et $\mathcal{SK}$ est la classe des graphes simples signés dont le graphe sous-jacent (de maille arbitrairement grande) est k-colorable. En particulier, pour la classe des graphes 2-dégénérés signés, le supremum de la borne n'est pas atteinte et plus précisément, pour tout graphe 2-dégénéré signé (G, σ) sur n sommets, on a $\chi_c(G, \sigma) \leqslant 4 - \frac{2}{\lfloor \frac{n+1}{2} \rfloor}$.

Flot circulaire dans les graphes signés

Une autre stratégie pour attaquer la Conjecture 2 ou son analogue est la théorie des flot.

Dans le Chapitre 5, nous introduisons la notion de flot circulaire dans les graphes signés, en tant que notion duale de la coloration circulaire des graphes signés. Ceci est différent du concept largement étudié du flot dans les graphes signés, basé sur la notion de bi-orientation. Comme le nombre chromatique circulaire d'un graphe signé est invariant sous l'opération de coupe-inversion, nous introduisons l'opération duale "cycle-inversion" et montrons que l'indice de flot circulaire d'un graphe signé est invariant par cycle-inversion. Il existe une connexion naturelle selon laquelle un graphe planaire signé (G, σ) admet une r-coloration circulaire si et seulement si le graphe planaire signé dual (G^*, σ^*) admet un r-flot circulaire. Semblable à l'étude de la coloration circulaire, nous introduisons l'indicateur de flot signé pour aider au calcul de l'indice de flot circulaire. Pour deux classes spéciales de graphes signés, nous définissons l'orientation modulo k sur les graphes signés et fournissons une caractérisation de l'existence d'une telle orientation.

Dans le Chapitre 6, nous adaptons les idées de la théorie des flots sur les graphes aux graphes signés. En utilisant les résultats de [JLPT92] sur les graphes $\mathbb{Z}_6$-connectés et $\mathbb{Z}_4$-connectés, nous montrons que chaque graphe signé 3-arête-connexe (respectivement, 4-arête-connexe) admet un 6-flot circulaire (respectivement, un 4-flot circulaire). De plus, nous prouvons que tout graphe signé 6-arête-connexe admet un r-flot circulaire où $r < 4$. En utilisant des résultats et des méthodes développés dans [LWZ20], nous montrons que tout graphe signé $(6p-1)$-arête-connexe (respectivement, $(6p+2)$-arête-connexe) admet un $\frac{4p}{2p-1}$-flot circulaire (respectivement, $\frac{2p+1}{p}$-flot circulaire). Aussi, nous prouvons que tout graphe Eulérien signé $(6p-2)$-arête-connexe admet un $\frac{4p}{2p-1}$-flot circulaire. Dans le cas des graphes planaires bipartis signés, nous prouvons un résultat plus fort : tout graphe planaire biparti signé de maille négative au moins $6p-2$ admet une $\frac{4p}{2p-1}$-coloration circulaire. C'est à ce jour la borne de maille négative la plus connue pour l'analogue biparti de la conjecture de Jaeger-Zhang.

Analogue de la conjecture de Jaeger-Zhang

Dans la Part IV, nous nous concentrerons sur le problème d'homomorphisme des graphes planaires bipartis signés.

Dans le Chapitre 7, nous fournissons une borne améliorée sur le nombre chromatique circulaire de graphes simples planaires bipartis signés en utilisant le nombre de sommets comme un paramètre. Plus précisément, pour tout graphe simple biparti signé (G, σ) sur n sommets, $\chi_c(G, \sigma) \leqslant 4 - \frac{4}{\lfloor \frac{n+2}{2} \rfloor}$. Nous construisons une série de graphes planaires bipartis signés montrant que cette borne supérieure est atteinte pour chaque valeur de $n \geqslant 2$.

Dans le Chapitre 8, motivé par une reformulation du théorème des quatre couleurs dans le langage de l'homomorphisme au 4-cycle négatif C_{-4}, nous étudions le problème de l'homomorphisme des graphes planaires bipartis signés à C_{-4}. Nous définissons d'abord la notion de graphes signés (H, π)-critiques, comme une généralisation de la notion de graphes k-critiques. Dans un cas particulier où $(H, \pi) = C_{-4}$, un graphe signé (G, σ) est dit C_{-4}-critique s'il est biparti, (G, σ) n'admet pas d'homomorphisme à C_{-4} mais chacun de ses sous-graphes propres le fait. Nous adaptons la méthode de potentiel utilisée dans l'étude des graphes critiques (voir [KY14a; DP17; PS22]) et fournissons la borne inférieure de $\frac{4}{3}$ pour la densité d'arêtes de tous les graphes C_{-4}-critiques signés sauf un. En conséquent, nous prouvons que tout graphe planaire biparti signé de maille négative au moins 8 admet un homomorphisme à C_{-4}. De plus, par une généralisation de la T_{k-2}-construction aux graphes signés, nous construisons un graphe planaire biparti signé de maille négative 6 qui n'admet aucun homomorphisme à C_{-4}. Au total, nous montrons que la maille 8 est la meilleure possible et réfutons le premier cas de l'analogue biparti de la conjecture de Jaeger-Zhang [NRS15].

Dans le Chapitre 9, nous montrons que tout graphe planaire biparti signé (G, σ) de maille négative au moins 6 admet un homomorphisme à $(K_{3,3}, M)$, de manière équivalente, $\chi_c(G, \sigma) \leqslant 3$.

De plus, cette maille négative est la meilleure possible. La preuve est basée sur un résultat de [NRS13], lui-même basé sur un résultat de [DKK16], que tout le graphe planaire biparti signé de maille négative supérieure ou égale à 6 admet un homomorphisme au cube projectif signé $SPC(5)$, et le théorème de Grötzsch. Ce résultat s'appuie sur un travail non publié [Gue03] qui prouve le cas de $k = 4, 5$ de la même conjecture en utilisant le théorème des quatre couleurs.

Dans le Chapitre 10, nous étudions l'homomorphisme des graphes signés creux aux (K_6, M), qui pourrait être considéré comme un analogue du théorème de Grötzsch. Dans ce chapitre, nous prouvons que tout graphe signé (G, σ) avec $mad(G) < \frac{14}{5}$ admet un homomorphisme à (K_6, M) et, en corollaire, on conclut que tout graphe planaire signé de maille supérieure ou égale à 7 est circulaire 3-colorable. C'est le résultat le plus connu pour le Question 2 quand $\ell = 3$.

Contents

Part I

Introduction

1 | Introduction

One of the most celebrated results in graph theory is the 4-color theorem. It dates back to 1852 when Francis Guthrie, a mathematician, tried to color a map of the counties of England such that no two regions sharing a common boundary received the same color. He postulated that the maximum number of colors required to color such maps is 4. Later, after Francis's brother Frederick inquired this question with his teacher Augustus De Morgan who promoted the visibility of this problem, it became the well-known 4-color problem. Over the next century, various attempts to study the problem were made and this problem was resolved in 1976 when two mathematicians, Kenneth Appel and Wolfgang Haken published a computer-aided proof of the 4-color theorem. However, their proof cannot be verified by hand and relies crucially on a computer. In 1996, Neil Robertson, Daniel P. Sanders, Paul Seymour, and Robin Thomas gave a simpler but still computer-assisted proof that has fewer configurations and discharging rules.

Even after 170 years, finding an alternate proof to the 4-color theorem is an attractive question to graph theorists. An elegant and human-checkable proof of the 4-color theorem is still elusive. In graph theory, the 4-color theorem has a simple and beautiful formulation: Every planar graph is 4-colorable. This simple statement has led to the notion of chromatic number which is the most widely-studied graph parameter. Following the 4-color problem, graph coloring has been vastly developed with many branches and generalizations. The theory of graph homomorphisms is a natural generalization of the notion of the proper coloring of graphs.

The study in this thesis is motivated by the 4-color theorem and its different reformulations and generalizations. The development of minor theory and the famous Hadwiger's conjecture piqued the interest to study the class of graphs forbidding specific graphs as minors. Tait's result about the 3-edge-colorability of planar cubic bridgeless graphs inspired the more general conjecture about edge-coloring of planar graphs proposed by Paul Seymour. Furthermore, Jaeger's circular flow question captured and generalized the 4-color problem from the view of flow theory. All of these conjectures and problems remain widely open.

In this thesis, we consider generalizations of these questions to signed graphs. That would serve toward better understanding of the structure and properties of graphs. A signed graph (G, σ) is a graph G together with an assignment σ of signs $\{+, -\}$ to the edges of G. A graph can be simply seen as a signed graph with all the edges being positive. The notion of signed graphs gives new perspectives to the conjectures mentioned above: We can reformulate a strengthening of Hadwiger's conjecture, odd Hadwiger's conjecture, using the notion of minor of signed graphs. The connection, between Seymour's edge-coloring conjecture and a conjecture of homomorphism to signed projective cubes, both as natural generalizations of the 4-color theorem, gives us more ideas of using the induction method. We note that odd cycles play important roles in the study of graph colorings and homomorphisms but even cycles are relatively pointless. The study of signed graphs helps to fill the gap between the parity and make full use of even cycles. In this line of the study, we can capture some graph coloring problems through signed graph coloring problems even restricted to the class of signed bipartite graphs. These motivations and extensions are presented in more detail in the following sections.

1.1 Odd Hadwiger's conjecture

In 1930, K. Kuratowski [Kur30] characterized the class of planar graphs with the concept of a topological minor. A graph is planar if and only if it contains neither K_5 nor $K_{3,3}$ as a topological minor. Soon after in 1937, K. Wagner introduced the notion of edge-contraction and graph minor, and furthermore, showed that a graph is planar if and only if it contains neither K_5 nor $K_{3,3}$ as a minor. This paper [Wag37] is regarded as a significant achievement and the theory of graph minors has developed vastly since then.

One of the most famous and motivating conjectures, connecting the theory of graph minors and colorings, is Hadwiger's conjecture, introduced by H. Hadwiger [Had43] in 1943.

Conjecture 1.1.1. [Hadwiger's conjecture] *For every integer $k \geqslant 0$, every graph with no K_{k+1}-minor is k-colorable.*

The appeal of Hadwiger's conjecture lies in not only its simple formalization using the concept of minors but also the fact that it represents a far-reaching generalization of the 4-color theorem. In 1943, H. Hadwiger proved the trivial cases $k \leqslant 2$ in his talk. For $k = 3$, there are three different proofs that successfully verified the statement separately in [Had43; Dir52; Duf65]. For $k = 4$, it has been shown to be equivalent to the 4-color theorem [Wag37]. Later in 1993, the case for $k = 5$ has been verified to be true and proved to be equivalent to the 4-color theorem as well in [RST93]. However, for $k \geqslant 6$, the conjecture remains unresolved.

For small values, a stronger variant of the conjecture has been proved. Especially for $k = 3$, in 1979, P. A. Catlin proved in [Cat79] that every graph with no odd-K_4 is 3-colorable. Here odd-K_4 is a subdivision of K_4 such that each original triangle of K_4 becomes an odd cycle after the subdivisions. Generalizing the result of Catlin, a famous analogue of Hadwiger's conjecture, called odd Hadwiger's conjecture, was proposed by B. Gerard and P. Seymour ([JT95] page 115). To better describe the "odd" structure, we need the notion of the minor of signed graphs, in which we allow switching, deleting vertices and edges, and contracting only positive edges. See Section 2.3.1 for the detailed definition.

Conjecture 1.1.2. [Odd Hadwiger's conjecture] *Given a graph G and an integer $k \geqslant 0$, if $(G, -)$ has no $(K_{k+1}, -)$-minor, then G is k-colorable.*

This conjecture is indeed a strengthening of Hadwiger's conjecture. It is easily observed that given a graph G, if G contains no K_k-minor, then the signed graph $(G, -)$ contains no $(K_k, -)$-minor. Moreover, a graph G without a K_k-minor needs to be sparse while its corresponding signed graph $(G, -)$ without $(K_k, -)$-minor could be partially dense. For example, the latter one may contain a complete bipartite graph as a subgraph while the former one cannot. The case for $k = 4$ is announced to be true by B. Guenin (see [Sey16]) but the proof has not yet been published and for $k \geqslant 5$ it is still unknown.

1.2 Seymour's edge-coloring conjecture

An equivalent formulation of the 4-color theorem in terms of edge-coloring was introduced in 1880 by P. G. Tait [Tai80]. Even though his "proof" of 4-color theorem was incomplete, his work is informative and remarkable as it provided a nontrivial equivalence of the 4-color theorem and moreover, the idea of its proof is the origin of the study of the nowhere-zero flows of graphs.

Theorem 1.2.1. [4-color theorem restated] *Every planar cubic bridgeless graph is* 3*-edge-colorable.*

The condition of "planarity" in the theorem is crucial because the Petersen graph is known to be cubic and bridgeless but admits no 3-edge-coloring. P. Seymour proposed a generalized conjecture in 1975:

Conjecture 1.2.2. [Seymour's edge-coloring conjecture] [Sey75] *Every planar k-graph is k-edge-colorable.*

Here, a k-graph is a k-regular multigraph with no odd-cut of size less than k. A cut (X, Y) is an odd-cut if $|X|$ is odd and the size of (X, Y) is the number of edges with one end in X and another in Y. It can be observed that if a k-regular multigraph G has an odd-cut of size less than k, then it does not admit a k-edge-coloring. In other words, "no odd-cut of size less than k" is a necessary condition for the k-edge-colorability of k-regular multigraphs. Seymour's conjecture claims that with the added condition of planarity, this necessary condition is also sufficient.

Conjecture 1.2.2 has been studied for some small values. For $k = 3$, it is true as it is equivalent to the 4-color theorem. P. Seymour proved that the case $k = 4$ implies the case $k = 3$ [Sey75]. B. Guenin [Gue03] verified the cases $k = 4, 5$, using the result of the case $k = 3$ (thus the 4-color theorem). Furthermore, based on the results and proofs given by B. Guenin, the cases $k = 6$, $k = 7$, and $k = 8$ have been verified in [DKK16], [CEKS15], and [CES15], respectively. This conjecture remains open for larger values $k \geqslant 9$.

In another direction, W. T. Tutte in 1966 (before the 4-color theorem was proved) proposed a different extension of Theorem 1.2.1, by replacing planarity with the condition of no Petersen graph as a minor.

Conjecture 1.2.3. *Every* 2*-edge-connected cubic graph with no Petersen minor is* 3*-edge-colorable.*

This conjecture is claimed to be proved following several works on the subject, one of which is not published yet, we refer to [ESST16] for further details.

1.3 Homomorphism to signed projective cube

From the homomorphism point of view, the 4-color theorem is equivalent to saying that every planar graph maps to K_4. To introduce a conjecture, generalizing the 4-color theorem, proposed by R. Naserasr, we need the notion of projective cube. A projective cube of dimension k, denoted by $PC(k)$, is built from a hypercube $H(k)$ by adding diagonal edges which connect each vertex with its furthest non-neighbor in $H(k)$. Note that the complete graph K_4 is just a projective cube of dimension 2. Furthermore, we note that for every integer k, $PC(2k)$ has odd-girth $2k + 1$ and $PC(2k - 1)$ is bipartite. Following the line of the study of homomorphism to projective cubes, R. Naserasr conjectured the following, whose first case is the 4-color theorem.

Conjecture 1.3.1. [$PC(2k)$ conjecture] [Nas07] *Given a positive integer k, every planar graph of odd-girth at least $2k + 1$ admits a homomorphism to $PC(2k)$.*

To make use of the bipartite projective cubes and fill the parity gap, B. Guenin introduced the notion of homomorphism of signed graphs and signed projective cubes. A special signed projective cube, denoted by $SPC(k)$, is defined as follows: we assign the signature to $PC(k)$ such that edges of the hypercube $H(k)$ are all positive and the additional diagonal edges are all negative. Note that the length of the shortest negative cycle, called negative-girth, of $SPC(2k - 1)$ is $2k$. B. Guenin proposed the following conjecture.

Conjecture 1.3.2. [$SPC(2k)$ conjecture] [Gue05] *Given a positive integer k, every signed bipartite planar graph of negative-girth $2k$ admits a homomorphism to $SPC(2k-1)$.*

The above two conjectures are of independent interest and are also strongly connected to the Seymour's edge-coloring conjecture (Conjecture 1.2.2). Note that the two conjectures are uniformly stated in Conjecture 10.1.2.

Theorem 1.3.3. [NRS13] *Let k be a given positive integer.*

- *Seymour's edge-coloring conjecture holds for $k = 2\ell + 1$ if and only if $PC(2\ell)$ conjecture holds for the given ℓ.*
- *Seymour's edge-coloring conjecture holds for $k = 2\ell$ if and only if $SPC(2\ell - 1)$ conjecture holds for the given ℓ.*

As mentioned before, it is a big challenge to resolve Conjecture 1.2.2 (and as well Conjectures 1.3.1 and 1.3.2). We note that the proofs of the cases $k \leqslant 8$ of Conjecture 1.2.2 are based on the induction on k. Thus the introduction of the $SPC(2k-1)$ conjecture helps to tackle Conjecture 1.3.1 by an inductive method and vice versa.

1.4 Jaeger's flow problem and its dual

Another notable attempt to generalize the 4-color problem is Tutte's 4-flow conjecture [Tut66] proposed by W.T. Tutte in 1966:

4-flow conjecture. *Every bridgeless Petersen-minor-free graph admits a nowhere-zero 4-flow.*

If true, this would imply Conjecture 1.2.3. Further developing the theory of nowhere-zero flows, W. T. Tutte proposed the following two conjectures:

3-flow conjecture. *Every 4-edge-connected graph admits a nowhere-zero 3-flow.*

5-flow conjecture. *Every bridgeless graph admits a nowhere-zero 5-flow.*

Towards 5-flow conjecture, P.D. Seymour has proved in [Sey81] that every 2-edge-connected graph admits a 6-flow. The best edge-connectivity bound for the 3-flow conjecture is given by L. M. Lovász, C. Thomassen, Y. Wu, and C.Q. Zhang in [LTWZ13] and one particular case of their results implies that every 6-edge-connected graph admits a nowhere-zero 3-flow.

Generalizing Tutte's 3-flow conjecture, F. Jaeger [Jae84; Jae88] proposed the following, known as Jaeger's circular flow conjecture nowadays: Given a positive integer k, every $4k$-edge-connected multigraph admits a circular $\frac{2k+1}{k}$-flow. A *circular $\frac{p}{q}$-flow* is a p-flow (D, f) such that for each edge e, $q \leqslant |f(e)| \leqslant p - q$. On the one hand, M. Han, J. Li, Y. Wu, and C.Q. Zhang have disproved this conjecture in 2018 [HLWZ18]. The $4k$-edge-connectivity is not a sufficient condition for a multigraph to admit a circular $\frac{2k+1}{k}$-flow. On the other hand, in supporting the conjecture, it has been verified in 2013 [LTWZ13] that every $6k$-edge-connected multigraph G admits a circular $\frac{2k+1}{k}$-flow. Until now, this is the best-known sufficient condition. It is natural to ask what is the best edge-connectivity that is sufficient for a (multi)graph to admit a circular $\frac{2k+1}{k}$-flow.

Question 1.4.1. *Given an integer $k \geqslant 1$, what is the smallest integer $f(k)$ such that every $f(k)$-edge-connected multigraph admits a circular $\frac{2k+1}{k}$-flow?*

Circular coloring and circular flow on graphs are dual notions that each helps the development of the other. The circular chromatic number is a widely studied concept for graphs. A *circular $\frac{p}{q}$-coloring* of a graph G is a mapping $\varphi : V(G) \to \{1, 2, \ldots, p\}$ such that for each edge uv, $q \leqslant$

$|\varphi(u)-\varphi(v)| \leqslant p-q$. We refer to Section 2.1 for more definitions and details. Restriction of Jaeger's circular flow conjecture to planar graphs, by duality, is of special interest because it is about giving the best possible upper bound on the circular chromatic number of planar graphs of high girth.

Conjecture 1.4.2. *Every planar graph of girth at least $4k$ admits a circular $\frac{2k+1}{k}$-coloring.*

Moreover, C.Q. Zhang strengthened Conjecture 1.4.2 and proposed the following which is known as the Jaeger-Zhang conjecture.

Conjecture 1.4.3. [Jaeger-Zhang conjecture][Zha02] *Every planar graph of odd-girth at least $4k+1$ admits a circular $\frac{2k+1}{k}$-coloring.*

Supporting Conjecture 1.4.2, in 2001, J. Nešetřil and X. Zhu [NZ96], and independently, A. Galluccio, L.A. Goddyn, and P. Hell [GGH01] proved that the girth condition of $10k-4$ is sufficient for a planar graph to admit a circular $\frac{2k+1}{k}$-coloring. In [KZ00], W. Klostermeyer and C.Q. Zhang introduced the folding lemma, using which they proved the odd-girth at least $10k-3$ to be sufficient for a planar graph to admit a circular $\frac{2k+1}{k}$-coloring. This odd-girth condition has been improved to $8k-3$ by X. Zhu in [Zhu01a]. In 2004, it has been improved further by O.V. Borodin, S.-J. Kim, A.V. Kostochka, and D.B. West to $\frac{20k-2}{3}$ in [BKKW04]. The best result for the general value of k, today, is obtained by applying the result of [LTWZ13] to the dual of planar graphs. It implies that every planar graph of odd-girth at least $6k+1$ admits a circular $\frac{2k+1}{k}$-coloring.

There are also attempts and improvements on Conjecture 1.4.3 for specific small values of k. For $k=1$, it is a restatement of the famous Grötzsch's theorem. For some proofs of this theorem, for example, see [DKT11; DKT20; KY14b; Tho03]. For $k=2$, the best known result follows from the edge-density of C_5-critical graphs proved in [DP17]. It follows from this result that every planar graph of odd-girth at least 11 admits a circular $\frac{5}{2}$-coloring. Extending the techniques of [DP17], for $k=3$, it is proved in [PS22] that every planar graph of odd-girth at least 17 admits a circular $\frac{7}{3}$-coloring. Independent proofs of these two results (for $k=2,3$) are given in [CL20].

Note that a graph G admits a circular $\frac{2k+1}{k}$-coloring if and only if it admits a homomorphism to C_{2k+1}. Inspired by Jaeger-Zhang conjecture and the curiosity of the absence of the role of even-values in the problem of homomorphism to cycles, in this thesis, we extend the notion of the circular coloring from graphs to signed graphs which is a refinement of 0-free $2k$-coloring of signed graphs defined by Zaslavsky [Zas82a]. See Section 3.1 for details. Based on the notion of the circular coloring of signed graphs, we studied an analogous conjecture that fills in the parity gap. It was conjectured in [NRS15] that every signed bipartite planar graph of negative-girth at least $4k-2$ admits a circular $\frac{4k}{2k-1}$-coloring. A supporting result in [CNS20] shows that negative-girth $8k-2$ suffices for a signed bipartite planar graph to admit a circular $\frac{4k}{2k-1}$-coloring.

A more general question extending Conjecture 1.4.3 to signed planar graphs is as follows:

Question 1.4.4. *Given a positive integer $\ell \geqslant 2$, what is the smallest value $g(\ell)$ such that every signed planar graph of girth at least $g(\ell)$ admits a circular $\frac{2\ell}{\ell-1}$-coloring?*

We note that Conjecture 1.4.2 and its bipartite analogue are restrictions of Question 1.4.4 on the class of signed planar graphs with all positive edges for odd ℓ and the class of signed bipartite planar graphs for even ℓ, respectively. Furthermore, in the Jaeger-Zhang conjecture and its bipartite analogue, the girth condition has been strengthened to be odd-girth condition and negative-girth condition, respectively.

1.5 Homomorphism to negative cycles

Besides the complete graphs, the most studied graphs in the theory of graph homomorphisms are odd cycles. In 1990, P. Hell and J. Nešetřil [HN90] showed that the $(2k+1)$-coloring problem of graphs can be captured by the problem of homomorphism of graphs to C_{2k+1}, through a simple graph subdivision operation. Also, a graph G admits a homomorphism to C_{2k+1} if and only if the circular chromatic number of G is bounded by $\frac{2k+1}{k}$. Thus the class of odd cycles plays an important role in the study of homomorphism and circular coloring problems of graphs. Motivated by the above observation and Conjecture 1.4.3, homomorphisms to odd cycles have been studied a lot, for example, see [Cat88; BHI+08; DP17; PS22]. Even cycles are less interesting as homomorphism targets because they are homomorphically equivalent to K_2.

We try to fill the parity gap using signed cycles. Given a graph G, we define $T_{k-2}(G)$ to be the signed graph obtained from G by replacing each edge of G with a negative path of length $k-2$. We have the following generalized result, which will be proved in Section 2.3.3.

Proposition 1.5.1. *For any integer k, a graph G is k-colorable if and only if $T_{k-2}(G)$ admits a homomorphism to a negative cycle C_{-k} of length k.*

In particular, when k=4, we could restate the 4-color theorem.

Theorem 1.5.2. [4-color theorem restated] *Given a planar graph G, the signed bipartite graph $T_2(G)$ admits a homomorphism to C_{-4}.*

For an even value k, the signed graph $T_{k-2}(G)$ is always bipartite. We are especially interested in the homomorphism problem of signed bipartite planar graphs to negative even cycles. With the notion of the circular coloring of signed graphs, there is a similar connection between the circular coloring of signed bipartite graphs and the homomorphism to negative even cycles, that a signed bipartite graph (G, σ) admits a homomorphism to C_{-2k} if and only if (G, σ) is circular $\frac{4k}{2k-1}$-colorable.

Question 1.5.3. [Bipartite analogue of Jaeger-Zhang conjecture] *Given a positive integer $k \geqslant 2$, what is the smallest value $g(k)$ such that every signed bipartite planar graph of negative-girth at least $g(k)$ admits a homomorphism to C_{-2k}?*

1.6 Contributions and organization

1.6.1 Circular coloring of signed graphs

In Chapter 3, we extend the notion of the circular coloring of graphs to signed graphs, as a refinement of Zaslavsky's $2k$-coloring of signed graphs. The circular chromatic number of signed graphs, denoted by $\chi_c(G, \sigma)$, is a parameter worth investigating. We show that the circular chromatic number is invariant under switching and provide several equivalent definitions of our circular coloring. Especially, we introduce the signed circular cliques $K^s_{p;q}$ and show that a signed graph is circular $\frac{p}{q}$-colorable if and only if it admits an edge-sign preserving homomorphism to $K^s_{p;q}$. Since these notions are relatively new, we develop some tools, for example, signed indicators, to calculate the circular chromatic number of signed graphs. Moreover, restricted to the class of signed bipartite graphs, we introduce signed bipartite circular cliques.

Given a class $\mathcal{C}$ of signed graphs, $\chi_c(\mathcal{C}) = \max\{\chi_c(G, \sigma) : (G, \sigma) \in \mathcal{C}\}$. In Chapter 4, we determine bounds for the circular chromatic number of several classes of signed graphs: $\chi_c(\mathcal{SD}_d) = 2\lfloor \frac{d}{2} \rfloor + 2$, $\chi_c(\mathcal{SBP}) = 4$, $\frac{14}{3} \leqslant \chi_c(\mathcal{SP}) \leqslant 6$ and $\chi_c(\mathcal{SK}) = 2k$ where $\mathcal{SD}_d$ is the class of signed d-degenerate simple graphs, $\mathcal{SBP}$ is the class of signed bipartite planar simple graphs, $\mathcal{SP}$ is the class of signed planar simple graphs and $\mathcal{SK}$ is the class of signed simple graphs whose underlying graph is k-colorable (with arbitrarily large girth). Especially, for the class of signed 2-degenerate graphs, the supremum of the bound is not achievable and more precisely, for any signed 2-degenerate graph (G, σ) on n vertices, we have that $\chi_c(G, \sigma) \leqslant 4 - \frac{2}{\lfloor \frac{n+1}{2} \rfloor}$.

1.6.2 Circular flows in signed graphs

Another strategy to approach Conjecture 1.4.3 or its analogue is through the theory of flows.

In Chapter 5, we introduce the notion of the circular flows in signed graphs, as a dual notion of the circular colorings of signed graphs. This is different from the widely-studied concept of the flow on signed graphs, based on the notion of bidirection. As the circular chromatic number of a signed graph is invariant under the switching operation, we introduce the dual operation "inversing" and show that the circular flow index of a signed graph is invariant under inversing. There is a natural connection that a signed planar graph (G, σ) admits a circular r-coloring if and only if the dual signed planar graph (G^*, σ^*) admits a circular r-flow. Similarly to the study of circular coloring, we introduce the signed flow indicator to help with calculating the circular flow index. For two special classes of signed graphs, we define the modulo k-orientation on signed graphs and provide a characterization of the existence of such orientation.

In Chapter 6, we adapt the ideas from the theory of flows on graphs to signed graphs. Using results of [JLPT92] about $\mathbb{Z}_6$-connected and $\mathbb{Z}_4$-connected graphs, we show that every 3-edge-connected (respectively, 4-edge-connected) signed graph admits a circular 6-flow (respectively, circular 4-flow). Also, we prove that every 6-edge-connected signed graph admits a circular r-flow where $r < 4$. Using some results and methods developed in [LWZ20], we show that every $(6p-1)$-edge-connected (respectively, $(6p+2)$-edge-connected) signed graph admits a circular $\frac{4p}{2p-1}$-flow (respectively, $\frac{2p+1}{p}$-flow). Also, we prove that every $(6p-2)$-edge-connected signed Eulerian graph admits a circular $\frac{4p}{2p-1}$-flow. When restricted to signed bipartite planar graphs, we prove a stronger result that every signed bipartite planar graph of negative-girth at least $6p-2$ admits a circular $\frac{4p}{2p-1}$-coloring. This is so far the best known negative-girth bound of the bipartite analogue of Jaeger-Zhang conjecture.

1.6.3 Analogue of Jaeger-Zhang conjecture

In Part IV, we will focus on the homomorphism problem of signed bipartite planar graphs.

In Chapter 7, we provide an improved bound on the circular chromatic number of signed bipartite planar simple graphs using the number of vertices as a parameter. More precisely, for any signed bipartite planar simple graph (G, σ) on n vertices, $\chi_c(G, \sigma) \leqslant 4 - \frac{4}{\lfloor \frac{n+2}{2} \rfloor}$. We construct a series of signed bipartite planar graphs showing this upper bound is tight for each value of $n \geqslant 2$.

In Chapter 8, motivated by a restatement of the 4-color theorem in the language of homomorphism to the negative 4-cycle C_{-4}, we study the homomorphism problem of signed bipartite planar graphs to C_{-4}. We first define the notion of (H, π)-critical signed graphs, as a generalization of the notion of k-critical graphs. In a special case when $(H, \pi) = C_{-4}$, a signed graph (G, σ) is said to be C_{-4}-critical if it is bipartite, (G, σ) admits no homomorphism to C_{-4} but each of its proper subgraphs does. We adapt the potential method used in the study of critical graphs (see [KY14a; DP17; PS22]) and provide the tight lower bound of $\frac{4}{3}$ for the edge-density of all but one C_{-4}-critical signed graphs. Consequently, we prove that every signed bipartite planar graph of negative-girth at least 8 maps to C_{-4}. Moreover, by a generalization of T_{k-2}-construction to signed graphs, we construct a signed bipartite planar graph of negative-girth 6 that admits no homomorphism to C_{-4}. Altogether, we show that the girth 8 is the best possible and disprove the first case of the bipartite analogue of Jaeger-Zhang conjecture [NRS15].

In Chapter 9, we show that every signed bipartite planar graph (G, σ) of negative-girth at least 6 admits a homomorphism to $(K_{3,3}, M)$, equivalently, $\chi_c(G, \sigma) \leqslant 3$. Moreover, this girth bound is the best possible. The proof is based on a result of [NRS13], which itself is based on a result of [DKK16] (the case of $k = 6$ of Conjecture 1.2.2), that every signed bipartite planar graph of negative-girth at least 6 admits a homomorphism to the signed projective cube $SPC(5)$, and Grötzsch's Theorem. This result relies on an unpublished work [Gue03] which proves the case of $k = 4, 5$ of the same conjecture using the 4-color theorem.

In Chapter 10, we study the homomorphism of sparse signed graphs to (K_6, M), which could be regarded as an analogue of the Grötzsch Theorem. In this chapter, we prove that every signed graph (G, σ) with $mad(G) < \frac{14}{5}$ admits a homomorphism to (K_6, M) and, as a corollary, we conclude that every signed planar graph of girth at least 7 is circular 3-colorable. This is the best-known result for Question 1.4.4 when $\ell = 3$.

2 | Preliminary

A *graph* is a pair $G = (V(G), E(G))$ where $V(G)$ is a finite set, called the set of *vertices*, and $E(G)$ is a multiset of pairs of (not necessarily distinct) elements of $V(G)$, called the set of *edges*. The elements of the pair forming an edge are called the *endpoints* of the edge. An edge whose endpoints are identical is called a *loop*. If two edges have the same endpoints, then they are *parallel*. Thus a graph in this work is allowed to have loops and parallel edges. A *simple* graph is a graph without loops and parallel edges and a *multigraph* is a graph without loops. The *order* of a graph G is the cardinality of $V(G)$, denoted by $|V(G)|$ or simply $v(G)$ while the *size* of G is the cardinality of $E(G)$, denoted by $|E(G)|$ or simply $e(G)$. A *subgraph* H of G is a graph with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and a subgraph is *proper* if either $V(H) \subsetneq V(G)$ or $E(H) \subsetneq E(G)$.

The *degree* of a given vertex v, denoted $d(v)$, is the number of edges incident to v. Moreover, given a graph G, we denote by $\delta(G)$ the minimum degree and $\Delta(G)$ the maximum degree, which is respectively the minimum and the maximum among all $d(v)$'s for $v \in V(G)$. The *average degree* of a graph G is defined to be $\frac{2e(G)}{v(G)}$ and the *maximum average degree* of G, denoted $\mathrm{mad}(G)$, is the maximum average degree of H taken over all the subgraphs H of G. An *even-degree* graph is a graph where the degree of each vertex is even. A connected even-degree graph is called an *Eulerian* graph.

A *walk* W of a graph G is an alternating sequence $v_1, e_1, v_2, \ldots, e_{k-1}, v_k$ of vertices and edges (allowing repetition) such that for $1 \leqslant i \leqslant k$, the edge e_i has endpoints v_i and v_{i+1}. A walk of a graph is said to be *closed* if v_1 and v_k are identical. A *path* in a graph is a walk with all the vertices v_i's being distinct. We say a path P in a graph G is a *thread* if all of its internal vertices are of degree 2 in G. A *cycle* of a graph is a closed walk with all the vertices v_i's (except $v_1 = v_k$) being distinct. The *length* of a walk is the number of its edges. A path (or thread, or cycle) of length k is simply called a *k-path* (or a *k-thread*, or a *k-cycle*). We denote the k-path or k-thread by P_k and denote the k-cycle with C_k. An *odd* (or *even*) cycle is a cycle of odd (or even) length.

A *cut* of a graph G, denoted by $C = (X, Y)$, is a partition of $V(G)$ into two disjoint subsets X and Y such that $X \cup Y = V(G)$. The *cut-set* or *edge-cut* of a cut $C = (X, Y)$, denoted by $E(X, Y)$, is the set of edges (of $E(G)$) that have one endpoint in X and have the other endpoint in Y. A graph is *k-edge-connected* if for any cut (X, X^c) of $V(G)$, we have that $|E(X, X^c)| \geqslant k$.

2.1 Circular coloring and circular flow of graphs

The graph coloring has various generalizations and refinements. In this thesis, we will focus on the circular coloring and its dual, circular flow, when restricted to planar graphs.

2.1.1 Circular coloring

Let r be a real number larger than 1. We denote by C^r the circle of circumference r, which could be obtained from an interval $[0, r]$ by identifying two endpoints 0 and r. Thus points in C^r are real numbers from $[0, r)$. For two points x, y on C^r, the *distance* between x and y on C^r, denoted by $d_{C^r}(x, y)$ or $d_{(\text{mod } r)}(x, y)$, is the length of the shorter arc of C^r connecting x and y. Given two real numbers a and b, the interval $[a, b]$ on C^r is a closed interval of C^r in clockwise orientation of the circle whose first point is $a \pmod r$ and whose end point is $b \pmod r$. For example, for a given $r > 4$, $[4, 1] = \{t : 4 \leqslant t < r, \text{ or } 0 \leqslant t \leqslant 1\}$. Intervals $[a, b)$, $(a, b]$ and (a, b) are defined similarly. The length of the interval $[a, b]$ is denoted by $\ell([a, b])$. Thus $d_{(\text{mod } r)}(x, y) = \min\{\ell([x, y]), \ell([y, x])\}$.

Given a graph G, a *circular r-coloring* of G is a mapping $f : V(G) \to C^r$ such that for any edge $uv \in E(G)$, $d_{(\text{mod } r)}(f(u), f(v)) \geqslant 1$. The *circular chromatic number* of G is defined as

$$\chi_c(G) = \inf\{r \mid G \text{ admits a circular } r\text{-coloring}\}.$$

The concept of circular coloring of graphs was introduced by Vince in 1988 in [Vin88], where a different but equivalent definition was given and the parameter was called the "star chromatic number". Later, the above definition was given in [Zhu92] and the term "circular chromatic number" was coined in [Zhu01b]. One important feature of the circular chromatic number is that for any graph G, $\chi(G) - 1 < \chi_c(G) \leqslant \chi(G)$ and hence $\chi(G) = \lceil \chi_c(G) \rceil$. In other words, the invariant $\chi_c(G)$ is a refinement of $\chi(G)$ and it contains more information about the structure of G. The circular chromatic number of graphs has been studied extensively in the literature. We refer to surveys [Zhu01b; Zhu06] for more on this subject.

A *homomorphism* of a graph G to a graph H is a mapping $f : V(G) \to V(H)$ such that for every edge uv of G, $f(u)f(v)$ is an edge of H. It is a folklore fact that a graph G is k-colorable if and only if G admits a homomorphism to K_k (the complete graph on k vertices). Similarly, circular chromatic number of graphs could also be defined through graph homomorphisms. For integers $p \geqslant 2q > 0$, the *circular clique* $K_{p;q}$ has vertex set $[p] = \{0, 1, \ldots, p-1\}$ and edge set $\{ij : q \leqslant |i - j| \leqslant p - q\}$. A graph G admits a circular $\frac{p}{q}$-coloring if and only if it admits a homomorphism to $K_{p;q}$.

2.1.2 Flow and circular flow

Given a graph $G = (V, E)$, an *orientation* D of G is an assignment of a direction to each edge of G. For each edge $e = uv$, if e is *oriented* from u to v in D, then we write $(u, v) \in D$.

Given a graph G and an Abelian group A, an *A-flow* of G is a pair (D, f) where D is an orientation on G and $f : E(G) \to A$ such that the total out-flow equals to the total in-flow at each vertex v, i.e.,

$$\sum_{(v,w) \in D} f(vw) - \sum_{(u,v) \in D} f(uv) = 0$$

where $\sum$ refers to the addition in A.

Sometimes, for the simplicity, we write $\partial_D f(v) := \sum\limits_{(v,w) \in D} f(vw) - \sum\limits_{(u,v) \in D} f(uv)$ and when the orientation is clear from the context, we simply write $\partial f(v)$. So here the condition of each vertex is $\partial f(v) = 0$. Moreover, a *nowhere-zero A-flow* of G is an *A-flow* such that $f : E(G) \to A \backslash \{0\}$.

There are two kinds of flows that are widely studied. A *nowhere-zero (integer) k-flow* of a graph G is a $\mathbb{Z}$-flow (D, f) of G satisfying that $1 \leqslant |f(e)| \leqslant k - 1$. A *modulo k-flow* of G is a nowhere-zero $\mathbb{Z}_k$-flow in G. A nowhere-zero integer k-flow in G is naturally a modulo k-flow and conversely, the existence of a modulo k-flow also implies the existence of a nowhere-zero integer k-flow. The latter fact follows from a stronger result of [Tut54].

Lemma 2.1.1. [Tut54] *If a graph admits a modulo k-flow (D, f), then it admits an integer k-flow (D, f') such that $f'(e) \equiv f(e) \pmod k$ for every edge e.*

When we work with the nowhere-zero k-flow notion (or equivalently, nowhere-zero $\mathbb{Z}_k$-flow notion), we have values $\{1, \ldots, k-1\}$ to assign to each of edges such that only a single value 0 is forbidden. A more general view is to forbid an arbitrary value on each edge. This is studied in the literature and is strongly related to the notion of $\mathbb{Z}_k$-connectivity defined below.

A mapping $\beta : V(G) \to \mathbb{Z}_k$ is called a $\mathbb{Z}_k$*-boundary* if it satisfies

$$\sum_{v \in V(G)} \beta(v) \equiv 0 \pmod k.$$

A graph G is said to be $\mathbb{Z}_k$*-connected* if for every $\mathbb{Z}_k$-boundary β, there is an orientation D on G and a mapping $f : E(G) \to \mathbb{Z}_k$ such that for each vertex $v \in V(G)$,

$$\partial_D f(v) \equiv \beta(v) \pmod k.$$

The connection between $\mathbb{Z}_k$-boundary and forbidding one value on each edge is given in [JLPT92] as follows:

Proposition 2.1.2. [JLPT92] *Let G be a connected graph. The following claims are equivalent:*

- *G is $\mathbb{Z}_k$-connected.*
- *For any function $g : E(G) \to \mathbb{Z}_k$, there exists a modulo k-flow (D, f) such that for each edge e of G, $f(e) \not\equiv g(e) \pmod k$.*
- *For any $\mathbb{Z}_k$-boundary $\beta : V(G) \to \mathbb{Z}_k$ and any mapping $g : E(G) \to \mathbb{Z}_k$, there exists a function $f : E(G) \to \mathbb{Z}_k$ which satisfies that $\partial_D f(v) \equiv \beta(v) \pmod k$ for each $v \in V(G)$ and $f(e) \not\equiv g(e) \pmod k$ for each $e \in E(G)$.*

Circular flow and modulo ℓ-orientation

For more restrictions on the flow values, F. Jaeger first considered the $\frac{2k+1}{k}$-flow [Jae84; Jae88], which was further generalized by L.A. Goddyn, M. Tarsi, and C.Q. Zhang [GTZ98] to the notion of *circular flow* of graphs. For any positive integers p, q with $p \geqslant 2q$, a $\frac{p}{q}$*-flow* of a graph G is a pair (D, f) where D is an orientation on G and $f : E(G) \to \mathbb{Z}$ satisfies $q \leqslant |f(e)| \leqslant p - q$ and for each vertex v, $\partial_D f(v) = 0$.

Given an orientation D, we denote by $\overleftarrow{d_D}(v)$ and $\overrightarrow{d_D}(v)$ the *out-degree* and the *in-degree* at vertex v, which is the number of edges oriented to v and oriented from v, respectively.

Given a graph G, a *modulo ℓ-orientation* of G is an orientation D satisfying that

$$\overleftarrow{d_D}(v) - \overrightarrow{d_D}(v) \equiv 0 \pmod \ell.$$

Based on the parity of ℓ, we have the following two cases:

- For $\ell = 2k$, a graph admits a modulo $2k$-orientation if and only if it is an even-degree graph.
- [Jae84] For $\ell = 2k+1$, a graph admits a modulo $(2k+1)$-orientation if and only if it admits a $\frac{2k+1}{k}$-flow.

Here we provide another necessary and sufficient condition for a graph G to admit a modulo $(2k+1)$-orientation.

Theorem 2.1.3. *A graph G admits a modulo $(2k+1)$-orientation if and only if we can partition the edges of G into $2k+1$ classes $E_1, E_2, \dots, E_{2k+1}$ such that we can orient all the edges so that for each vertex v and any i, j,*

$$\overleftarrow{d_{E_i}}(v) - \overrightarrow{d_{E_i}}(v) = \overleftarrow{d_{E_j}}(v) - \overrightarrow{d_{E_j}}(v)$$

where $\overleftarrow{d_{E_k}}(v)$ and $\overrightarrow{d_{E_k}}(v)$ represent respectively the in-degree and out-degree of vertex v in the class E_k.

Proof. One direction is easy. If there is a partition $\{E_1, E_2 \dots, E_{2k+1}\}$ of $E(G)$ and an orientation D satisfying the conditions of the theorem, then we have that

$$\overleftarrow{d_D}(v) - \overrightarrow{d_D}(v) = \sum_{i=1}^{2k+1} \overleftarrow{d_{E_i}}(v) - \overrightarrow{d_{E_i}}(v) = (2k+1)(\overleftarrow{d_{E_1}}(v) - \overrightarrow{d_{E_1}}(v)) \equiv 0 \pmod{2k+1}.$$

For the other direction, suppose that a graph G admits a modulo $(2k+1)$-orientation D. We will apply some operations on G to get a new graph. As the operation does not change the orientation, with a minor abuse of notation we will use D to denote the orientation on the new construction. We keep on lifting a pair of incident edges with one ingoing and one outgoing, that is to say, in one turn, we delete two directed edges $(u, w), (w, v)$, and add a new directed edge (u, v). After all those lifting operations, in the new graph with the orientation D, each vertex is either all-in or all-out, and its degree is a multiple of $2k+1$. Now we split each vertex to some vertices of degree exactly $2k+1$ and denote the resulting graph by H. Note that H is $(2k+1)$-regular and every vertex is either all-in or all-out in D. This means that H is a $(2k+1)$-regular bipartite graph.

We consider a proper $(2k+1)$-edge-coloring of this bipartite graph H and thus we can partition the edge set into $2k+1$ classes, say $C_1, C_2 \dots, C_{2k+1}$. We define a partition $\{E_1, E_2 \dots, E_{2k+1}\}$ of $E(G)$ as follows: For $uv \in E(G) \cap E(H)$, if $uv \in C_i$, then let $uv \in E_i$; For $uv \in E(H)$ which is obtained by lifting some pair of edges $uw_1, w_1w_2, \dots, w_nv \in E(G)$, if $uv \in C_i$, then let $uw_1, w_1w_2, \dots, w_nv \in E_i$. Now we can check that $E_1, E_2, \dots, E_{2k+1}$ is a desired edge-partition of $E(G)$: We can see that the reverse operation of splitting vertex is the vertex identification, which does not affect this property, and the reverse operation of lifting a pair of incident edges does not affect this property either. □

2.2 Signed graphs and homomorphisms

A *signed graph* (G, σ) is a graph $G = (V, E)$ together with an assignment $\sigma : E(G) \to \{+, -\}$, which is called a *signature*. When the signature is clear from the context or can be omitted, sometimes we denote the signed graph by $\hat{G}$. An edge with the sign $-$ is called a *negative* edge and an edge with the sign $+$ is called a *positive* edge. Given a signed graph (G, σ), we denote by $E^+(G, \sigma)$ and $E^-(G, \sigma)$ the sets of positive and negative edges of (G, σ), respectively. A signed graph with all edges being negative is denoted by $(G, -)$ while a signed graph with all edges being positive is denoted by $(G, +)$. Given a signed graph $\hat{G}$, we denote by $-\hat{G}$ the signed graph where the sign of each edge is opposite of what is in $\hat{G}$. For example, $(G, +) = -(G, -)$.

A signed multigraph on two vertices with two parallel edges of different signs is called a *digon*. We say a signed graph (H, π) is an *(induced) subgraph* of (G, σ) if H is an (induced) subgraph of G and σ' is a signature on H such that for every $e \in E(H)$: $\sigma'(e) = \sigma(e)$. For simplicity, we may write (H, σ) as a subgraph of (G, σ) if H is a subgraph of G.

In this thesis, for drawing a signed graph, we use solid or blue lines to represent positive edges and dashed or red lines to represent negative edges. For underlying graphs with no signature, we use gray color.

Given a signed graph (G, σ) and a vertex v of (G, σ), a *switching at the vertex* v is to multiply each of the signs of all edges incident to v by $-$. This operation is based on the mathematical relation between the signs $+$ and $-$. Note that if there is a loop at v, then the multiplication will be done twice, thus nullified. Given a set $A \subseteq V(G)$, a *switching at the set* A is to switch at each vertex in A. That is equivalent to switching the signs of all edges in the edge-cut $E(A, V(G)\backslash A)$. We say a signed graph (G, σ') is *switching equivalent* to (G, σ) if it is obtained from (G, σ) by a series of switchings at vertices. In that case, we also simply say σ' is a switching equivalent signature of σ. It is easily observed that given a graph G, the relation "switching equivalent" is an equivalence class on the set of all signatures on G. In the study of coloring and homomorphism of signed graphs, switching-equivalent signed graphs are viewed as the same signed graph.

The *sign* of a structure in (G, σ) is the product of the signs of all edges in the given structure, counting multiplicity. One of the most important notions we use is the *sign of a closed walk* (or *a cycle*) which is the product of signs of all its edges (allowing repetition). The sign of some structures, such as a closed walk or a cycle, is invariant under a switching, while for some other structures, such as a path, the sign of it may change (for example, switching at one of the endpoints of a path). Thus we may relax or restrict our use accordingly. For example, when speaking of the sign of a cycle, we may refer to any equivalent signature, but when speaking of the sign of a path, we are restricted to the signature in hand. As the sign of a cycle is invariant under the switching, for a given cycle, there are only two switching equivalent signatures on it. For a positive integer ℓ, we denote a negative cycle of length ℓ to be $C_{-\ell}$ and a positive cycle of length ℓ to be $C_{+\ell}$.

A result of Zaslavsky, fundamental in the study of signed graphs, shows that a switching equivalent class to which (G, σ) belongs to is determined by signs of the cycles of (G, σ).

Proposition 2.2.1. [Zas82b] *Two signed graphs (G, σ) and (G, σ') are switching equivalent if and only if they have the same set of negative cycles.*

Closed walks are the key structures of a signed graph. There are four possible types based on the parities of their lengths and signs (i.e., the parity of the number of negative edges). One may use elements of $\mathbb{Z}_2^2$ to denote these four types. For any $ij \in \mathbb{Z}_2$, we say a closed walk W of (G, σ) is of *type* ij if the number of negative edges of W (counting multiplicity) is congruent to i (mod 2) and the total number of edges (counting multiplicity) is congruent to j (mod 2). More precisely, a *positive odd closed walk* is of *type* 01, a *negative odd closed walk* is of *type* 11, a *positive even closed walk* is of *type* 00, and a *negative even closed walk* is of *type* 10.

Given a signed graph (G, σ), for each closed walk of type ij for $ij \in \mathbb{Z}_2^2$, we denote the length of a shortest closed walk of type ij in (G, σ) to be $g_{ij}(G, \sigma)$. When there is no such closed walk, we write $g_{ij}(G, \sigma) = \infty$. Note that g_{00} is trivially 2 for any graph with at least one edge. The other three values, $g_{01}(G, \sigma)$, $g_{10}(G, \sigma)$ and $g_{11}(G, \sigma)$, play an important role in the study of homomorphisms of signed graphs.

Proposition 2.2.2. [NSZ21] *Given a connected signed graph (G, σ), if two of $g_{01}(G, \sigma)$, $g_{10}(G, \sigma)$ and $g_{11}(G, \sigma)$ are finite, then the third one is also finite.*

This leads to three special sub-classes of signed graphs each of which is of special importance in the study of signed graphs.

- The class $\mathcal{G}_{01}$ where for each $(G, \sigma) \in \mathcal{G}_{01}$, $g_{01}(G, \sigma)$ might be bounded but the other two girth values are ∞. Every element of this class, after a switching, can be presented as $(G, +)$, thus this class is the most natural embedding of graphs into larger class of signed graphs.

- The class $\mathcal{G}_{10}$ where for each $(G, \sigma) \in \mathcal{G}_{10}$, $g_{10}(G, \sigma)$ might be bounded but the other two girth values are ∞. That is to say the underlying graph G has no odd-cycle. Thus this is the class of all signed bipartite graphs. The importance of this class will be further mentioned in Section 2.3.4.

- The class $\mathcal{G}_{11}$ where for each $(G, \sigma) \in \mathcal{G}_{11}$, $g_{11}(G, \sigma)$ might be bounded but the other two girth values are ∞. Every element of this class, after a switching, can be presented as $(G, -)$. This class is of high importance specially in connection to minor theory.

The *girth* of a signed graph (G, σ) is the length of a smallest cycle of the underlying graph G, and the *odd-girth* or *negative-girth* of (G, σ) is, respectively, the smallest g_{i1} or the smallest g_{1j} for any $i, j \in \{0, 1\}$.

2.2.1 Coloring of signed graphs

One of the most natural extensions of proper colorings of graphs to signed graphs is the notion of 0-free coloring introduced by Zaslavsky in [Zas82b]. Given a signed graph (G, σ) and a positive integer k, a *0-free 2k-coloring* of (G, σ) is a mapping $c : V(G) \to \{\pm 1, \pm 2, \dots, \pm k\}$ such that for any edge $e = uv$ of (G, σ), $c(u) \neq \sigma(e)c(v)$.

One observes immediately that if (G, σ) contains a positive loop, then for no value of k it admits a proper coloring. However, for any signed graph (G, σ) without a positive loop, there exists an integer k such that (G, σ) admits a proper 0-free $2k$-coloring. Moreover, k can be chosen to be (at most) the number of vertices, because assigning a distinct absolute value for each vertex satisfies the condition.

The specialty of the definition is that (G, σ) admits a 0-free $2k$-coloring if and only if for every switching-equivalent signature σ', (G, σ') also admits 0-free $2k$-coloring. That is because if c is such a $2k$-coloring for (G, σ), then after a switching at a vertex v one may change the color of v from $c(v)$ to $-c(v)$ to preserve the property of being proper.

Note that the number of colors used in the 0-free coloring is always even. There have been several attempts to introduce an analogue coloring which uses an odd number of colors. The term "0-free" indeed identifies this coloring from a similar coloring where 0 is added to the set of colors and the set of vertices colored with 0 induces an independent set. The next definition was introduced in [Zas82b] and is further developed in [MRŠ16]. Given a signed graph (G, σ) and a positive integer k, a *$(2k+1)$-coloring* of (G, σ) is a mapping $c : V(G) \to \{0, \pm 1, \dots, \pm k\}$ such that for any edge $e = uv$ of (G, σ), $c(u) \neq \sigma(e)c(v)$.

Combining these two notions of colorings of signed graphs, there is a notion of *chromatic number* of a signed graph (G, σ), denoted by $\chi(G, \sigma)$, defined to be the smallest n such that (G, σ) admits an n-coloring and we refer to [MRŠ16] for more details. In this thesis, we normally work with signed graphs without positive loops. For simplicity, when (G, σ) contains a positive loop, one may define $\chi(G, \sigma) = \infty$.

2.2.2 Homomorphism of signed graphs

Extending the notion of homomorphisms of graphs to signed graphs, regarding the signs of edges as two different colors, we have the following concept.

Definition 2.2.3. Given two signed graphs (G, σ) and (H, π), an *edge-sign preserving homomorphism* of (G, σ) to (H, π) is a mapping of $V(G)$ and $E(G)$ to $V(H)$ and $E(H)$ (respectively) such that the adjacencies, the incidences and the signs of edges are preserved. When there exists an edge-sign preserving homomorphism of (G, σ) to (H, π), we write $(G, \sigma) \xrightarrow{s.p.} (H, \pi)$.

Note that for two signed simple graphs, an edge-sign preserving homomorphism of one to another could be simplified to a vertex mapping which preserves the adjacency and the signs of edges.

One of the important operations, which distinguishes the signed graphs from the 2-edge-colored graphs, is switching. Thus we have the following definition which more faithfully represents the homomorphism relation between two signed graphs up to switching equivalence.

Definition 2.2.4. [NSZ21] Given two signed graphs (G, σ) and (H, π), a *(switching) homomorphism* of (G, σ) to (H, π) is a mapping of $V(G)$ and $E(G)$ to $V(H)$ and $E(H)$ respectively such that the adjacencies, the incidences and the signs of closed walks are preserved. When there is a (switching) homomorphism of (G, σ) to (H, π), we write $(G, \sigma) \to (H, \pi)$.

If there exists a switching homomorphism of (G, σ) to (H, π), then we may say that (G, σ) admits an (H, π)-*coloring* or that (G, σ) is (H, π)-*colorable.*

An easy observation is that if $(G, \sigma) \xrightarrow{s.p.} (H, \pi)$, then $(G, \sigma) \to (H, \pi)$ but the converse is not necessarily true.

Theorem 2.2.5. [NSZ21] *Given two signed graphs (G, σ) and (H, π), $(G, \sigma) \to (H, \pi)$ if and only if there exists a switching-equivalent signature σ' of σ such that $(G, \sigma') \xrightarrow{s.p.} (H, \pi)$.*

The above theorem could be viewed as the equivalent definition of the (switching) homomorphism of signed graphs. It is convenient to use this definition in practice. Thus a homomorphism ϕ of (G, σ) to (H, π) consists of three parts: $\phi_1 : V(G) \to \{+, -\}$ which decides for each vertex v whether a switching is done at v; $\phi_2 : V(G) \to V(H)$ which decides to which vertex of (H, π) the vertex v is mapped; $\phi_3 : E(G) \to E(H)$ which decides the image of each edge. However, as most of time we will only consider simple graphs, ϕ_3 is induced by ϕ_2 and, therefore, the mapping ϕ is composed of ϕ_1 and ϕ_2, i.e. $\phi = (\phi_1, \phi_2)$. We note that since switching at X is the same as switching at $V \backslash X$, the two mappings (ϕ_1, ϕ_2) and $(-\phi_1, \phi_2)$ are identical.

Each of the notions (of homomorphism) leads to a corresponding notion of *isomorphism*: that is a homomorphism ϕ where ϕ_2 and ϕ_3 are one-to-one and onto. This, furthermore, leads to two notions of automorphism. For example, the negative 4-cycle with only one negative edge, as a 2-edge-colored graph, has only one non-trivial automorphism. Whereas, it is both vertex-transitive and edge-transitive with respect to the notion of (switching) homomorphism. It will be clear from the context which notion of isomorphism or automorphism we refer to. Following this notion of isomorphism, if (G_1, σ_1) is a subgraph of (G, σ') where σ' is equivalent to σ, then we may refer to (G_1, σ_1) as a subgraph of (G, σ) as well.

Given two signed graphs, how to prove whether there is a homomorphism or not from one to another? The parameters g_{ij}, for $ij \in \mathbb{Z}_2^2$, can help us to determine the nonexistence of the homomorphism.

Lemma 2.2.6. [No-homomorphism Lemma] [NSZ21] *Given two signed graphs (G, σ) and (H, π), if $(G, \sigma) \to (H, \pi)$, then $g_{ij}(G, \sigma) \geqslant g_{ij}(H, \pi)$.*

We note that, algorithmically, it is not difficult to determine $g_{ij}(G, \sigma)$, we refer to [NSZ21] and [CNS20] for more on this.

The *core* of a graph G is a smallest subgraph H of G to which G admits a homomorphism. Given a signed graph (G, σ), we define the *edge-sign preserving core* of (G, σ) to be a smallest signed subgraph (H, σ) to which (G, σ) admits an edge-sign preserving homomorphism. Similarly, the *switching core* of (G, σ) is a smallest signed subgraph (H, σ) to which (G, σ) admits a switching homomorphism. Note that the edge-sign preserving core and the switching core of a finite signed graph is unique up to isomorphism, which was shown in [NRS15].

2.2.3 Double switching graphs

A strong relation between the homomorphism of signed graphs and the edge-sign preserving homomorphism of signed graphs is provided in Theorem 2.2.5. In this section, we will use the following notion to build more connections between these two notions of homomorphisms of signed graphs.

Definition 2.2.7. [NSZ21] Given a signed graph (G, σ) with vertex set $V(G) = \{v_1, \dots, v_n\}$, the *Double Switching Graph* of (G, σ), denoted by $\text{DSG}(G, \sigma)$, is a signed graph with vertex set $V^+ \cup V^-$ with $V^+ = \{v_1^+, \dots, v_n^+\}$ and $V^- = \{v_1^-, \dots, v_n^-\}$, satisfying the following conditions: Each set of vertices V^+, V^- induces a copy of (G, σ), that is to say, for any positive (or negative) edge v_iv_j, $v_i^+v_j^+$ and $v_i^-v_j^-$ are two positive (respectively, negative) edges of $\text{DSG}(G, \sigma)$. For any positive (or negative) edge v_iv_j, $v_i^+v_j^-$ and $v_i^-v_j^+$ are two negative (respectively, positive) edges of $\text{DSG}(G, \sigma)$.

Note that a vertex v_i^- connects to vertices in V^+ as it is obtained from a switching on v_i. This construction was originally defined in [BG09] in a study of homomorphism of 2-edge-colored graphs, where a connection to homomorphism of signed graphs was established (see Theorem 2.2.8). The notion of the double switching graph is from [NSZ21]. It refers to the fact that for any switching-equivalent signed graph (G, σ') of (G, σ), there are two disjoint edge-sign preserving isomorphic copies of (G, σ') in $\text{DSG}(G, \sigma')$.

The next theorem builds another connection between the switching homomorphism and the edge-sign preserving homomorphism. For the sake of completeness, we give a direct proof using Definition 2.2.4.

Theorem 2.2.8. [BG09] *Given two signed graphs (G, σ) and (H, π), $(G, \sigma) \to (H, \pi)$ if and only if $(G, \sigma) \xrightarrow{s.p.} \text{DSG}(H, \pi)$.*

Proof. An identity mapping of the subgraph of $\text{DSG}(H, \pi)$ induced on V^+ can be extended to a mapping of $\text{DSG}(H, \pi)$ to (H, π) as follows. Each vertex v_i^- is mapped to v_i^+, each edge $v_i^+v_j^-$ is mapped to an edge $v_i^+v_j^+$ of the opposite sign and each edge $v_i^-v_j^-$ is mapped to an $v_i^+v_j^+$ of the same sign. It is easily verified that this mapping is a homomorphism of $\text{DSG}(H, \pi)$ to (H, π). Thus any edge-sign preserving homomorphism of (G, σ) to $\text{DSG}(H, \pi)$ induces a homomorphism of (G, σ) to (H, π).

For the converse, assume (G, σ) maps to (H, π) and let ϕ be such a mapping. Let σ' be a signature on G where the sign of each edge uv is the same as the sign of $\phi(u)\phi(v)$. Then, clearly, the image of each cycle C of (G, σ) in (H, ϕ) is a closed walk whose sign (with respect to π) is the same as the sign of C in (G, σ'). As this is also the case for (G, σ), it follows from Proposition 2.2.1 that σ' is switching equivalent to σ. Thus there is a set X of vertices such that σ' is obtained from σ by a switching at X. We modify ϕ to a mapping ψ of (G, σ) to $\text{DSG}(H, \pi)$ as follows. If $v \notin X$ and $\phi(v) = v_i$, then $\psi(v) = v_i^+$. If $v \in X$ and $\phi(v) = v_i$, then $\psi(v) = v_i^-$. One may now easily verify that ψ is an edge-sign preserving homomorphism of (G, σ) to $\text{DSG}(H, \pi)$. □

2.3 Signed graph operations

2.3.1 Minor

The notion of signed graphs, through independent development from the direction of minor theory, would build a better connection between the graph coloring theory and the graph minor theory.

Definition 2.3.1. Given two signed graphs (G, σ) and (H, π), (H, π) is said to be a *minor* of (G, σ) if it can be obtained from (G, σ) by switching, deleting vertices and edges, and contracting positive edges.

If (H,π) is a minor of (G,σ), for simplicity, we may say (G,σ) contains (H,π)-*minor*, and otherwise, (G,σ) is (H,π)-*minor-free*.

Given graphs G and H, we say H is an *odd-minor* of G if $(G,-)$ contains $(H,-)$ as a minor. For example, a K_3-minor-free graph is a forest, while in contrast, a graph containing no odd-K_3-minor is bipartite. The latter one forms a larger class even though both of them are 2-colorable. Similarly, the class of graphs G where $(G,-)$ has no $(K_k,-)$-minor is much larger than the class of graphs without K_k-minor.

2.3.2 Signed indicator

In the study of coloring and homomorphism of graphs, using gadgets to construct new graphs from old ones is a fruitful tool. In this section, we extend this notion to signed graphs.

Definition 2.3.2. A *signed indicator* $\mathcal{I}$ is a triple $\mathcal{I}=(\hat{G},u,v)$ such that $\hat{G}$ is a signed graph (with at least one edge) and u,v are two distinct vertices of $\hat{G}$.

Assume $\hat{G}$ is a signed graph, $\mathcal{I}=(\hat{G},u,v)$ is a signed indicator and $e=xy$ is an (either positive or negative) edge of $\hat{H}$. By *replacing* $e=xy$ *with a copy of* $\mathcal{I}=(\hat{G},u,v)$, we mean the following operation: We delete the edge e from $\hat{H}$, identify the vertex x with u and identify the vertex y with v.

There is a subtle issue in the above definition. An edge $e=xy$ is an unordered pair. So we can write it as $e=yx$ as well. However, by identifying y with u and identifying x with v, the resulting signed graph is different from the one as defined above. To avoid such confusion, it is safer to first orient the edges of $\hat{H}$ and then replace the directed edge e with $\mathcal{I}$. However, for our usage, the difference does not affect our discussion, so we just say that we replace the edge e with $\mathcal{I}$.

Definition 2.3.3. For a graph G and a signed indicator $\mathcal{I}$, we denote by $G(\mathcal{I})$ the signed graph obtained from G by replacing each edge with a copy of $\mathcal{I}$.

For a signed graph $\hat{G}$ and two signed indicators $\mathcal{I}_+$ and $\mathcal{I}_-$, we denote by $\hat{G}(\mathcal{I}_+,\mathcal{I}_-)$ the signed graph obtained from $\hat{G}$ by replacing each positive edge with a copy $\mathcal{I}_+$ and replacing each negative edge with a copy of $\mathcal{I}_-$.

An easy indicator we use often is the digon. Normally, given a graph G, we denote by $\tilde{G}$ the signed graph obtained from G by replacing each edge with a digon. Given a graph G, we denote by kG the multigraph obtained from G by replacing each edge e of G with k parallel edges. Similarly, we denote by $k\hat{G}$ the signed multi-graph obtained from $\hat{G}$, say $\hat{G}=(G,\sigma)$ by replacing each edge e with k parallel edges and assigning a signature such that the product of the signs of all the k parallel edges is $-\sigma(e)$. Note that here $k\hat{G}$ might be not unique but up to inversing operation (introduced in Section 5.2), all these $k\hat{G}$ are (inversing) equivalent.

2.3.3 $T_k(G,\sigma)$ and C_{-k}-coloring of signed graphs

As mentioned in Section 1.5, based on recent development of the theory of homomorphism of signed graphs, we show that by replacing odd cycles with negative cycles, we can fill the parity gap in this study. In order to connect the k-coloring problem with the C_{-k}-coloring problem, we introduce the following construction.

Definition 2.3.4. Given a signed graph (G,σ), we define $T_k(G,\sigma)$ to be the signed graph (G_k,π) where G_k is obtained from G by subdividing each edge so that it becomes a path of length k and π is an assignment of signs on the edges of G_k so that the sign of the $u-v$ path, corresponding to the edge $uv\in E(G)$, is the same as $-\sigma(uv)$.

In other words, given a signed graph $\hat{G}$ and two signed indicators $\mathcal{I}_+ = (P_k^-, u, v)$ and $\mathcal{I}_- = (P_k^+, u', v')$ where P_k^- is a negative path of length k with two endpoints u and v while P_k^+ is a positive path of length k with two endpoints u' and v', $T_k(G, \sigma) = \hat{G}(\mathcal{I}_+, \mathcal{I}_-)$. Using this T_k-construction, the following lemma then shows the importance of the study of C_{-k}-coloring.

Theorem 2.3.5. *A graph G is k-colorable if and only if $T_{k-2}(G, +)$ admits a homomorphism to C_{-k}.*

Proof. Since we can prove the theorem independently in each connected component of G, we may assume that G is connected. We consider two cases based on the parity of k. If k is an odd number, then in $T_{k-2}(G, +)$ a cycle is negative if and only if it is of odd length. Let G_{k-2} denote the underlying graph of $T_{k-2}(G, +)$. Thus, this signed graph is switching equivalent to $(G_{k-2}, -)$. Then, the problem of mapping $T_{k-2}(G, +)$ to C_{-k} is reduced to a graph homomorphism problem of mapping G_{k-2} to C_k. The equivalence then can be easily checked and we refer to [HN90] for a proof.

We now assume that $k = 2\ell$ is an even number, in which case $T_{k-2}(G, +)$ is a signed bipartite graph.

We first show that if $T_{k-2}(G, +) \to C_{-k}$, then G is k-colorable. Observe that, regarded as a signed graph equipped with switching, C_{-k} is both vertex-transitive and edge-transitive. Let $x_1, x_2, \ldots x_{2\ell}$ be the vertices of C_{-k} in the cyclic order. Let $X_1 = \{x_1, x_3, \ldots x_{2\ell-1}\}$ and $X_2 = \{x_2, x_4, \ldots x_{2\ell}\}$ be the two parts of C_{-k}. Let ϕ be a homomorphism of $T_{k-2}(G, +)$ to C_{-k}. Observe that as G, and therefore $T_{k-2}(G, +)$, is connected, the mapping ϕ preserves the bipartition of $T_{k-2}(G, +)$. Thus we may assume, without loss of generality, that the vertices of $T_{k-2}(G, +)$ which correspond to the vertices of G map to the vertices in X_1. Furthermore, recall that the homomorphism ϕ consists of two components $\phi_1 : V(T_{k-2}(G, +)) \to \{+, -\}$, and $\phi_2 : V(T_{k-2}(G, +)) \to X_1 \cup X_2$. Thus the restriction of ϕ onto $V(G)$ is a mapping to the set $\{+, -\} \times X_1$ which is of order 2ℓ. We claim that ϕ is a proper coloring of G. That is simply because if ϕ maps two adjacent vertices to the same element of $\{+, -\} \times X_1$, the negative $(k-2)$-path that connects them in $T_{k-2}(G, +)$ is mapped to a negative closed walk of length at most $k-2$, but that contradicts the no-homomorphism lemma.

The converse then is easier. Assume $\chi(G) \leqslant 2\ell$ and let ψ be a 2ℓ-coloring of G where $\{+, -\} \times X_1$ is the color set. Therefore the coloring ψ can be viewed as $\psi = (\psi_1, \psi_2)$ where $\psi_1 : V(G) \to \{+, -\}$ and $\psi_2 : V(G) \to X_1$. We claim that ψ can be extended as a homomorphism of $T_{k-2}(G, +)$ to C_{-k}. For any edge uv in G, noting that $\psi(u) = \psi(v)$ is not possible because ψ is a proper coloring, we consider two possibilities:

- $(\psi_1(u), \psi_2(u)) = (-\psi_1(v), \psi_2(v))$. The mapping ψ then has applied a switching only in one end of the $u - v$ path, and thus switches it to a positive (even) path. After identifying its end points the resulting positive even cycle can be mapped to just an edge of any sign.

- $\psi_2(u) \neq \psi_2(v)$. The two $\psi_2(u) - \psi_2(v)$ paths in C_{-k} are even, exactly one is negative, and each has length at most $k-2$. The $u - v$ path then can be mapped to the path of the same sign where the sign is taken after applying possible switching by ψ_1 at its end points.

It completes the proof. □

Corollary 2.3.6. *A graph G is 4-colorable if and only if $T_2(G, +)$ maps to C_{-4}.*

In particular, the 4-color theorem can be restated as:

Theorem 2.3.7. [4-color theorem restated] *For any planar graph G, the signed bipartite planar graph $T_2(G, +)$ maps to C_{-4}.*

2.3.4 $S(G)$ and signed bipartite graphs

Our special interest in signed bipartite graphs is raised by the following definition and theorem.

Definition 2.3.8. [NRS15] Let $\mathcal{I}^* = (C_{-4}, u, v)$ be a signed indicator where u and v are two non-adjacent vertices of the 4-cycle. We denote by $S(G)$ the signed graph $G(\mathcal{I}^*)$, which is obtained from replacing each edge of G with the signed indicator $\mathcal{I}^*$.

Note that there is more than one choice of signature here, and moreover, having vertices already labeled, not every two such signatures are switching equivalent: Which of the two sides of a 4-cycle is chosen to be negative makes a difference here. However, up to a switching isomorphism, any two signature choices of $S(G)$ are the same. In Figure 2.1, a planar embedding of $S(K_4)$ is depicted.

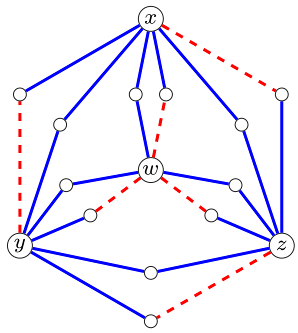


Figure 2.1. $S(K_4)$

The first easy observation on $S(G)$ is that it is a signed bipartite graph where in one part all vertices are of degree 2. The next two theorems show the importance of this construction. The first one shows how to naturally embed the homomorphism order of graphs into homomorphism order of signed bipartite graphs, and the second one shows how to capture the notion of the chromatic number of a graph in this homomorphism order.

Theorem 2.3.9. [NRS15] *Given graphs G and H, G admits a homomorphism to H if and only if $S(G)$ admits a homomorphism to $S(H)$.*

We denote by $(K_{2,2}, e)$ the signed graph on $K_{2,2}$ with only one negative edge and by $(K_{k,k}, M)$ the signed graph on $K_{k,k}$ with its negative edges forming a perfect matching.

Theorem 2.3.10. [NRS15] *Given a graph G, we have the following claims.*

- $\chi(G) \leqslant 2$ *if and only if* $S(G) \to (K_{2,2}, e)$;
- $\chi(G) \leqslant k$ *if and only if* $S(G) \to (K_{k,k}, M)$ *for* $k \geqslant 3$;

As the problem of mapping signed graphs to $(K_{k,k}, M)$ could capture the problem of the coloring of ordinary graphs, the signed bipartite graphs $(K_{k,k}, M)$ are of special importance in the study of homomorphism of signed graphs.

Note that the planarity is preserved when we construct $S(G)$ from a planar graph G. Let $\mathcal{SPB}_2$ be the class of signed bipartite planar simple graphs in which one partite set has maximum degree of at most 2. It is clear that for each planar graph G, $S(G)$ is in $\mathcal{SPB}_2$ and that core of each signed bipartite graph in $\mathcal{SPB}_2$ is a subgraph of $S(G)$ for some planar graph G.

As a direct corollary of Theorem 2.3.9, we have the following reformulation of 4-color theorem.

Theorem 2.3.11. [4-color theorem restated] *For any planar graph G, we have $S(G) \to S(K_4)$.*

Applying the case $k = 4$ of Theorem 2.3.10, we have another restatement of the 4-color theorem as follows.

Theorem 2.3.12. [4-color theorem restated] *For any planar graph G, $S(G) \to (K_{4,4}, M)$.*

One can check that with an appropriate labeling of vertices, $(K_{4,4}, M)$ is the signed projective cube of dimension 3, i.e., $SPC(3)$. The next theorem is a strengthening of the above restated 4-color theorem. Its proof is, in [NRS13], based on an edge-coloring result of B. Guenin [Gue03] which in turn is based on the 4-color theorem.

Theorem 2.3.13. *Every signed bipartite planar simple graph admits a homomorphism to $(K_{4,4,}M)$.*

Viewing the 4-color theorem as the homomorphisms of planar graphs to K_4, one may consider the only three core (proper) subgraphs K_1, K_2, K_3 of K_4 and for each of them ask: which planar graphs map to these subgraphs? While the homomorphism problem to K_1 is a matter of triviality and to K_2 is rather easy, the homomorphism problem of planar graphs to K_3 has been a subject of extensive study. On one hand, it is proved to be an NP-complete problem. On the other hard, starting with Grötzsch's theorem, an extensive family of planar graphs are proved to be 3-colorable. We refer to [DKT11; DKT20; Dvo13; KY14b; Tho03] and to the references there for some of the work on this subject.

Motivated by this observation, Theorem 2.3.10, and Theorem 2.3.13, it is natural to ask for each core subgraph of $(K_{4,4}, M)$, which families of signed planar graphs map to it. Of such subgraphs of $(K_{4,4}, M)$ there are two notable ones to consider: (1) the negative 4-cycle C_{-4}. We refer to Chapter 8 for progress on this problem; (2) $(K_{3,3}, M)$. We refer to Chapter 9 for results on the homomorphism problem to this signed graph.

2.3.5 Folding lemma on planar graphs

When we study the homomorphism and coloring problem of planar graphs, we have a useful lemma that enable us to make "uniform" the type of the facial cycles of a plane graph without changing its odd-girth. It has been introduced as the folding lemma in [KZ00].

Lemma 2.3.14. [Folding lemma] [KZ00] *Let G be a plane graph of odd-girth g and let $C = v_1 \cdots v_k$ be a face whose boundary is not a cycle of length g. Then there is an integer $i \in \{1, \ldots, k\}$ such that the graph G' obtained from G by identifying two vertices v_i and v_{i+2}, where the index is taken modulo k, is still of odd-girth g.*

Moreover, there is a bipartite analogue of the folding lemma on signed bipartite planar graphs proved in [NRS13]. We will apply this lemma frequently when we study the circular coloring and homomorphism problems of signed bipartite planar graphs in Part IV.

Lemma 2.3.15. [Bipartite folding lemma] [NRS13] *Let $\hat{G}$ be a signed bipartite plane graph and let $2k$ be the length of its shortest negative cycle. Assume that C is a facial cycle that is not a negative $2k$-cycle. Then there are vertices v_{i-1}, v_i, v_{i+1}, consecutive in the cyclic order of the boundary of C, such that identifying v_{i-1} and v_{i+1}, after a possible switching at one of the two vertices, the resulting signed graph remains a signed bipartite plane graph whose shortest negative cycle is still of length $2k$.*

We observe that by applying this lemma repeatedly, we get a homomorphic image of $\hat{G}$ which is also a signed bipartite plane graph in which every facial cycle is a negative cycle of length exactly $2k$.

2.4 (H, π)-critical signed graphs

2.4.1 k-critical graphs

One of the key notions in the study of proper coloring is the concept of *k-critical* graphs, which is defined to be a graph of chromatic number k whose proper subgraphs are all $(k-1)$-colorable. An extension of the notion to homomorphism was proposed in 1980's by Catlin [Cat88], but the concept was not drawn much attention until recently.

Definition 2.4.1. Given a graph H, a graph G is said to be *H-critical*, if G does not admit a homomorphism to H but each proper subgraph of it does.

Note that a graph is k-critical if and only if it is K_{k-1}-critical. Next to the complete graphs, the most studied target graphs are odd cycles. One of the key directions of the study of k-critical graphs or C_{2k+1}-critical graphs is to bound from below the number of their edges as a function of k and n (the number of vertices). Kostochka and Yancy gave a nearly tight lower bound in [KY14b], almost settling a conjecture of Gallai. Observing that being a 4-critical graph is the same as being C_3-critical, it follows from the special case presented in [KY14a] that any C_3-critical graph on n vertices has at least $\lceil \frac{5n-2}{3} \rceil$ edges. Their approach is extended to the study of C_5-critical graphs in [DP17] and to C_7-critical graphs in [PS22]. In [DP17], it is proved that any C_5-critical graph on n vertices has at least $\frac{5n-2}{4}$ edges and they conjecture that the bound can be improved to $\frac{14n-9}{11}$. Similarly, in [PS22], it is proved that any C_7-critical graph on n vertices has at least $\frac{17n-2}{15}$ edges and they conjecture that the bound can be improved to $\frac{27n-20}{23}$.

2.4.2 (H, π)-critical signed graphs

We extend this notion to signed graphs.

Definition 2.4.2. Given a signed graph (H, π), a signed graph (G, σ) is said to be *(H, π)-critical* if the following conditions are satisfied:

- $g_{ij}(G, \sigma) \geqslant g_{ij}(H, \pi)$ for $ij \in \mathbb{Z}_2^2$, (the condition of the no-homomorphism lemma),
- $(G, \sigma) \nrightarrow (H, \pi)$,
- $(G', \sigma) \to (H, \pi)$ for every proper subgraph G' of G.

The notion captures and extends the notion of k-critical graphs as follows: A graph G is k-critical if the signed graph $(G, +)$ is $(K_{k-1}, +)$-critical, here the condition of no-homomorphism lemma implies that G has no loop. The notion of H-critical graphs is also captured by viewing H as the signed graph $(H, +)$ but with a minor revision. If G is an H-critical graph in the sense of [Cat88] and it has an odd cycle C_{2k+1}, where odd-girth$(H) > 2k+1$, then G is the odd cycle C_{2k+1}. Our first condition then eliminates these trivial cases.

For the particular case when $(H, \pi) = C_{-\ell}$, we identify two cases based on the parity of ℓ:

- $\ell = 2k+1$. In this case, in order for (G, σ) to satisfy the conditions of no-homomorphism lemma, in particular, we must have $(G, -)$ switching equivalent to (G, σ). After a switching of (G, σ) to $(G, -)$ and $C_{-\ell}$ to $(C_{2k+1}, -)$, the problem is reduced to the study of C_{2k+1}-critical graphs (of odd-girth at least $2k+1$).
- $\ell = 2k$. In this case, in order for (G, σ) to satisfy the conditions of no-homomorphism lemma, G must, in particular, be bipartite. This is the case of main interest of Part IV of this thesis.

We note that in the first case, to determine if (G, σ) is switching equivalent to $(G, -)$ can be done in polynomial time and quite efficiently, but to determine if $G \to C_{2k+1}$ is an NP-complete problem. In contrast, in the second case, to find an equivalent signature under which we can map (G, σ) to $C_{-\ell}$ is the hard part, and given a fixed signature, we can determine, in polynomial time, if there exists an edge-sign preserving homomorphism to $C_{-\ell}$ with only one negative edge(see Theorem 8.1.1).

Strengthening Theorem 2.3.5, we proved the following.

Theorem 2.4.3. *A graph G is $(k+1)$-critical if and only if $T_{k-2}(G, +)$ is C_{-k}-critical.*

Proof. First we assume G is $(k+1)$-critical. We need to show that $T_{k-2}(G, +)$ is C_{-k}-critical. Let e be an edge of $T_{k-2}(G, +)$ and assume it is on the path corresponding to the edge uv of G. Then since G is critical, $G - uv$ admits a k-coloring which can be transformed into a mapping of $T_{k-2}(G - uv, +)$ to C_{-k}. This mapping could then be extended to the remaining vertices of the corresponding uv-path.

Conversely, assuming that $T_{k-2}(G, +)$ is C_{-k}-critical, we need to show that G is $(k+1)$-critical. This follows from the fact that $T_{k-2}(G - uv, +)$ is a proper subgraph of $T_{k-2}(G, +)$ for any edge uv and Theorem 2.3.5. □

Part II

Circular Colorings of Signed Graphs

3 | Circular colorings of signed graphs

This chapter is based on the following papers:

[NWZ21] R. Naserasr, Z. Wang, and X. Zhu. "Circular chromatic number of signed graphs". In: *Electron. J. Combin.* 28.2 (2021), Paper No. 2.44, 40. DOI: 10.37236/9938

[NW21] R. Naserasr and Z. Wang. *Signed bipartite circular cliques and a bipartite analogue of Grötzsch's theorem.* 2021. arXiv: 2109.12618 [math.CO]

Motivated by Jaeger-Zhang conjecture and its bipartite analogue discussed in Section 1.4, in this chapter, we extend the notion of circular coloring from graphs to signed graphs. It is a refinement of the 0-free coloring of signed graphs, which was introduced by Zaslavsky in [Zas82b]. In this chapter, we only consider signed graphs without positive loops. In section 3.1, we prove the basic properties of this new notion. Especially, we provide a tight cycle argument in Section 3.1.2, using which we show that the circular chromatic number of a finite signed graph is a rational number. This idea allows us to find a finite set of candidates for the circular chromatic number of a given signed graph.

In Section 3.2, we show several equivalent definitions of circular r-coloring. Moreover, given a positive even integer p and positive integer q with $p \geqslant 2q$, we build the signed circular clique $K^s_{p;q}$ and show that a signed graph admits a circular $\frac{p}{q}$-coloring if and only if it admits a homomorphism to $K^s_{p;q}$. Combining the tight-cycle argument, we know that there exists an (exponential time) algorithm which could determine the circular chromatic number of a finite signed graph.

In Section 3.3, we apply some graph operations on signed graphs and investigate the relationship between the circular chromatic number of the original (signed) graphs and the circular chromatic number of the resulting (signed) graphs. Lemma 3.3.11 tells us that if indicators $\mathcal{I}_+, \mathcal{I}_-$ satisfy certain circular coloring properties, then the circular chromatic numbers of $\hat{G}$ and $\hat{G}(\mathcal{I}_+, \mathcal{I}_-)$ can be determined by each other. Especially, Theorem 3.3.12 indicates that via a signed graph operation, the circular chromatic number of a graph G can be determined by the circular chromatic number of $S(G)$ and vice versa. In Lemma 3.3.18, we give the formula of the circular chromatic number of certain subdivision of a signed graph (G, σ) in terms of the circular chromatic number of (G, σ).

When restricted to the class of signed bipartite graphs, in Section 3.4, we define signed bipartite circular cliques $B_{p;q}$, which play an important role in the homomorphism order of signed bipartite graphs. More precisely, we prove in Theorem 3.4.2 that the circular chromatic number of a signed bipartite graph is bounded by $\frac{p}{q}$ if and only if it admits a homomorphism to $B_{p;q}$. In particular, some interesting homomorphism targets are shown to be essentially signed bipartite circular cliques, for example, $(K_{3,3}, M)$ and C_{-2k}, etc. Based on the discussion on the class of signed bipartite graphs, we give some new restatements of the 4-color theorem at the end of this section.

3.1 Circular coloring of signed graphs

Let C^r be a circle of circumference r obtained from the interval $[0, r]$ by identifying the points 0 and r. For each point x on C^r, the unique point of distance $\frac{r}{2}$ from x is called the *antipodal* of x and is denoted by $\bar{x}$. Given a set A of points on C^r, the *antipodal* of A, denoted by $\bar{A}$, is the set of antipodals of points in A. In particular, if I is an open unit interval of C^r, then $\bar{I}$ is its antipodal, which is also an open unit interval of C^r.

Definition 3.1.1. Given a signed graph (G, σ) and a real number r, a *circular r-coloring* of (G, σ) is a mapping $f : V(G) \to C^r$ satisfying the followings.

- For each positive edge uv of (G, σ), $d_{(\text{mod } r)}(f(u), f(v)) \geqslant 1$.
- For each negative edge uv of (G, σ), $d_{(\text{mod } r)}(f(u), \overline{f(v)}) \geqslant 1$.

The *circular chromatic number* of (G, σ) is defined as

$$\chi_c(G, \sigma) = \inf\{r \mid (G, \sigma) \text{ admits a circular } r\text{-coloring}\}.$$

Note that for any negative edge uv, the condition $d_{(\text{mod } r)}(f(u), \overline{f(v)}) \geqslant 1$ is equivalent to

$$d_{(\text{mod } r)}(f(u), f(v)) \leqslant \frac{r}{2} - 1.$$

Equivalently, we have a geometrical interpretation of circular coloring in the language of an assignment of unit intervals to the vertices of signed graphs. Let $\mathcal{I}(C^r)$ be the set of all the open unit intervals of C^r. Given a signed graph (G, σ) and a real number r, a *circular r-coloring* of (G, σ) is an assignment $\varphi : V(G) \to \mathcal{I}(C^r)$ such that for each positive edge uv of (G, σ), $\varphi(u) \cap \varphi(v) = \varnothing$, and for each negative edge uv of (G, σ), $\varphi(u) \cap \overline{\varphi(v)} = \varnothing$.

For finite (signed) graphs which are the main concerns of this work, noting that C^r is a compact set and by a basic application of the notion of limits, we observe that the "inf" in the definition is attained and can be replaced by the minimum. It implies that if $\chi_c(G, \sigma) = r$, then there exists a circular r-coloring of (G, σ).

3.1.1 Some basic properties

The first proposition is to show that our definition of the circular chromatic number is invariant under the switching isomorphism.

Proposition 3.1.2. *Let (G, σ) and (G, σ') be two switching-equivalent signed graphs. Then every circular r-coloring of (G, σ) corresponds to a circular r-coloring of (G, σ'). In particular, $\chi_c(G, \sigma') = \chi_c(G, \sigma)$.*

Proof. As signed graphs (G, σ) and (G, σ') are switching equivalent, without loss of generality, we assume that (G, σ') is obtained from (G, σ) by switching at a vertex set A. Let f be a circular r-coloring of (G, σ). Define $g : V(G) \to C^r$ as

$$g(v) = \begin{cases} f(v), & \text{if } v \in V(G) \backslash A, \\ \overline{f(v)}, & \text{if } v \in A. \end{cases}$$

It is easy to verify that g is a circular r-coloring of (G, σ'). □

The next observation tells us that the circular chromatic number of a signed graph is indeed a generalization of the circular chromatic number of a graph.

Observation 3.1.3. *If G is a graph with no loop, then $\chi_c(G,+) = \chi_c(G)$.*

The circular chromatic number of a signed graph is a refinement of its chromatic number, defined based on the notion of 0-free coloring define by Zaslavsky [Zas82b]. Recall that a *0-free 2k-coloring* of (G,σ) is a mapping $\varphi : V(G) \to \{\pm 1, \pm 2, \dots, \pm k\}$ such that for any edge $e = uv$ of (G,σ), $\varphi(u) \neq \sigma(e)\varphi(v)$.

Proposition 3.1.4. *Given a positive integer k and a signed graph (G,σ), (G,σ) is 0-free $2k$-colorable if and only if (G,σ) is circular $2k$-colorable.*

Let $f : V(G) \to \{\pm 1, \pm 2, \dots, \pm k\}$ be a 0-free $2k$-coloring of (G,σ). Let

$$g(v) = \begin{cases} f(v) - 1, & \text{if } f(v) \in \{1, 2, \dots, k\} \\ -f(v) + k - 1, & \text{if } f(v) \in \{-1, -2, \dots, -k\}. \end{cases}$$

It is straightforward to verify that g is a circular $2k$-coloring of (G,σ). Another direction follows directly from Lemma 3.2.6 which will be introduced later.

Note that our definition of circular coloring is quite symmetric, meaning that each point on the circle plays the same role as others. Moreover, comparing our definition to the chromatic number defined in [MRŠ16], when the chromatic number is an odd integer, it is completely different. For example, it has been showed in [MRŠ16] that signed planar simple graphs are 5-colorable but later in Section 4.4, we will show that for signed planar simple graph (G,σ), $\chi_c(G,\sigma) \leqslant 6$.

Another key observation is that the circular chromatic number of signed graphs we defined plays well with the notion of homomorphisms of signed graphs. This will lead to the development of signed circular cliques in Section 3.2. Here we mention the no-homomorphism lemma first and observe that the circular chromatic number parameter follows the homomorphic order.

Lemma 3.1.5. *If $(G,\sigma) \to (H,\pi)$, then we have $\chi_c(G,\sigma) \leqslant \chi_c(H,\pi)$.*

Proof. As $(G,\sigma) \to (H,\pi)$, by Theorem 2.2.5, there is a switching-equivalent σ' and an edge-sign preserving homomorphism φ, regarded as a vertex mapping, of (G,σ') to (H,π). Given a circular r-coloring f of (H,π), $f \circ \varphi$ is a circular r-coloring of (G,σ'), which implies a circular r-coloring of (G,σ). □

Observe that $\chi_c(G,\sigma) = 1$ if and only if (G,σ) has no edge. If (G,σ) has one (either positive or negative) edge, then (G,σ) is not circular r-colorable for any r strictly smaller than 2. Thus we always assume that $r \geqslant 2$. Moreover, we have the characterization of signed graphs whose circular chromatic numbers equal to 2.

Proposition 3.1.6. *Given a signed graph (G,σ), $\chi_c(G,\sigma) = 2$ if and only if (G,σ) is switching equivalent to $(G,-)$.*

Proof. Without loss of generality, assume that (G,σ) is connected. For one direction, assume that $\chi_c(G,\sigma) = 2$. Let C be the circle of circumference 2 and let $f : V(G) \to C$ be a circular 2-coloring. Assume there is a positive edge uv. Let $f(u) = a$ and $f(v) = b$. As $d_{(\text{mod } 2)}(f(u), f(v)) = 1$ and C is of circumference 2, all the vertices will be mapped to either a or b. We denote A the set of vertices which are mapped to a. Switching at A, we obtain $(G,-)$. The inverse is quite obvious. If (G,σ) is switching equivalent to $(G,-)$, then assigning the same point to all the vertices is a circular 2-coloring of $(G,-)$. □

For graphs, parallel edges do not have effect on (circular) chromatic number. However, for signed graphs, the presence or absence of parallel edges of different signs influences the circular chromatic number.

Proposition 3.1.7. *Let D be a digon. We have that $\chi_c(D) = 4$.*

Proof. Let x and y be the two vertices of D. It is easy to find a circular 4-coloring of D. That $\chi_c(D) \geqslant 4$ holds because in any circular r-coloring ψ of D, the four points $\psi(x)$, $\overline{\psi(x)}$, $\psi(y)$ and $\overline{\psi(y)}$ must be pairwise at distance at least one. □

Thus when we work with a signed graph (G, σ), it easily follows that $\chi_c(G, \sigma) \geqslant 4$ if (G, σ) contains a digon (as a subgraph).

3.1.2 Tight cycle in a circular coloring

In this part, we extend the idea of the tight cycle to signed graphs. Using this concept, we show that the circular chromatic number of any finite (signed) graph is a rational number, and, moreover, it can be computed for any given signed graph. We start with the following definition.

Definition 3.1.8. Assume (G, σ) is a signed graph and $\phi : V(G) \to [0, r)$ is a circular r-coloring of (G, σ). The *partial orientation* $D_\phi(G, \sigma)$ of (G, σ) with respect to ϕ is defined as follows: (u, v) is an arc of $D_\phi(G, \sigma)$ if and only if one of the following holds:

- uv is a positive edge and $\phi(v) - \phi(u) = 1 \pmod r$.
- uv is a negative edge and $\overline{\phi(v)} - \phi(u) = 1 \pmod r$.

Furthermore, arcs in $D_\phi(G, \sigma)$ are called *tight arcs* of (G, σ) with respect to ϕ. A directed path (respectively, a directed cycle) in $D_\phi(G, \sigma)$ is called a *tight path* (respectively, a *tight cycle*) with respect to ϕ.

For any given circular r-coloring of a signed graph (G, σ), the existence of a tight cycle in its partial orientation $D_\phi(G, \sigma)$ is determined by the value of r. More precisely, we have the next lemma.

Lemma 3.1.9. *Let (G, σ) be a signed graph and let ϕ be a circular r-coloring of (G, σ). If $D_\phi(G, \sigma)$ is acyclic, then there exists an r_0 with $r_0 < r$, such that (G, σ) admits a circular r_0-coloring.*

Proof. For a given signed graph (G, σ) and a circular r-coloring ϕ of (G, σ), suppose that $D_\phi(G, \sigma)$ is acyclic. Moreover, we assume among all such ϕ, $D_\phi(G, \sigma)$ has minimum number of arcs.

First we show that $D_\phi(G, \sigma)$ has no arc. Otherwise, since $D_\phi(G, \sigma)$ is acyclic, $D_\phi(G, \sigma)$ has an arc (v, u) such that u is a sink. Thus for every positive edge uw, $(\phi(w) - \phi(u)) \pmod r > 1$ and for every negative edge uw, $(\overline{\phi(w)} - \phi(u)) \pmod r > 1$. As G is finite, there exists an $\epsilon > 0$ such that for every positive edge uw of (G, σ), $(\phi(w) - \phi(u)) \pmod r > 1 + \epsilon$ and for every negative edge uw, $(\overline{\phi(w)} - \phi(u)) \pmod r > 1 + \epsilon$. Let $\psi(x) = \phi(x)$ for $x \neq u$ and $\psi(u) = \phi(u) + \epsilon$. Then ψ is a circular r-coloring of (G, σ) and $D_\psi(G, \sigma)$ is a sub-digraph of $D_\phi(G, \sigma)$, in which (v, u) is not an arc and no new arc is created. So $D_\psi(G, \sigma)$ is acyclic and has fewer arcs than $D_\phi(G, \sigma)$, a contradiction.

Since $D_\phi(G, \sigma)$ has no arc, it follows from the definition that there exists an $\epsilon > 0$ such that for any positive edge uv, $1 + \epsilon \leqslant |\phi(u) - \phi(v)| \leqslant r - (1 + \epsilon)$ and for any negative edge uv, $1 + \epsilon \leqslant |\overline{\phi(u)} - \phi(v)| \leqslant r - (1 + \epsilon)$. Let $r_0 = \frac{r}{1+\epsilon}$ and let $\psi : V(G) \to [0, r_0)$ be defined as $\psi(v) = \frac{\phi(v)}{1+\epsilon}$ for each vertex $v \in V(G)$. Then ψ is a circular r_0-coloring of (G, σ). □

It implies that for a signed graph (G, σ), if $\chi_c(G, \sigma) = r$, then every circular r-coloring of (G, σ) has a tight cycle. Moreover, we have the following necessary and sufficient conditions based on the tight cycle argument for $\chi_c(G, \sigma)$ being r. It indicates that the notion of the tight cycle is a key tool in the study of the circular coloring of signed graphs.

Lemma 3.1.10. *Given a signed graph (G, σ), $\chi_c(G, \sigma) = r$ if and only if (G, σ) is circular r-colorable and every circular r-coloring ϕ of (G, σ) has a tight cycle.*

Proof. One direction follows from Lemma 3.1.9. It remains to show that if (G, σ) is circular r-colorable and every circular r-coloring ϕ of (G, σ) has a tight cycle, then $\chi_c(G, \sigma) = r$. Suppose to the contrary that $\chi_c(G, \sigma) < r$, we would prove that there is a circular r-coloring ϕ of (G, σ) such that $D_\phi(G, \sigma)$ is acyclic.

Assume $\chi_c(G, \sigma) = r' < r$. Let $\psi : V(G) \to [0, r)$ be a circular r'-coloring of (G, σ). We define $\phi(v) = \frac{r}{r'}\psi(v)$. Then it is easy to verify that ϕ is a circular r-coloring of (G, σ) and $D_\phi(G, \sigma)$ contains no arc. Hence, it is acyclic. □

We remark here that given a circular coloring of a signed graph, the tight cycle is independent of a switching-equivalent signature. Assume that $\chi_c(G, \sigma) = r$ and $f : V(G) \to [0, r)$ is a circular r-coloring of (G, σ). Given a vertex set A, let (G, σ') be a signed graph obtained from (G, σ) by switching at A. We define

$$g(v) = \begin{cases} f(v) - \frac{r}{2}, & \text{if } v \in A, \\ f(v), & \text{if } v \notin A. \end{cases}$$

Then g is a circular r-coloring of (G, σ'). A tight cycle $C = v_1 v_2 \cdots v_\ell$ with respect to f is also a tight cycle with respect to g.

As shown in the tight cycle argument, cycles play an important role in determining the circular chromatic number. Thus we provide a plausible set of values for the circular chromatic number of (G, σ).

Proposition 3.1.11. *Every signed graph (G, σ), which is not a forest, has a cycle with s positive edges and t negative edges such that*

$$\chi_c(G, \sigma) = \frac{2(s + t)}{2a + t} \tag{3.1}$$

for some integer a.

Proof. Let C^r denote a circle of circumference r, which is obtained by identifying the end points 0 and r of an interval $[0, r]$. Let (G, σ) be a signed graph with $\chi_c(G, \sigma) = r$ and let $\psi : V(G) \to [0, r)$ be a circular r-coloring of (G, σ). By Lemma 3.1.10, $D_\psi(G, \sigma)$ contains a directed cycle C.

Suppose that C consists of s positive edges and t negative edges and $C = v_1 v_2 \cdots v_{s+t}$. If $v_i v_{i+1}$ is a positive edge, then traversing from the point $\psi(v_i)$, one unit along the clockwise direction of C^r, we arrive at the point $\psi(v_{i+1})$. If $v_i v_{i+1}$ is a negative edge, then from the point $\psi(v_i)$, by first traversing $\frac{r}{2}$ unit along the anti-clockwise direction of C^r then traversing along the clockwise direction a unit distance, we arrive at the point $\psi(v_{i+1})$.

Therefore, the directed cycle C represents a traverse along the circle C^r for the distance of $s - (\frac{r}{2} - 1) \cdot t$, at end of which one must come back to the starting point. Intuitively, the cycle C (regarded as a curve) winds around the circle C^r clockwise some number of times, say a times. So

$$1 \times s - (\frac{r}{2} - 1) \times t = r \times a$$

for some integer a. Hence $r = \frac{2(s+t)}{2a+t}$. □

As $r \geqslant 2$, a is an integer satisfying that $|a| \leqslant \frac{|V(G)|}{2}$. Since $s+t \leqslant |V(G)|$ and $r \geqslant 2$, given the number of vertices of (G, σ), there is a finite number of candidates for the circular chromatic number of (G, σ). More precisely, the values of $\chi_c(G, \sigma)$ is limited to one of at most $2|V(G)|^2$ possibilities. Thus we have the following corollary.

Corollary 3.1.12. *Assume (G, σ) is a signed graph on n vertices. Then $\chi_c(G, \sigma) = \frac{p}{q}$ for some $p \leqslant 2n$ and $q \leqslant n$. In particular,*

$$\chi_c(G, \sigma) = \min\{\frac{p}{q} \mid (G, \sigma) \text{ admits a circular } \frac{p}{q}\text{-coloring}\}.$$

It follows from Corollary 3.1.12 that there is an exponential-time algorithm ($O(n^2 p^n)$) that determines the circular chromatic number of a finite signed graph. As the 3-colorability of graphs is one of the well-known NP-complete problems, the problem of determining the circular chromatic number of a given signed graph falls into the class of NP-complete problems and, thus, unless P=NP, one does not expect to find an algorithm that runs in less than the exponential time in order of a graph and that works for all signed graphs. But with a restriction on the input class, one may find better algorithms and approximations.

3.2 Equivalent definitions

The notions of the circular coloring and the chromatic number of signed graphs have various equivalent restatements. Here we introduce some of them. For $s, t \in [0, r)$, let $d_{(\text{mod } r)}(s, t) = \min\{|s-t|, r-|s-t|\}$. A circular r-coloring of a signed graph can be reformulated as follows, which is sometimes more convenient for the computation.

Definition 3.2.1. Given a signed graph (G, σ) and a real number r, a *circular r-coloring* of a signed graph (G, σ) is a mapping $f : V(G) \to [0, r)$ such that for each positive edge uv,

$$1 \leqslant |f(u) - f(v)| \leqslant r - 1$$

and for each negative edge uv,

$$\text{either } |f(u) - f(v)| \leqslant \frac{r}{2} - 1 \text{ or } |f(u) - f(v)| \geqslant \frac{r}{2} + 1.$$

We remark that if (G, σ) is circular r-colorable, then there always exists a circular r-coloring $f : V(G) \to [0, r)$ satisfying that $f(v) \leqslant \frac{r}{2}$ for every vertex v. Assume that f is a circular r-coloring of (G, σ). Let $A = \{v \mid f(v) \geqslant \frac{r}{2}\}$. We define

$$g(v) = \begin{cases} f(v), & \text{if } v \notin A \\ f(v) - \frac{r}{2}, & \text{if } v \in A. \end{cases}$$

By Proposition 3.1.2, g is a circular r-coloring of (G, σ') which is obtained from (G, σ) by switching at a vertex set A. Such g is a circular r-coloring satisfying that $g(v) \leqslant \frac{r}{2}$ for every vertex v.

We can also define the circular coloring of signed graphs through the homomorphism. We first need some nice homomorphism targets. Normally, we only consider finite graphs but here we introduce infinite signed graphs based on the notion of circular r-coloring.

Definition 3.2.2. For a real number $r \geqslant 2$, let K_r^s be an infinite signed graph which has the vertex set $[0, r)$, in which for two vertices x and y,

- xy is a positive edge if $1 \leqslant |x-y| \leqslant r-1$ and
- xy is a negative edge if either $|x-y| \leqslant \frac{r}{2}-1$ or $|x-y| \geqslant \frac{r}{2}+1$.

Note that when $r \geqslant 4$, there are some pair of vertices are connected by one positive edge and one negative edge. The next lemma, viewed as one of equivalent definitions of circular coloring of signed graphs, follows directly from Definition 3.2.2.

Lemma 3.2.3. *Given a signed graph (G,σ) and a real number $r \geqslant 2$, (G,σ) admits a circular r-coloring if and only if $(G,\sigma) \xrightarrow{s.p.} K_r^s$.*

By the natural projection of $[0,r)$ to $[0,r')$ ($f(x) = \frac{r'x}{r}$ for $x \in [0,r)$), we have the following.

Lemma 3.2.4. *Given real numbers r' and r, $r' \geqslant r \geqslant 2$, $K_r^s \xrightarrow{s.p.} K_{r'}^s$.*

Let $r = \frac{p}{q}$ and assume that p is an even positive integer. We claim that any signed (finite) graph (G,σ) which is circular r-colorable admits a coloring where the colors used are in the form of $\frac{i}{q}$ with i being integer. As (G,σ) is circular r-colorable, assume that f is a circular r-coloring of (G,σ). We partition the interval $[0,\frac{p}{q})$ into p parts as follows: for each $i \in \{0,\dots,p-1\}$, let $I_i = [\frac{i}{q},\frac{i+1}{q})$ and thus $\bigcup_{i=0}^{p-1} I_i = [0,r)$. For each vertex v of (G,σ), we define a mapping $g: V(G) \to \{\frac{i}{q} \mid i \in \{0,1,\dots,p-1\}\}$ satisfying that $g(v) = \frac{i}{q}$ if $f(v) \in I_i$. We now show that such a mapping g is also a circular $\frac{p}{q}$-coloring of (G,σ). We discuss two possibilities based on the signs of the edges.

- For a positive edge uv, we have $1 \leqslant |f(u)-f(v)| \leqslant \frac{p}{q}-1$. It implies that
$$1-\frac{1}{q} < |g(u)-g(v)| < \frac{p}{q}-1+\frac{1}{q}.$$
Since $q|g(u)-g(v)|$ is an integer, we conclude that $1 \leqslant |g(u)-g(v)| \leqslant \frac{p}{q}-1$.
- For a negative edge uv, we know that either $|f(u)-f(v)| \leqslant \frac{p}{2q}-1$ or $|f(u)-f(v)| \geqslant \frac{p}{2q}+1$. It implies that either
$$|g(u)-g(v)| < \frac{p}{2q}-1+\frac{1}{q} \text{ or } |g(u)-g(v)| > \frac{p}{2q}+1-\frac{1}{q}.$$
Since p is even, $\frac{p}{2}$ is an integer. As $q|g(u)-g(v)|$ is an integer, we conclude that either $|g(u)-g(v)| \leqslant \frac{p}{2q}-1$ or $|g(u)-g(v)| \geqslant \frac{p}{2q}+1$.

Note that it is crucial that p is an even integer. For otherwise $\frac{p}{2}$ is not an integer, and we cannot conclude that $|g(u)-g(v)| \leqslant \frac{p}{2q}-1$ or $|g(u)-g(v)| \geqslant \frac{p}{2q}+1$. Indeed, if p is odd, then the set $\{0,\frac{1}{q},\dots,\frac{p-1}{q}\}$ is not closed under taking antipodal points.

The above observation leads to the following equivalent definition of the circular chromatic number of signed graphs after a scaling of the values. For $i,j \in \{0,1,\dots,p-1\}$, the *modulo-$p$ distance* between i and j is defined to be $d_{(\text{mod } p)}(i,j) = \min\{|i-j|, p-|i-j|\}$. Given an even integer p and an integer $x \in \{0,1,\dots,p-1\}$, the *p-antipodal* of x is $\bar{x} = x + \frac{p}{2} \pmod{p}$. When the set of integers that we work with is clear from the context, we just call $\bar{x}$ the *antipodal* of x.

Definition 3.2.5. Given a signed graph (G, σ) and integers $p \geqslant 2q > 0$ such that p is even, a *(p, q)-coloring* of (G, σ) is a mapping $f : V(G) \to \{0, 1, \ldots, p-1\}$ such that for any positive edge uv,

$$d_{(\text{mod } p)}(f(u), f(v)) \geqslant q,$$

and for any negative edge uv,

$$d_{(\text{mod } p)}(f(u), \overline{f(v)}) \geqslant q.$$

Note that $d_{(\text{mod } p)}(i, j) \geqslant q$ is equivalent to $q \leqslant |i - j| \leqslant p - q$. The next lemma follows directly from the discussion above.

Lemma 3.2.6. *Given a signed graph (G, σ) and integers $p \geqslant 2q > 0$ such that p is even, (G, σ) admits a circular $\frac{p}{q}$-coloring if and only if (G, σ) admits a (p, q)-coloring. Thus the circular chromatic number of (G, σ) is*

$$\chi_c(G, \sigma) = \min\{\frac{p}{q} \mid p \textit{ is even and } (G, \sigma) \textit{ admits a } (p, q)\textit{-coloring}\}.$$

Based on the notion of (p, q)-coloring of signed graphs, defined above, we define a signed circulant graph to connect this coloring problem with the homomorphism problems.

Definition 3.2.7. For integers p and q satisfying that $p \geqslant 2q > 0$ and p is even, the *signed circular clique* $K^s_{p;q}$ has the vertex set $[p] = \{0, 1, \ldots, p-1\}$, where

- ij is a positive edge if $q \leqslant |i - j| \leqslant p - q$ and
- ij is a negative edge if either $|i - j| \leqslant \frac{p}{2} - q$ or $|i - j| \geqslant \frac{p}{2} + q$.

For $q = 1$, $K^s_{p;1}$ is also denoted by K^s_p.

Note that when $\frac{p}{q} \geqslant 4$, there are some pair of vertices are connected by a digon. In practice, we may take a circle of circumference $\frac{p}{q}$ and choose p points on this circle such that the length of the arc between any two consecutive points is $\frac{1}{q}$. In this presentation of $K^s_{p;q}$, vertices i and j are connected by a positive edge if $d_{(\text{mod } r)}(i, j) \geqslant 1$ and they are connected by a negative edge if $d_{(\text{mod } r)}(i, \bar{j}) \geqslant 1$. Equivalently, we may easily take a circle of circumference p, choose p points where consecutive points are at distance 1 and define positive and negative adjacencies based on whether $d_{(\text{mod } r)}(i, j) \geqslant q$ or $d_{(\text{mod } r)}(i, \bar{j}) \geqslant q$ (respectively). Two examples $K^s_{8;3}$ and $K^s_{4;1}$ are showed in Figures 3.1 and 3.2 respectively.

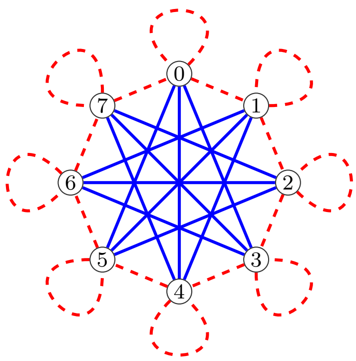


Figure 3.1. $K^s_{8;3}$

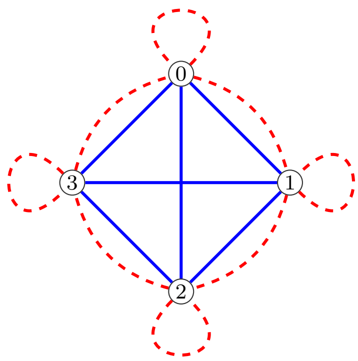


Figure 3.2. $K^s_{4;1}$

If $r = \frac{p}{q}$ is a rational and p is an even integer, then it is easily observed that $K^s_{p;q}$ is a subgraph of K^s_r. Another observation on $K^s_{p;q}$ is that the antipodal $\bar{v}$ of a vertex v (as a point of the circle) is among the vertices of $K^s_{p;q}$ because we choose p to be an even number. This is a key reason behind the fact that our definition is a refinement of the 0-free coloring of signed graphs but not other similar colorings of signed graphs. Moreover, each vertex i in $K^s_{p;q}$ is incident to itself with a negative loop. As we mentioned before, when $\frac{p}{q} \geqslant 4$, there are digons in $K^s_{p;q}$. Furthermore, the subgraph induced by all the positive edges of $K^s_{p;q}$ is exactly the circular clique $K_{p;q}$, which is known to be of circular chromatic number $\frac{p}{q}$. Combining with the easy observation that there is an obvious (p,q)-coloring of $K^s_{p;q}$, we have

$$\chi_c(K^s_{p;q}) = \frac{p}{q}. \tag{3.2}$$

Naturally, we have the next lemma, which gives another equivalent definition of the circular chromatic number of a signed graph.

Lemma 3.2.8. *Given a signed graph (G,σ) and integers $p \geqslant 2q > 0$ such that p is even, (G,σ) admits a (p,q)-coloring if and only if (G,σ) admits an edge-sign preserving homomorphism to $K^s_{p;q}$. Thus the circular chromatic number of (G,σ) is*

$$\chi_c(G,\sigma) = \min\{\frac{p}{q} \mid p \text{ is even and } (G,\sigma) \xrightarrow{s.p.} K^s_{p;q}\}.$$

As the homomorphism relation is transitive, we have the following.

Corollary 3.2.9. *For $r = \dfrac{p}{q}$, $K^s_r \xrightarrow{s.p.} K^s_{p;q}$.*

Next we state some basic properties of signed circular cliques.

Lemma 3.2.10. *Given positive integers p, p', q and q' with p, p' being even, if $\frac{p}{q} \leqslant \frac{p'}{q'}$, then $K^s_{p;q} \xrightarrow{s.p.} K^s_{p';q'}$.*

Proof. Let $r = \frac{p}{q}, r' = \frac{p'}{q'}$. Combining Lemma 3.2.4 and Corollary 3.2.9, we have that

$$K^s_{p;q} \xrightarrow{s.p.} K^s_r \xrightarrow{s.p.} K^s_{r'} \xrightarrow{s.p.} K^s_{p';q'}.$$

This completes the proof. □

The next lemma follows directly from the above.

Lemma 3.2.11. *Given positive integers p, q, and k satisfying that $p \geqslant 2q$, and assuming that p is even, we have that $K^s_{kp;kq} \xrightarrow{s.p.} K^s_{p;q}$.*

In a signed circular clique $K^s_{p;q}$, each vertex v has its antipodal $\bar{v}$, which behaves oppositely compared to itself, that means all the positive (or negative) neighbors of v are the negative (or positive, respectively) neighbors of $\bar{v}$. We can switch at half of vertices of $K^s_{p;q}$, say $\frac{p}{2}, \frac{p}{2}+1, \ldots, p-1$ and then map them to $0, 1, \ldots, \frac{p}{2}-1$ correspondingly. That's the way we obtain the next definition of $\hat{K}^s_{p;q}$.

Definition 3.2.12. Let $\hat{K}^s_{p;q}$ be the signed subgraph of $K^s_{p;q}$ induced by vertices $\{0, 1, \ldots, \frac{p}{2}-1\}$.

Then we have the following lemma, which can be viewed as another definition of the circular chromatic number of signed graphs.

Lemma 3.2.13. *Given a signed graph (G, σ) and integers $p \geqslant 2q > 0$ such that p is even, (G, σ) admits a (p, q)-coloring if and only if (G, σ) admits a switching homomorphism to $\hat{K}^s_{p;q}$. Thus the circular chromatic number of (G, σ) is*

$$\chi_c(G, \sigma) = \min\{\frac{p}{q} \mid p \text{ is even and } (G, \sigma) \to \hat{K}^s_{p;q}\}.$$

Analogously, we have the following.

Lemma 3.2.14. *Given positive integers p, p', q, and q' with p, p' both being even, if $\frac{p}{q} \leqslant \frac{p'}{q'}$, then $\hat{K}^s_{p;q} \to \hat{K}^s_{p';q'}$.*

The core of signed circular clique

One may view the class of signed circular cliques $K^s_{p;q}$ or $\hat{K}^s_{p;q}$ as a representation of rational numbers at least 2 in the homomorphism order of the class of all signed graphs. Then the circular chromatic number of a signed graph (G, σ) is determined by the first element of this chain (representing rational numbers) which is larger than (G, σ) with respect to the homomorphism order. Therefore, we need to know that what is the core of such important signed graphs.

Lemma 3.2.15. *Assume $r = \frac{p}{q}$ is a rational, p is an even positive integer and with respect to this condition, $\frac{p}{q}$ is in its simplest form. Then $\hat{K}^s_{p;q}$ is the switching core of K^s_r.*

Proof. Since $\hat{K}^s_{p;q}$ is a subgraph of K^s_r and $K^s_r \to \hat{K}^s_{p;q}$, it suffices to show that $\hat{K}^s_{p;q}$ is a switching core, i.e., it is not switching homomorphic to any of its proper signed subgraphs.

Assume to the contrary that there is a switching homomorphism of $\hat{K}^s_{p;q}$ to a proper signed subgraph of $\hat{K}^s_{p;q}$, say (H, σ). As $(H, \sigma) \to \hat{K}^s_{p;q}$ and $\hat{K}^s_{p;q} \to (H, \sigma)$, we have $\chi_c(H, \sigma) = \chi_c(\hat{K}^s_{p;q}) = \frac{p}{q}$. Let ϕ be a switching homomorphism of (H, σ) to $\hat{K}^s_{p;q}$. By Lemma 3.1.10, there is a tight cycle C with respect to ϕ. Assume $C = v_1 v_2 \cdots v_\ell$ is a cycle of length ℓ. It follows from the definition of the tight cycle that $\phi(v_{i+1}) - \phi(v_i) = q \pmod{\frac{p}{2}}$. Thus $\ell q = \frac{mp}{2}$ for some positive integer m. Since $(\frac{p}{2}, q) = 1$, we conclude that $\ell \geqslant \frac{p}{2}$. So $|V(H)| \geqslant |V(C)| = \ell \geqslant \frac{p}{2} = |V(\hat{K}^s_{p;q})|$. Noting that (H, σ) is a core, the homomorphism of $\hat{K}^s_{p;q} \to (H, \sigma)$ is surjective and onto. Hence $(H, \sigma) = \hat{K}^s_{p;q}$. □

Lemma 3.2.16. *Assume $r = \frac{p}{q}$ is a rational, p is an even positive integer and with respect to this condition $\frac{p}{q}$ is in its simplest form. Then $K^s_{p;q}$ is the edge-sign preserving core of K^s_r.*

Proof. As $K^s_r \xrightarrow{s.p.} K^s_{p;q}$, it is enough to prove that $K^s_{p;q}$ is an edge-sign preserving core. Let (H, σ) be the edge-sign preserving core of $K^s_{p;q}$ which is a proper subgraph and let φ be an edge-sign preserving homomorphism of $K^s_{p;q}$ to (H, σ). Since any edge-sign preserving homomorphism is, in particular, a switching homomorphism and by Lemma 3.2.15, $\hat{K}^s_{p;q}$ is a subgraph of (H, σ). Observe that for each vertex u of $\hat{K}^s_{p;q}$ there are two corresponding vertices u_1 and u_2 of $K^s_{p;q}$ such that a switching at u_1 gives u_2. Furthermore, there exists a positive edge $u_1 u_2$ in $K^s_{p;q}$. So $\varphi(u_1) \neq \varphi(u_2)$. Moreover, $\varphi(v_i) \neq \varphi(u_j)$, for any $i, j \in \{1, 2\}$ and for any other vertex v of $\hat{K}^s_{p;q}$, as otherwise we have an edge-sign preserving homomorphism of $\hat{K}^s_{p;q}$ to its proper subgraph by mapping u to v. It is a contradiction. □

3.3 Operations on signed graphs

In this section, we will develop graph operations that help us in the study of the circular chromatic number. Recall that the circular chromatic number of a graph G is equal to the circular chromatic number of the signed graph $(G,+)$. Are there other relations of this sort? Our goal here is to develop a technique to provide such relations. We begin with an easy observation.

Observation 3.3.1. *Given integers p,q satisfying $p \geqslant 2q \geqslant 2$, for any $i,j \in [p]$, ij is an edge of $K_{p;q}$ if and only if ij is a digon of $\hat{K}^s_{2p;q}$.*

Recall that $\tilde{G}$ is a signed graph obtained from G by replacing each edge of G with a digon. The following then is implied by Observation 3.3.1.

Corollary 3.3.2. *For any simple graph G, $\chi_c(\tilde{G}) = 2\chi_c(G)$.*

Since for any signature σ on a graph G the signed graph (G,σ) is a subgraph of $\tilde{G}$, we have the following upper bound on the circular chromatic number of (G,σ).

Corollary 3.3.3. *For any graph G and any signature σ of G, $\chi_c(G,\sigma) \leqslant 2\chi_c(G)$.*

By Corollary 3.3.2, there is a large family of signed multigraphs for which the circular chromatic number bound of this corollary is tight. In Section 4.3, we present constructions of signed graphs of arbitrarily large girth, in particular simple ones, for which the bound of this corollary is tight.

Observe that $\hat{K}^s_{2k}$ is the signed graph on k vertices where each pair of distinct vertices are adjacent by a digon and each vertex has a negative loop. It follows from the structure of these signed graphs that in any edge-sign preserving homomorphism of a signed graph (G,σ) to $\hat{K}^s_{2k}$, negative edges impose no restriction and a positive edge poses only one restriction, more precisely, vertices connected by a positive edge cannot be mapped to the same vertex. In other words, any such a mapping is a proper k-coloring of the subgraph induced by the set of positive edges of (G,σ), denoted by G^+_σ. Therefore, based on the following definition we have the next theorem. Let $\chi_+(G,\sigma) = \min\limits_{\sigma'\equiv\sigma}\{\chi(G^+_{\sigma'})\}$.

Theorem 3.3.4. *For any signed graph (G,σ), $2\chi_+(G,\sigma) - 2 < \chi_c(G,\sigma) \leqslant 2\chi_+(G,\sigma)$.*

3.3.1 Signed indicators

In Section 2.3.2, we have mentioned how we can use an indicator to build new signed graphs from a graph or a signed graph. Here we provide conditions on the indicators which help us to claim a formula. That is to say, one can determine the exact value of the circular chromatic number of the resulting signed graph from the original one.

Given signed indicators $\mathcal{I}_+$ and $\mathcal{I}_-$, recall that the signed graph $\Omega(\mathcal{I}_+,\mathcal{I}_-)$ is obtained from Ω by replacing each positive edge with a copy of the indicator $\mathcal{I}_+$ and replacing each negative edge with a copy of the indicator $\mathcal{I}_-$.

Definition 3.3.5. Assume $\mathcal{I} = (\Gamma, u, v)$ is a signed indicator and $r \geqslant 2$ is a real number. For $a,b \in [0,r)$, we say the color pair (a,b) is *feasible for $\mathcal{I}$* with respect to r if there is a circular r-coloring ϕ of Γ such that $\phi(u) = a$ and $\phi(v) = b$.

Note that if a color pair (a,b) is feasible for $\mathcal{I}$, then for any $t \in [0,r)$, $(a+t,b+t)$ and $(-a,-b)$ are also feasible for $\mathcal{I}$. Here the calculation is taken modulo r. Then as soon as we know all feasible pairs of the form $(0,b)$ for $b \in [0,\frac{r}{2}]$, we know all the feasible pairs.

Definition 3.3.6. Assume $\mathcal{I} = (\Gamma, u, v)$ is a signed indicator and $r \geqslant 2$ is a real number. Let

$$Z(\mathcal{I}, r) = \{b \in [0, \frac{r}{2}] : (0, b) \text{ is feasible for } \mathcal{I} \text{ with respect to } r \}.$$

Observe that for $\mathcal{I} = (\Gamma, u, v)$, $Z(\mathcal{I}, r) \neq \varnothing$ if and only if $\chi_c(\Gamma) \leqslant r$. One useful interpretation of $Z(\mathcal{I}, r)$ is that this is the set of possible distances in C^r between the two colors assigned to u and v in a circular r-coloring of Γ. Assume that $\mathcal{I} = (\Gamma, u, v)$ is a signed indicator.

- $\mathcal{I}$ is said to be a *plus indicator* if for each $r \geqslant \chi_c(\Gamma)$, there is a value $f(r)$ such that

$$Z(\mathcal{I}, r) = [f(r), \frac{r}{2}].$$

- $\mathcal{I}$ is said to be a *minus indicator* if for each $r \geqslant \chi_c(\Gamma)$, there is a value $g(r)$ such that

$$Z(\mathcal{I}, r) = [0, \frac{r}{2} - g(r)].$$

- $\mathcal{I}$ is said to be a *plus-minus indicator* if for each $r \geqslant \chi_c(\Gamma)$, there is a value $h(r)$ such that

$$Z(\mathcal{I}, r) = [h(r), \frac{r}{2} - h(r)].$$

Intuitively speaking, after a proper projection of the circle of circumference r such that 1 becomes $f(r)$, the role of a positive edge can be passed into a plus indicator. Similarly, one can pass the role of a negative edge into a minus indicator. To do both at the same time then we must have $f(r) = g(r)$. Note that given a plus indicator $\mathcal{I}_1 = (\Gamma_1, u_1, v_1)$ and a minus indicator $\mathcal{I}_2 = (\Gamma_2, u_2, v_2)$ satisfying that $f(r) = g(r)$, we may build a plus-minus indicator from $\mathcal{I}_1$ and $\mathcal{I}_2$ by identifying vertices u_1 and v_1 with u_2 and v_2, respectively. In such cases perhaps we will have more restrictions on the domain of the function h than the domain of f and g, but when all three are defined, we will have $f(r) = g(r) = h(r)$. Let us first mention some examples.

Example 3.3.7. Assume that $\mathcal{I} = (\Gamma, u, v)$ is a signed indicator.

(1) If Γ is a negative 2-path connecting two vertices u and v, then $\mathcal{I}$ is a plus indicator and moreover, $f(r) = 2 - \frac{r}{2}$ for $2 \leqslant r < 4$ and $f(r) = 0$ for $r \geqslant 4$, i.e.,

$$Z(\mathcal{I}, r) = [2 - \frac{r}{2}, \frac{r}{2}] \text{ for } 2 \leqslant r < 4 \text{ and } Z(\mathcal{I}, r) = [0, \frac{r}{2}] \text{ for } r \geqslant 4.$$

(2) If Γ is a positive 2-path connecting two vertices u and v, then $\mathcal{I}$ is a minus indicator and moreover, $g(r) = 2 - \frac{r}{2}$ for $2 \leqslant r < 4$ and $g(r) = 0$ for $r \geqslant 4$, i.e.,

$$Z(\mathcal{I}, r) = [0, r - 2] \text{ for } 2 \leqslant r < 4 \text{ and } Z(\mathcal{I}, r) = [0, \frac{r}{2}] \text{ for } r \geqslant 4.$$

(3) If Γ consists of a negative 2-path and a positive 2-path connecting two vertices u and v, then $\mathcal{I}$ is a plus-minus indicator and moreover, $h(r) = 2 - \frac{r}{2}$ for $\frac{8}{3} \leqslant r < 4$ and $h(r) = 0$ for $r \geqslant 4$, i.e.,

$$Z(\mathcal{I}, r) = [2 - \frac{r}{2}, r - 2] \text{ for } \frac{8}{3} \leqslant r < 4 \text{ and } Z(\mathcal{I}, r) = [0, \frac{r}{2}] \text{ for } r \geqslant 4.$$

The following is easily proved by projections between circles.

Proposition 3.3.8. *Given a plus, minus or plus-minus indicator $\mathcal{I} = (\Gamma, u, v)$, the corresponding function, f, g, or h, defined on $[\chi_c(\Gamma), +\infty)$, is a continuous non-increasing function. In particular, for r large enough, the value of each function at r will be 0.*

In the next two lemmas, we will see, for a given graph G, the relation between $\chi_c(G)$ and $\chi_c(G(\mathcal{I}))$ for the cases when $\mathcal{I}$ is either a plus indicator or a plus-minus indicator.

Lemma 3.3.9. *Assume that $\mathcal{I} = (\Gamma, u, v)$ is a plus indicator and r, r' are two real numbers satisfying that $r > \chi_c(\Gamma)$ and $r' > 2$. If $Z(\mathcal{I}, r) = [\frac{r}{r'}, \frac{r}{2}]$, then given a graph G,*

$$\chi_c(G(\mathcal{I})) = r \quad \textit{if and only if} \quad \chi_c(G) = r'.$$

Proof. Let $t = \frac{r}{r'}$. We first assume that $\chi_c(G) = r'$ and prove that $\chi_c(G(\mathcal{I})) = r$. To prove that $\chi_c(G(\mathcal{I})) = r$, we show that $\chi_c(G(\mathcal{I})) \leqslant r$ and $\chi_c(G(\mathcal{I})) \geqslant r$. To see that $\chi_c(G(\mathcal{I})) \leqslant r$, let φ be a circular r'-coloring of G. Then we define $\psi : V(G) \to [0, r)$ as $\psi(u) = t\varphi(u)$. This mapping ψ satisfies that for any edge uv of G, $d_{(\text{mod } r)}(\psi(u), \psi(v)) \geqslant t$. So $d_{(\text{mod } r)}(\psi(u), \psi(v)) \in Z(\mathcal{I}, r)$, and the mapping ψ can be extended to a circular r-coloring of the copy of Γ that was used to replace the edge uv. Therefore, this mapping ψ can be extended to a circular r-coloring of $G(\mathcal{I})$.

To prove that $\chi_c(G(\mathcal{I})) \geqslant r$, we assume to the contrary that $\chi_c(G(\mathcal{I})) = r_0 < r$ where $r_0 \geqslant \chi_c(\Gamma)$. Let ψ be a circular r_0-coloring of $G(\mathcal{I})$. As $\mathcal{I}$ is a plus indicator, there exists a t_0 such that $Z(\mathcal{I}, r_0) = [t_0, \frac{r_0}{2}]$ and thus $t_0 \leqslant d_{(\text{mod } r_0)}(\psi(u), \psi(v)) \leqslant \frac{r_0}{2}$ for each edge $uv \in E(G)$. Since $\chi_c(\Gamma) \leqslant r_0 < r$ and by Proposition 3.3.8, $t_0 \geqslant t$. We define $\varphi(u) = \frac{1}{t_0}\psi(u)$ for each vertex u of G and let $r'_0 = \frac{r_0}{t_0}$. Note that for any edge uv of G, $d_{(\text{mod } r'_0)}(\varphi(u), \varphi(v)) \geqslant 1$. Hence, φ is a circular r'_0-coloring of G. But $r'_0 = \frac{r_0}{t_0} < \frac{r}{t} = r'$, a contradiction.

We now assume that $\chi_c(G(\mathcal{I})) = r$ and prove that $\chi_c(G) = r'$. To prove that $\chi_c(G) = r'$ we show that $\chi_c(G) \leqslant r'$ and $\chi_c(G) \geqslant r'$. To see that $\chi_c(G) \leqslant r'$, let ψ be a circular r-coloring of $G(\mathcal{I})$. So $d_{(\text{mod } r)}(\psi(u), \psi(v)) \in [t, \frac{r}{2}]$. Then we define $\varphi : V(G) \to [0, r')$ as $\varphi(u) = \frac{\psi(u)}{t}$. The mapping φ satisfies that for any edge uv of G, $d_{(\text{mod } r')}(\varphi(u), \varphi(v)) \geqslant 1$. Thus φ is a circular r'-coloring of the graph G.

To prove that $\chi_c(G) \geqslant r'$, we assume to the contrary that $\chi_c(G) = r'_0 < r' = \frac{r}{t}$. Let φ be a circular r'_0-coloring of G. Thus, for any edge uv of G, $1 \leqslant d_{(\text{mod } r'_0)}(\varphi(u), \varphi(v)) \leqslant \frac{r'_0}{2}$. For real values of x satisfying $\frac{\chi_c(\Gamma)}{r'_0} \leqslant x < \frac{r}{r'_0}$, we define $f(x)$ such that $Z(\mathcal{I}, xr'_0) = [f(x), \frac{xr'_0}{2}]$. By Proposition 3.3.8, f is a continuous non-increasing function in terms of x. As $f(\frac{r}{r'_0}) = t$, and since $r > \chi_c(\Gamma)$, there is a choice of $f(x) = t_0$ such that $\chi_c(\Gamma) \leqslant t_0 r'_0 < r$ and $t_0 \geqslant t$. Let us now define $\psi(u) = t_0\varphi(u)$ for each vertex u of G. We have that $t_0 \leqslant d_{(\text{mod } t_0 r'_0)}(\psi(u), \psi(v)) \leqslant \frac{t_0 r'_0}{2}$. Thus, by the definition of $Z(\mathcal{I}, t_0 r'_0)$, for each edge uv of G, the coloring ψ can be extended to a circular $(t_0 r'_0)$-coloring of Γ. This results in a circular $(t_0 r'_0)$-coloring of $G(\mathcal{I})$. However, this is in contradiction with the fact that $t_0 r'_0 < r$. □

Lemma 3.3.10. *Assume that $\mathcal{I} = (\Gamma, u, v)$ is a plus-minus indicator and r, r' are two real numbers satisfying that $r > \chi_c(\Gamma)$ and $r' > 2$. If $Z(\mathcal{I}, r) = [\frac{r}{2r'}, \frac{r}{2} - \frac{r}{2r'}]$, then given a graph G,*

$$\chi_c(G(\mathcal{I})) = r \quad \textit{if and only if} \quad \chi_c(G) = r'.$$

Proof. Let $t = \frac{r}{2r'}$. We first assume that $\chi_c(G) = r'$ and prove that $\chi_c(G(\mathcal{I})) = r$. To see that $\chi_c(G(\mathcal{I})) \leqslant r$, let φ be a circular r'-coloring of G and thus $1 \leqslant |\varphi(u) - \varphi(v)| \leqslant r' - 1$ for each edge $uv \in E(G)$. Then we define $\psi(u) = t\varphi(u)$ for $u \in V(G)$. This mapping ψ satisfies that for each edge uv of G, $t \leqslant |\psi(u) - \psi(v)| \leqslant \frac{r}{2} - t$. As $d_{(\text{mod } r)}(\psi(u), \psi(v)) \in Z(\mathcal{I}, r)$, ψ can be extended to a circular r-coloring of each copy of Γ and furthermore, can be extended to a circular r-coloring of $G(\mathcal{I})$.

To prove that $\chi_c(G(\mathcal{I})) \geqslant r$, we assume to the contrary that $\chi_c(G(\mathcal{I})) = r_0 < r$ where $r_0 \geqslant \chi_c(\Gamma)$. Let ψ be a circular r_0-coloring of $G(\mathcal{I})$. By certain switchings, we may assume that $\psi(v) \in [0, \frac{r_0}{2})$ for each vertex v of $G(\mathcal{I})$. Since $\chi_c(\Gamma) \leqslant r_0 < r$ and $\mathcal{I}$ is a plus-minus indicator, there exists a t_0 such that $t_0 \leqslant d_{(\text{mod } r_0)}(\psi(u), \psi(v)) \leqslant \frac{r_0}{2} - t_0$ for each edge uv of G. Since $r_0 < r$, it follows from Proposition 3.3.8 that $t_0 \geqslant t$. We define $\varphi(u) = \frac{1}{t_0}\psi(u)$ for each vertex $u \in V(G)$ and let $r_0' = \frac{r_0}{2t_0}$. Note that for any edge uv of G, $1 \leqslant |\varphi(u) - \varphi(v))| \leqslant r_0' - 1$. Hence, φ is a circular r_0'-coloring of G, noting that $r_0' = \frac{r_0}{2t_0} < \frac{r}{2t} = r'$, a contradiction.

We now assume that $\chi_c(G(\mathcal{I})) = r$ and prove that $\chi_c(G) = r'$. To see that $\chi_c(G) \leqslant r'$, let ψ be a circular r-coloring of $G(\mathcal{I})$. By certain switchings, we may assume that $\psi(v) \in [0, \frac{r}{2})$ for each vertex v of $G(\mathcal{I})$. So $d_{(\text{mod } r)}(\psi(u), \psi(v)) \in [t, \frac{r}{2} - t]$. Then we define $\varphi : V(G) \to [0, r')$ as $\varphi(u) = \frac{\psi(u)}{t}$ and thus for any $uv \in E(G)$, $1 \leqslant |\varphi(u) - \varphi(v))| \leqslant r' - 1$. Such a mapping φ is a circular r'-coloring of G.

To prove that $\chi_c(G) \geqslant r'$, we assume to the contrary that $\chi_c(G) = r_0' < r' = \frac{r}{2t}$. Let φ be a circular r_0'-coloring of G. Thus, for any $uv \in E(G)$, $1 \leqslant |\varphi(u) - \varphi(v))| \leqslant r_0' - 1$. As $\mathcal{I}$ is a plus-minus indicator, for real values of x satisfying $\frac{\chi_c(\Gamma)}{2r_0'} \leqslant x < \frac{r}{2r_0'}$, we define $h(x)$ such that $Z(\mathcal{I}, 2xr_0') = [h(x), xr_0' - h(x)]$. By Proposition 3.3.8, h is a continuous non-increasing function in terms of x. As $h(\frac{r}{2r_0'}) = t = \frac{r}{2r'}$ and $r > \chi_c(\Gamma)$, there is a choice of $h(x) = t_0$ such that $t_0 \geqslant t$ and $\chi_c(\Gamma) \leqslant 2t_0r_0' < r$. We define $\psi(u) = t_0\varphi(u)$ for each vertex u of G and thus $t_0 \leqslant |\psi(u) - \psi(v))| \leqslant t_0r_0' - t_0$. By the definition of $Z(\mathcal{I}, 2t_0r_0')$, this circular $(2t_0r_0')$-coloring ψ can be extended to each of Γ. We obtain a circular $(2t_0r_0')$-coloring of $G(\mathcal{I})$, noting that $2t_0r_0' < r$, a contradiction. □

Applying a similar proof, we obtain the following useful lemma, whose proof we do not include.

Lemma 3.3.11. *Assume $\mathcal{I}_+ = (\Gamma_+, u_1, v_1)$ is a plus indicator and $\mathcal{I}_- = (\Gamma_-, u_2, v_2)$ is a minus indicator. Assume that $r > \max\{\chi_c(\Gamma_+), \chi_c(\Gamma_-)\}$ and $r' > 2$ are two real numbers and let $t = \frac{r}{r'}$. If*

$$Z(\mathcal{I}_+, r) = [t, \frac{r}{2}] \text{ and } Z(\mathcal{I}_-, r) = [0, \frac{r}{2} - t],$$

then given a signed graph Ω,

$$\chi_c(\Omega(\mathcal{I}_+, \mathcal{I}_-)) = r \text{ if and only if } \chi_c(\Omega) = r'.$$

Remark. Note that in Lemmas 3.3.9, 3.3.10 and 3.3.11, the condition of "$r > \chi_c(\Gamma)$" or "$r > \max\{\chi_c(\Gamma_+), \chi_c(\Gamma_-)\}$" is crucial. We provide two indicator examples to show that if $r = \chi_c(\Gamma)$, then Lemmas 3.3.9 and 3.3.10 may fail. Similar examples can be provided for Lemma 3.3.11. Let $\mathcal{I}_1 = (\hat{G}_1, u, v)$ and $\mathcal{I}_2 = (\hat{G}_2, u, v)$ be two indicators depicted in Figures 3.3 and 3.4, respectively. It can be observed that $\chi_c(\hat{G}_1) = \chi_c(\hat{G}_2) = \frac{8}{3}$. Note that $Z(\mathcal{I}_1, r) = [5 - \frac{3r}{2}, \frac{r}{2}]$ for $\frac{8}{3} \leqslant r < \frac{10}{3}$ and $Z(\mathcal{I}_1, r) = [0, \frac{r}{2}]$ for $r \geqslant \frac{10}{3}$, and also note that $Z(\mathcal{I}_2, r) = [3 - r, \frac{3r}{2} - 3]$ for $\frac{8}{3} \leqslant r < 3$ and $Z(\mathcal{I}_2, r) = [0, \frac{r}{2}]$ for $r \geqslant 3$. Therefore, $\mathcal{I}_1$ is a plus indicator and $\mathcal{I}_2$ is a plus-minus indicator. In particular, when $r = \frac{8}{3}$, we have that $Z(\mathcal{I}_1, \frac{8}{3}) = [1, \frac{4}{3}]$ and $Z(\mathcal{I}_2, \frac{8}{3}) = [\frac{1}{3}, 1]$. For $\mathcal{I}_1$, we consider $C_5(\mathcal{I}_1)$. It is not hard to produce a circular $\frac{8}{3}$-coloring of $C_5(\mathcal{I}_1)$, which implies that $\chi_c(C_5(\mathcal{I}_1)) = \frac{8}{3}$. Noting that $\chi_c(C_5) = \frac{5}{2}$, we observe that Lemma 3.3.9 does not apply to this pair with $r = \frac{8}{3}$. For $\mathcal{I}_2$, we consider $K_3(\mathcal{I}_2)$. Similarly, $\chi_c(K_3(\mathcal{I}_2)) = \frac{8}{3}$ and $\chi_c(K_3) = 3$. Thus Lemma 3.3.10 with $r = \frac{8}{3}$ does not apply to this pair.

However, in our applications, for a plus indicator $\mathcal{I} = (\Gamma, u, v)$, if we have $Z(\mathcal{I}, \chi_c(\Gamma)) = \{\frac{\chi_c(\Gamma)}{2}\}$, then the conclusion of Lemma 3.3.9 still holds. Similarly, Lemma 3.3.10 also holds for a plus-minus indicator $\mathcal{I} = (\Gamma, u, v)$ satisfying that $Z(\mathcal{I}, \chi_c(\Gamma)) = \{\frac{\chi_c(\Gamma)}{4}\}$. For a plus indicator $\mathcal{I}_+ = (\Gamma_1, u_1, v_1)$

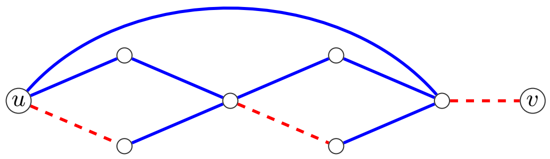


Figure 3.3. $\mathcal{I}_1 = (\hat{G}_1, u, v)$

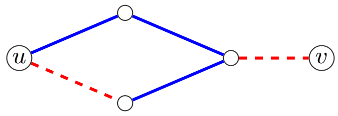


Figure 3.4. $\mathcal{I}_2 = (\hat{G}_2, u, v)$

and a minus indicator $\mathcal{I}_- = (\Gamma_2, u_2, v_2)$ satisfying the conditions of Lemma 3.3.11 and moreover, $Z(\mathcal{I}_+, \chi_c(\Gamma_1)) = \{\frac{\chi_c(\Gamma_1)}{2}\}$ and $Z(\mathcal{I}_-, \chi_c(\Gamma_2)) = \{0\}$, Lemma 3.3.11 still holds.

One basic example of a plus-minus indicator is the digon D with two endpoints u and v. It has been proved in Proposition 3.1.7 that $\chi_c(D) = 4$. Moreover, given $r \geqslant 4$, if ϕ is a circular r-coloring of D where $\phi(u) = 0$, then simply by the definition, we know that $\phi(v) \in [1, \frac{r}{2} - 1]$. Hence, $\mathcal{I}_D = (D, u, v)$ is a plus-minus indicator and note that $Z(\mathcal{I}_D, 4) = \{1\}$. Thus, by Lemma 3.3.10, for any given graph G, $\chi_c(G(\mathcal{I}_D)) = 2\chi_c(G)$. This is a restatement of Corollary 3.3.2. In particular, we have $\chi_c(K_4(\mathcal{I}_D)) = 8$. Noting that this is a signed planar multigraph and that, by the 4-color theorem, every signed planar multigraph without a loop admits an edge-sign preserving homomorphism to it.

Another important example of a plus-minus indicator is C_{-4} with u, v being two non-adjacent vertices (Example 3.3.7 (3)). Observe that for $\mathcal{I} = (C_{-4}, u, v)$, $G(\mathcal{I})$ is defined as $S(G)$ in Section 2.3.4. Applying Lemma 3.3.10 and noting that $Z(\mathcal{I}, \frac{8}{3}) = \{\frac{2}{3}\}$, we have:

Theorem 3.3.12. *Given a graph G, we have*

$$\chi_c(S(G)) = 4 - \frac{4}{\chi_c(G) + 1}.$$

In particular, we have that

(1) $\chi_c(G) \leqslant 4$ if and only if $\chi_c(S(G)) \leqslant \frac{16}{5}$,

(2) $\chi_c(G) \leqslant 3$ if and only if $\chi_c(S(G)) \leqslant 3$.

Proof. By Example 3.3.7 (3), given real number x with $2 \leqslant x < 4$, $Z(\mathcal{I}, x) = [2 - \frac{x}{2}, x - 2]$. Assume that $\chi_c(G) = r'$ with $r' \geqslant 2$. Then for a real number r, $2 \leqslant r < 4$, by Lemma 3.3.10, if $\frac{r}{2r'} = 2 - \frac{r}{2}$, then $\chi_c(S(G)) = r$. This condition translates to $r = 4 - \frac{4}{r'+1}$. □

It has been shown in Theorem 2.3.10 that by using $S(G)$ construction and the graph homomorphism, the chromatic number of graphs can be captured by switching homomorphisms of signed bipartite graphs. Theorem 3.3.12 shows, furthermore, that $\chi_c(S(G))$ also determines $\chi_c(G)$.

Recall that we denote by $\mathcal{SPB}_2$ the class of signed bipartite planar simple graphs in which one part vertex set has maximum degree of at most 2. A direct corollary of Theorem 3.3.12 is the following reformulation of the 4-color theorem.

Theorem 3.3.13. [4-color theorem restated] *Every signed graph in $\mathcal{SPB}_2$ admits a circular $\frac{16}{5}$-coloring.*

Note that $\mathcal{SPB}_2$ is a subclass of the class of signed bipartite planar 2-degenerate graphs. As proving this theorem directly is as hard as showing the 4-color theorem, this then naturally leads to two questions, each based on dropping one of the conditions.

Question 3.3.14. *What is the best upper bound on the circular chromatic number of signed 2-degenerate simple graphs?*

Question 3.3.15. *What is the best upper bound on the circular chromatic number of signed bipartite planar simple graphs?*

We will answer the first question in Theorem 4.1.1 in Section 4.1.1 and answer the second question in Chapter 7. Moreover, for both problems, in terms of the number of vertices, we provide a precise upper bound and we show that each of our bound is tight.

3.3.2 Subdivisions

A classic relation between the chromatic number of a graph and homomorphism from a certain subdivision of it to the odd cycle is extended to a relation between the circular chromatic number of signed graphs and homomorphism of its subdivision to negative cycles in section 2.3.3. Here we present a slightly stronger version and then figure out the circular chromatic number of signed graphs built using this operation.

Definition 3.3.16. Given a signed graph (G, σ) and a positive integer ℓ, we define $T^*_\ell(G, \sigma)$ to be the signed graph obtained from (G, σ) by replacing each edge e with a path P_ℓ of length ℓ where internal vertices of different paths are distinct and assigning a signature satisfying that P_ℓ contains an odd number of positive edges if e is a positive edge and P_ℓ contains an even number of positive edges if e is a negative edge.

We note that there are many choices for the signature in defining $T^*_\ell(G, \sigma)$, but, as all such choices are switching equivalent, one may take any. The relation between the circular chromatic number of (G, σ) and $T^*_\ell(G, \sigma)$ follows from two following lemmas.

We denote a path of length ℓ which contains an odd number of positive edges by P^o_ℓ and a path of length ℓ which contains an even number of positive edges by P^e_ℓ. Here we consider P^o_ℓ and P^e_ℓ to be the two signed indicators.

Lemma 3.3.17. [PZ22] *Given an integer $\ell \geqslant 1$ and a real number $r < \frac{2\ell}{\ell-1}$,*

$$Z(P^o_\ell, r) = [\ell - (\ell-1)\frac{r}{2}, \frac{r}{2}] \quad \textit{and} \quad Z(P^e_\ell, r) = [0, \ell\frac{r}{2} - \ell].$$

It follows from the lemma that P^o_ℓ is a plus indicator and P^e_ℓ is a minus indicator. Combining Lemmas 3.3.11 and 3.3.17, where we take $\Gamma_+ = P^o_\ell, \Gamma_- = P^e_\ell$ and $t = \ell - (\ell-1)\frac{r}{2}$, we have the following.

Lemma 3.3.18. *For any signed graph Ω,*

$$\chi_c(T^*_\ell(\Omega)) = \frac{2\ell\chi_c(\Omega)}{(\ell-1)\chi_c(\Omega) + 2}.$$

In a particular case of $\ell = 2$, we have that

$$\chi_c(T^*_2(G, \sigma)) = \frac{4\chi_c(G, \sigma)}{2 + \chi_c(G, \sigma)}.$$

Note that for each positive integer ℓ, and by considering a graph G as a signed graph where all edges are positive, the 4-colorability problem of G is equivalent to proving that $\chi_c(T^*_\ell(G)) \leqslant \frac{8\ell}{4\ell-2}$. For each choice of ℓ, we have a reformulation of the 4-color theorem: Every planar graph G satisfies that $\chi_c(T^*_\ell(G)) \leqslant \frac{8\ell}{4\ell-2}$. For each such ℓ, then one line of study is to introduce an interesting class of signed graphs that includes $T^*_\ell(G)$ for all planar graphs G and admits the same upper bound for the circular chromatic number. When ℓ is an even value, $T^*_\ell(\Omega)$ is a signed bipartite graph for any signed graph Ω. The last note is that the $S(G)$ construction is also a special case of this indicator construction. Given a graph G, one may first build the signed graph $\tilde{G}$ by replacing each edge with a digon, then naturally $T^*_2(\tilde{G})$ is the same as $S(G)$.

3.4 Signed bipartite circular clique

In Theorems 2.3.9, 2.3.10 and 3.3.12, we have seen the importance of the restriction of the study into the class $\mathcal{SB}$ of signed bipartite graphs. In this section, we will focus on the class $\mathcal{SB}$. The very first observation is that every signed bipartite graph admits a homomorphism to a digon. Combining this fact together with Proposition 3.1.7, we have $\chi_c(\mathcal{SB}) = 4$.

As the signed circular clique plays an important role in the homomorphism order of general signed graphs, a natural question to ask is if the homomorphism order restricted to the subclass of signed bipartite graphs behaves similarly? More precisely, we would like to know if there is a chain of signed bipartite graphs in the homomorphism order on $\mathcal{SB}$ which plays the role of the circular clique. We note that no signed circular clique $\hat{K}^s_{p;q}$ or $K^s_{p;q}$ is bipartite. Indeed each vertex in any of these cliques has a negative loop on it. In this section, for $2 \leqslant \frac{p}{q} \leqslant 4$, we introduce a bipartite subgraph of these circular cliques that plays the role of circular clique in the restricted class $\mathcal{SB}$.

Definition 3.4.1. Given a rational number $\frac{p}{q}$ where p is an even number, $2 \leqslant \frac{p}{q} \leqslant 4$ and subject to these conditions $\frac{p}{q}$ is in its simplest form, we define the signed graph $B_{p;q}$ to be the following subgraph of $K^s_{p;q}$:

- The vertex set $[p] = \{0, 1, \dots, p-1\}$ is partitioned to two parts X and Y where $X = \{0, 2, \dots, p-2\}$ and $Y = \{1, 3, \dots, p-1\}$.
- The edge set is formed by the edges of $K^s_{p;q}$ which have exactly one endpoint in X and another endpoint in Y.
- The signs of edges are also induced by $K^s_{p;q}$.

Next we show that $B_{p;q}$, which itself is a signed bipartite graph, plays the role of the circular clique in the subclass of signed bipartite graphs. However, such signed graphs $B_{p;q}$ are partitioned into two classes depending on whether p is a multiple of 4 or it is 2 (mod 4).

When p is a multiple of 4, then we will show that $B_{p;q}$ is a circular clique with respect to the edge-sign preserving homomorphism. It means that any signed bipartite graph of circular chromatic number at most $\frac{p}{q}$ admits an edge-sign preserving homomorphism to $B_{p;q}$. In this case, as we will show, the subgraph induced on the vertices $[\frac{p}{2}] = \{0, 1, \dots, \frac{p}{2} - 1\}$ forms the switching core of $B_{p;q}$ and will play the role of a signed bipartite clique with respect to the switching homomorphism. For example, $B_{16;5}$ is depicted in Figure 3.5 and its switching core, which is a signed graph on $K_{4,4}$, is depicted in Figure 3.12.

When $p \equiv 2 \pmod 4$, and again noting our assumption that $\frac{p}{q}$ is in its simplest form subject to p being even, the signed graph $B_{p;q}$ is already a core with respect to the switching homomorphism. For example, $B_{10;3}$ is depicted in Figure 3.6. In this case, to have a signed circular clique with respect to the edge-sign preserving homomorphism, we must consider DSG$(B_{p;q})$. To be more precise, when $p \equiv 2 \pmod 4$, for a signed bipartite graph (G, σ) to satisfy that $\chi_c(G, \sigma) \leqslant \frac{q}{q}$, it is necessary and sufficient that (G, σ) admits a switching homomorphism to $B_{p;q}$. However, for some choices of σ, a switching might be necessary. To be sure to have an edge-sign preserving homomorphism then we must consider DSG$(B_{p;q})$. For the example of $p = 6$ and $q = 2$, which

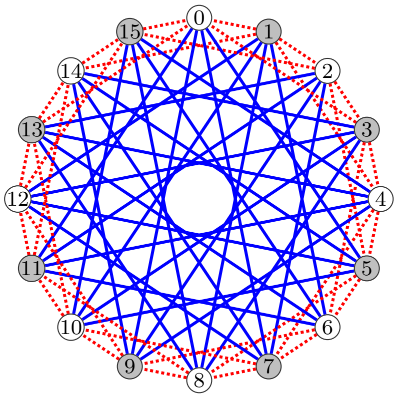


Figure 3.5. $B_{16;5}$

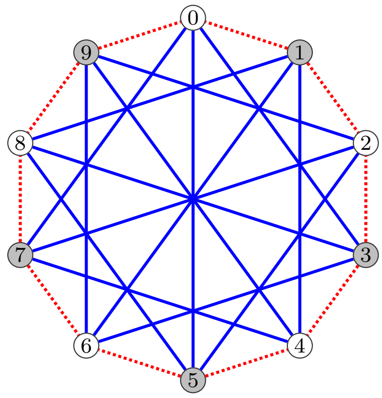


Figure 3.6. $B_{10;3}$

corresponds to circular chromatic number at most 3, see Figures 3.7, 3.8, 3.9 and 3.10. The first one, the signed graph of Figure 3.7 on three vertices, is the signed circular 3-clique with respect to the switching homomorphism. The second one, the signed graph of Figure 3.8, which is the Double Switch Graph of the first one, is the signed circular 3-clique with respect to the edge-sign preserving homomorphism. The third one, the signed graph of Figure 3.9, also on 6 vertices, is the signed bipartite circular 3-clique with respect to the switching homomorphism. Finally, the last one, the signed graph of Figure 3.10, on 12 vertices, is the Double Switch Graph of the previous one and is the signed bipartite circular 3-clique with respect to the edge-sign preserving homomorphism.

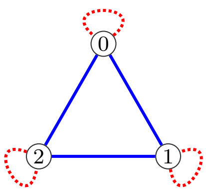


Fig 3.7. $\hat{K}^s_{6;2}$

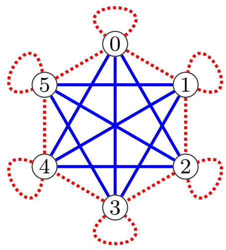


Fig 3.8. $K^s_{6;2}$

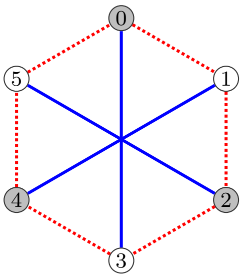


Fig 3.9. $B_{6;2}$

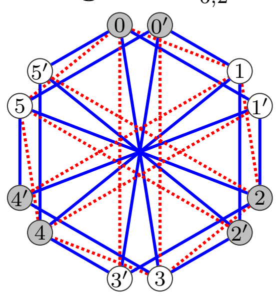


Fig 3.10. $\mathrm{DSG}(B_{6;2})$

To distinguish which of the two notions of homomorphisms we are working with and be consistent with the notation we used for the signed circular clique, we define $B^s_{p;q}$ and $\hat{B}^s_{p;q}$ as follows. Given a positive even integer p and a positive integer q such that subject to p being even, $\frac{p}{q}$ is in its simplest form and $\frac{p}{q} \geqslant 2$, we define $B^s_{p;q}$ to be $B_{p;q}$ when $4 \mid p$ and to be $\mathrm{DSG}(B_{p;q})$ when $4 \nmid p$. As mentioned before, these signed graphs play the role of signed bipartite circular cliques with respect to the edge-sign preserving homomorphism. For the switching homomorphism, we define $\hat{B}^s_{p;q}$ to be

$B_{p;q}$ when $4 \nmid p$ and to be the subgraph of $B_{p;q}$ induced on the vertices $\{0, \ldots, \frac{p}{2} - 1\}$ when $4 \mid p$.

We should also note that in defining $K^s_{p;q}$ and $\hat{K}^s_{p;q}$, we did not need to assume $\frac{p}{q}$ is in the simplest form. However, we note that $K^s_{ap;aq}$ and $\hat{K}^s_{ap;aq}$ map, respectively, to $K^s_{p;q}$ and $\hat{K}^s_{p;q}$. By taking such a homomorphism and then taking the pre-image of $B^s_{p;q}$ and $\hat{B}^s_{p;q}$, one may define $B^s_{ap;aq}$ and $\hat{B}^s_{ap;aq}$.

That $B^s_{p;q}$ and $\hat{B}^s_{p;q}$ play the role of circular cliques in the subclass of signed bipartite graphs is the subject of the next theorem. For simplicity, it is stated using $B_{p;q}$ and switching homomorphism but one can easily restate it by using $B^s_{p;q}$ or $\hat{B}^s_{p;q}$ and the associated notion of homomorphism.

Theorem 3.4.2. *Given a signed bipartite graph (G, σ) and a rational number $\frac{p}{q}$ in $[2, 4]$ where p is a positive even integer and subject to this, $\frac{p}{q}$ is in its simplest form, we have $\chi_c(G, \sigma) \leqslant \frac{p}{q}$ if and only if $(G, \sigma) \to B_{p;q}$.*

Proof. Let (G, σ) be a signed bipartite graph. One direction is quite trivial. As $B_{p;q}$ is a subgraph of $K^s_{p;q}$, $\chi_c(B_{p;q}) \leqslant \frac{p}{q}$. If $(G, \sigma) \to B_{p;q}$, then, by Lemma 3.1.5, we have $\chi_c(G, \sigma) \leqslant \frac{p}{q}$.

It remains to show that if $\chi_c(G, \sigma) \leqslant \frac{p}{q}$, then $(G, \sigma) \to B_{p;q}$. Since $B_{p;q}$ behaves differently depending on whether p divides 4 or not, we divide the proof into two cases based on this criteria: (1) $p = 4k$. (2) $p = 4k + 2$. We note that in the first case, q must be an odd number.

Case 1 $p = 4k$.

As $\frac{p}{q} \geqslant 2$, we know q is an odd number smaller or equal to $2k - 1$. Let (X, Y) be the bipartition of $B_{4k;q}$ and let (A, B) be the bipartition of (G, σ). Since $\chi_c(G, \sigma) \leqslant \frac{4k}{q}$, there is an edge-sign preserving homomorphism of (G, σ) to $K^s_{4k;q}$. Let φ be such a homomorphism. Our goal is to modify φ, if needed, so that we obtain a mapping of (G, σ) to $B_{4k;q}$. This would of course be based on the bipartition of G. One such modification is given as follows:

$$\phi(u) = \begin{cases} \varphi(u) + 1 & \text{either } u \in A \text{ and } \varphi(u) \in Y \text{ or } u \in B \text{ and } \varphi(u) \in X, \\ \varphi(u) & \text{otherwise.} \end{cases}$$

Intuitively, we aim at modifying the mapping such that the vertices in the part A of G are mapped to the vertices in the part X of $B_{4k;q}$ and the vertices in the part B are mapped to the vertices in the part Y. In defining ϕ, for vertices of G satisfying these conditions under the mapping φ, we give a same image under ϕ. If this condition is not met, then we shift the image by 1 in the clockwise direction of the circle. What remains is to show that ϕ is also an edge-sign preserving homomorphism of (G, σ) to $K^s_{4k;q}$. Then it would naturally be a homomorphism of (G, σ) to $B_{4k;q}$ as well.

Given an edge $e = uv$ of G, if both $\phi(u) = \varphi(u)$ and $\phi(v) = \varphi(v)$ hold, then e is already mapped to an edge of a same sign under φ and nothing left to show. If $\phi(u) = \varphi(u) + 1$ and $\phi(v) = \varphi(v) + 1$, then the claim follows from the circular structure of $K^s_{p;q}$, that is, if there is an edge ij of sign η in $B_{4k;q}$, then there is also an $(i + 1)(j + 1)$ (additions done modulo $4k$) edge of sign η. It remains to consider the case that only one endpoint of $e = uv$ has been shifted. By the symmetry, we may assume $\phi(u) = \varphi(u)$ and $\phi(v) = \varphi(v) + 1$. Moreover, noting that u and v must be in different parts of the bipartite graph G, and again by the symmetries, we assume $u \in A$ and $v \in B$ with $\varphi(u), \varphi(v) \in X$. Hence, by our assumption, $\phi(u) = \varphi(u)$ and $\phi(v) = \varphi(v) + 1 \in Y$. Depending on the signature of e, we consider two cases.

If e is a positive edge, then $\varphi(u)\varphi(v)$ is a positive edge of $K^s_{4k;q}$. Thus $q \leqslant |\varphi(u) - \varphi(v)| \leqslant p - q = 4k - q$. Observe that, as $\varphi(u)$ and $\varphi(v)$ are both in X, they have a same parity, and thus $|\varphi(u) - \varphi(v)|$ is an even number. However, since q is an odd number, both sides of the inequality (i.e., q and $4k-q$) are odd numbers and, therefore, equality cannot hold there. It is implied that if we change (only) one of $\varphi(u)$ and $\varphi(v)$ by a value of at most 1, then the inequality would still hold. Thus $\phi(u)\phi(v)$ is a positive edge of $K^s_{4k;q}$.

If e is a negative edges, then (only) one of the following must hold: either $|\varphi(u)-\varphi(v)| \leqslant \frac{p}{2}-q = 2k-q$ or $|\varphi(u)-\varphi(v)| \geqslant \frac{p}{2}+q = 2k+q$. As in the previous case, we conclude that $|\varphi(u)-\varphi(v)|$ is an even number. However, $\frac{p}{2}=2k$ is an even number while q must be an odd number. Thus both of $2k-q$ and $2k+q$ are odd numbers and once again the equality cannot hold. Therefore, after shifting only one of the values of $\varphi(u), \varphi(v)$ by 1, the corresponding inequality holds with respect to the new function which is ϕ, that is to say, either $|\phi(u)-\phi(v)| \leqslant 2k-q$ or $|\phi(u)-\phi(v)| \geqslant 2k+q$. Hence e is mapped to a negative edge $\phi(u)\phi(v)$ of $K^s_{4k;q}$.

Case 2 $p = 4k+2$.

Notice that in this case, $\frac{p}{2} = 2k+1$ is an odd number. Let (X,Y) be the bipartition of $B_{4k+2;q}$ and let (A,B) be a bipartition of (G,σ).

Since $\chi_c(G,\sigma) \leqslant \frac{4k+2}{q}$, there exists an edge-sign preserving homomorphism of (G,σ) to $K^s_{4k+2;q}$, say φ. Our goal is to modify φ to obtain a (switching) homomorphism of (G,σ) to $B_{4k+2;q}$. This would be based on the bipartition of G. Intuitively, we want a mapping that maps vertices in A to X and those in B to Y. We observe that for each pair of antipodal vertices of $K^s_{4k+2;q}$, one is in X and the other is in Y. Thus in the mapping φ, if one vertex is not mapped to the correct part, then we first apply a switching at that vertex and then map it to the antipodal of the original image. This is formalized as follows.

$$\phi(u) = \begin{cases} \overline{\varphi(u)} \text{ (switching at } u) & \text{either } u \in A \text{ and } \varphi(u) \in Y \text{ or } u \in B \text{ and } \varphi(u) \in X, \\ \varphi(u) & \text{otherwise.} \end{cases}$$

What remains is to show that ϕ is a switching homomorphism of (G,σ) to $K^s_{4k+2;q}$. Then it would naturally be a homomorphism of (G,σ) to $B_{4k+2;q}$ as well.

Given an edge $e = uv$ of G, if both $\phi(u) = \varphi(u)$ and $\phi(v) = \varphi(v)$ hold, then it follows easily that $\phi(u)\phi(v)$ is the required edge. If $\phi(u) = \overline{\varphi(u)}$ and $\phi(v) = \overline{\varphi(u)}$, then we switch at both of vertices u and v. Thus the sign of uv does not change. Moreover, vertices i and j are connected by an edge of sign η in $K^s_{4k+2;q}$, then their antipodals are also connected by an edge of the same sign. Therefore, $\phi(u)\phi(v)$ is an edge of $K^s_{4k+2;q}$ with the same sign as $\varphi(u)\varphi(v)$ and thus as uv. The final case is that only one endpoint of $e = uv$ has been switched and mapped to the antipodal. By the symmetries, we may assume that we switch at v and $\phi(v) = \overline{\varphi(v)}$. Moreover, noting that u and v must be in different parts of the bipartite graph G, and again by the symmetries, we assume $u \in A$ and $v \in B$ with $\varphi(u), \varphi(v) \in X$. Hence, by our assumption, $\phi(u) = \varphi(u)$ and $\phi(v) = \overline{\varphi(v)} \in Y$. Depending on the sign of e, we consider two cases.

If e is a positive edge, then $\varphi(u)\varphi(v)$ is a positive edge of $K^s_{4k+2;q}$. Thus $q \leqslant |\varphi(u)-\varphi(v)| \leqslant p-q = 4k+2-q$. As $|\varphi(v)-\overline{\varphi(v)}| = \frac{p}{2} = 2k+1$, we have $|\varphi(u)-\overline{\varphi(v)}| \leqslant 2k+1-q$ or $|\varphi(u)-\overline{\varphi(v)}| \geqslant 2k+1+q$. Note that now uv is a negative edge of (G,σ') where σ' is obtained from σ by switching at v. Since $\varphi(u)\overline{\varphi(v)}$ satisfies the condition for being a negative edge of $K^s_{4k+2;q}$, $\phi(u)\phi(v)$ is a negative edge that we required.

If e is a negative edges, then (only) one of the following must hold: either $|\varphi(u)-\varphi(v)| \leqslant \frac{p}{2}-q = 2k+1-q$ or $|\varphi(u)-\varphi(v)| \geqslant \frac{p}{2}+q = 2k+1+q$. As in the previous case, we have that $|\varphi(v)-\overline{\varphi(v)}| = \frac{p}{2} = 2k+1$. Thus $q \leqslant |\varphi(u)-\overline{\varphi(v)}| \leqslant 4k+2-q$. Switching at v makes uv become a positive edge. Now $\varphi(u)\overline{\varphi(v)}$ satisfies the condition for being a positive edge of $K^s_{4k+2;q}$, in other words, $\phi(u)\phi(v)$ is a positive edge. Therefore, we verify that ϕ is a (switching) homomorphism of (G,σ) to $K^s_{4k+2;q}$ and thus also to its signed bipartite subgraph $B_{4k+2;q}$. □

Hence, we show that for any rational number $2 \leqslant \frac{p}{q} \leqslant 4$, there exists a signed bipartite circular clique whose circular chromatic number is exactly $\frac{p}{q}$.

We note that the assumption $\frac{p}{q} \leqslant 4$ is not used explicitly in the proof. If $\frac{p}{q} > 4$, then the signed bipartite graph induced by odd versus even vertices will contain a digon that admits a homomorphism from any signed bipartite graph and provides the upper bound of 4 for the circular chromatic number of this class of signed graphs. That leaves us with circular 3-coloring as a special case. In this case, by switching at all vertices of one part of $B_{6;2}$, we get a signed graph $(K_{3,3}, M)$. Hence, as a special case we have:

Corollary 3.4.3. *Given a signed bipartite graph (G, σ), $\chi_c(G, \sigma) \leqslant 3$ if and only if $(G, \sigma) \to (K_{3,3}, M)$. In particular, we have that $\chi_c(K_{3,3}, M) = 3$.*

Another special case is when $p = 4k$ and $q = 2k - 1$. In this case, one may observe that the (switching) core of $B_{4k;2k-1}$ (on $2k$ vertices) is switching equivalent to the negative cycle C_{-2k}. Hence, we have the following corollary.

Corollary 3.4.4. *Given a signed bipartite graph (G, σ), $\chi_c(G, \sigma) \leqslant \frac{4k}{2k-1}$ if and only if $(G, \sigma) \to C_{-2k}$. In particular, we have that $\chi_c(C_{-2k}) = \frac{4k}{2k-1}$.*

This helps us to fill the parity gap in some studies of the circular coloring of graphs where homomorphism of a graph to odd cycle C_{2k+1} is known to be equivalent to a circular $\frac{4k+2}{2k}$-coloring. A uniform presentation of the two is as follows.

Theorem 3.4.5. *Given a positive integer ℓ, $\ell \geqslant 2$, and a signed graph (G, σ) satisfying that $g_{ij}(G, \sigma) \geqslant g_{ij}(C_{-\ell})$ for each $ij \in \mathbb{Z}_2^2$, we have $\chi_c(G, -\sigma) \leqslant \frac{2\ell}{\ell-1}$ if and only if $(G, \sigma) \to C_{-\ell}$.*

New perspectives of the 4-color theorem

Recall a particular case of Theorem 3.3.12 that for any planar graph G, we have $\chi_c(S(G)) \leqslant \frac{16}{5}$. Since every $S(G)$ is a signed bipartite graph, the claim of this theorem is equivalent to the existence of an edge-sign preserving mapping from $S(G)$ to $B^s_{16;5}$. Note that $\hat{B}^s_{16;5}$, the switching core of $B^s_{16;5}$, is a signed graph on $K_{4,4}$. With one choice of a signature, among all equivalent signatures, this core is presented in Figure 3.12. We recall that the 4-color theorem is also restated in Theorem 2.3.12 in the form of mapping $S(G)$, for every planar G, to $(K_{4,4}, M)$. To highlight the difference between the two reformulations, we note that $\chi_c(K_{4,4}, M) = 4$. To see this, we prove a stronger statement. Let $(K_{3,4}, M)$ be the signed graph obtained from $(K_{4,4}, M)$ by deleting one vertex.

Proposition 3.4.6. $\chi_c(K_{3,4}, M) = 4$.

Proof. Since $(K_{3,4}, M)$ is bipartite, it admits a homomorphism to the digon, thus $\chi_c(K_{3,4}, M) \leqslant 4$. As $(K_{3,3}, M) \subset (K_{3,4}, M)$, $\chi_c(K_{3,4}, M) \geqslant 3$. Following from Proposition 3.1.11 and noting that there is no odd cycle in $(K_{3,4}, M)$, we conclude that $\chi_c(K_{3,4}, M) \in \{3, 4\}$. It suffices to prove that $(K_{3,4}, M)$ is not circular 3-colorable. The signed graph $(K_{3,4}, M)$ consists of $(K_{3,3}, M)$ and a vertex w connecting to each vertex of one part of $(K_{3,3}, M)$ with a positive edge. Since the subgraph $(K_{3,3}, M)$ is switching equivalent to $B_{6;2}$, the vertex mapping of $(K_{3,3}, M)$ to $B_{6;2}$ is surjective and unique up to switching. The vertex w has nowhere to map, a contradiction. □

Corollary 3.4.7. *We have $\chi_c(K_{4,4}, M) = 4$. Furthermore, $\chi_c(K_{k,k}, M) = 4$ for $k \geqslant 4$.*

Let $\mathcal{P}$ be the class of all simple planar graphs and let $S(\mathcal{P}) = \{S(G) \mid G \in \mathcal{P}\}$. Then, by the discussions in Section 2.3.4 and also above, the 4-color theorem is equivalent to bounding the class $S(\mathcal{P})$ by one of the following signed bipartite graphs: $S(K_4)$ depicted in Figures 2.1 (also in Figure 3.11 for a better comparison), $\hat{B}^s_{16;5}$ depicted in 3.12 (the switching core of Figure 3.5), and $(K_{4,4}, M)$ depicted in 3.13. One may observe that $S(K_4)$ admits a homomorphism to both $(K_{4,4}, M)$ and $\hat{B}^s_{16;5}$ but that $(K_{4,4}, M)$ and $\hat{B}^s_{16;5}$ are homomorphically incomparable. The latter is a consequence of the following two facts:

- Any pair of nonadjacent vertices in $(K_{4,4}, M)$ or in $\hat{B}^s_{16;5}$ belongs to a negative 4-cycle which means identifying them would result in a digon;
- The two signed graphs are not switching isomorphic, since $\chi_c(K_{4,4}, M) = 4$ but $\chi_c(\hat{B}^s_{16;5}) = \frac{16}{5}$.

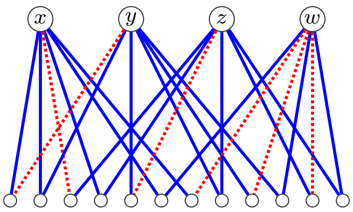


Figure 3.11. $S(K_4)$

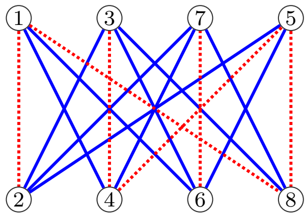


Figure 3.12. $\hat{B}^s_{16;5}$

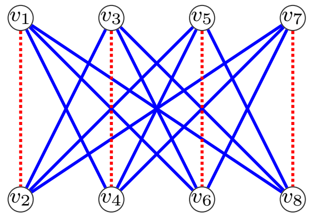


Figure 3.13. $(K_{4,4}, M)$

As $S(K_4)$ itself is in the family $S(\mathcal{P})$, one does not expect much room to strengthen the result regarding this one beyond expected strengthening of the 4-color theorem. For example, the statement holds when G is K_5-minor-free, and is expected to hold if $(G, -)$ has no $(K_5, -)$-minor.

Since $S(K_4)$ admits a homomorphism to each of $(K_{4,4}, M)$ and $\hat{B}^s_{16;5}$, it would not be a surprise if a stronger statement can be proved regarding these two targets. Indeed that is the case for $(K_{4,4}, M)$: it bounds the class of all signed bipartite planar simple graphs (see Theorem 2.3.13). As the limit of the circular chromatic number of signed bipartite planar simple graphs is 4 (see in Chapter 7), this cannot be the case for $\hat{B}^s_{16;5}$. Thus it remains an open question to bound a larger class of signed bipartite planar graphs with $\hat{B}^s_{16;5}$.

4 | Circular chromatic numbers of classes of signed graphs

This chapter is based on the following papers:

[NWZ21] R. Naserasr, Z. Wang, and X. Zhu. "Circular chromatic number of signed graphs". In: *Electron. J. Combin.* 28.2 (2021), Paper No. 2.44, 40. DOI: 10.37236/9938

[KNNW22] F. Kardoš, J. Narboni, R. Naserasr, and Z. Wang. "Circular $(4-\epsilon)$-coloring of some classes of signed graphs". In: *SIAM J. Discrete Math. Accepted subject to revision. arXiv:2107.12126* (2022)

Based on the notions of the circular coloring of signed graphs we have introduced in Chapter 3, in this chapter, we study the bounds of the circular chromatic number of some families of signed graphs.

Given a class $\mathcal{C}$ of signed graphs, we define

$$\chi_c(\mathcal{C}) = \sup\{\chi_c(G,\sigma) : (G,\sigma) \in \mathcal{C}\}.$$

As mentioned before, a signed graph with a positive loop does not admit any circular coloring and a negative loop does not affect the circular chromatic number. Though the presence of digons may affect the circular chromatic number, the restriction of the study on the class of signed simple graphs will be the focus of this work. Those families of signed graphs that we will investigate are listed as follows.

- $\mathcal{SD}_d$: the class of signed d-degenerate simple graphs,
- $\mathcal{SSP}$: the class of signed series-parallel simple graphs,
- $\mathcal{SO}$: the class of signed outerplanar simple graphs,
- $\mathcal{SK}$: the class of signed simple graphs whose underlying graph is k-colorable,
- $\mathcal{SP}$: the class of signed planar simple graphs.

We first study the circular chromatic number of signed d-degenerate graphs in Section 4.1. In Theorem 4.1.5, we show that $\chi_c(\mathcal{SD}_d) = 2\lfloor \frac{d}{2} \rfloor + 2$ for each integer d and that the supremum could be replaced by the maximum for $d \geqslant 3$. In the exceptional case of $d = 2$, answering Question 3.3.14, we prove that for any signed 2-degenerate graph, 4 is the upper bound but it is not reachable by a finite signed graph. More precisely, for a signed 2-degenerate graph (G,σ), we show, in Theorem 4.1.1, that $\chi_c(G,\sigma) \leqslant 4 - \frac{2}{\lfloor \frac{n+1}{2} \rfloor}$ where n is the number of vertices of G.

In Section 4.2, for the class $\mathcal{SSP}$ and $\mathcal{SO}$, it has been proved in [NRS15] that every signed series-parallel simple graph admits a homomorphism to the signed Paley graph $SPal_5$, which is proved to be isomorphic to a subgraph of the signed circular clique $K^s_{10;3}$. Thus by constructing a tight example that has circular chromatic number exactly $\frac{10}{3}$, we prove that $\chi_c(\mathcal{SSP}) = \chi_c(\mathcal{SO}) = \frac{10}{3}$ and that this bound is attainable. Moreover, Theorem 4.2.2 provides a tight bound of the circular chromatic number of the class of signed triangle-free series-parallel graph.

As we have seen in Corollary 3.3.3 that $\chi_c(G,\sigma) \leqslant 2\chi_c(G)$, and we are curious that whether this upper bound can be improved or not if the girth of the graph is large. The family of signed simple graphs whose underlying graph is k-colorable is of special interest. In Section 4.3, we show in Theorem 4.3.2 that $\chi_c(\mathcal{SK}) = 2k$ even when restricted to graphs of arbitrary large girth. The proof is by construction based on the notion of the augmented tree. The augmented tree was introduced in [AKR+16] to construct a graph of girth at least g whose chromatic number is larger than k for any given integers k and g.

In an attempt to generalize the 4-color theorem, E. Máčajová, A. Raspaud, and M. Škoviera conjectured in [MRŠ16] that every signed planar simple graph is $\{\pm1, \pm2\}$-colorable. By Proposition 3.1.4, this is equivalent to say that $\chi_c(G,\sigma) \leqslant 4$ for every signed planar graph (G,σ). However, recently, in [KN21], F. Kardoš and J. Narboni refuted this conjecture by constructing a non-4-colorable signed planar graph. In Section 4.4, we study this class of signed graphs. Based on an indicator, first presented in [KN21], we construct a signed planar simple graph with the circular chromatic number $4 + \frac{2}{3}$. This is, so far, the best-known lower bound for the circular chromatic number of $\mathcal{SP}$. The best-known upper bound, which is 6, is a straightforward consequence of the 5-degeneracy of planar graphs.

We present a selection of open problems in Section 4.5. Note that using the notion of signed graph minor, we can restate the odd Hadwiger's conjecture (see Conjecture 1.1.2). Furthermore, in Section 4.5.1, we propose more general questions about signed $(K_k, -)$-minor-free graphs. As we mention in Section 4.4, the exact value of the circular chromatic number of the class of signed planar simple graphs is still an open problem. We discuss questions related to this special class in Section 4.5.2. The spectrum questions for several families of signed graphs, for example, signed planar graphs, are also not complete yet and we mention the questions in Section 4.5.3. The topic we will discuss in Section 4.5.4 is one of the main interests of this thesis. We pose the general question but will study some cases of the restriction of this question to the class of signed bipartite planar graphs in Part IV.

4.1 Signed d-degenerate simple graphs

The first class we investigate is the class of signed d-degenerate simple graphs. As graphs, they are easily observed to be $(d+1)$-colorable. However, in the context of the circular chromatic number of signed d-degenerate simple graphs, we discuss two cases depending on whether $d = 2$ or not.

4.1.1 Signed 2-degenerate simple graphs

First, we consider the special case when $d = 2$ and obtain the next result.

Theorem 4.1.1. *If (G,σ) is a signed 2-degenerate simple graph on n vertices, then*

$$\chi_c(G,\sigma) \leqslant 4 - \frac{2}{\lfloor \frac{n+1}{2} \rfloor}.$$

Moreover, this upper bound is tight for each value of $n \geqslant 2$.

We first prove the following theorem which, in particular, implies that circular chromatic number of any signed 2-degenerate simple graph is strictly smaller than 4. Then using the notion of tight cycle and Proposition 3.1.11, we will conclude Theorem 4.1.1.

Theorem 4.1.2. *Let $\hat{G}$ be a signed simple graph with a vertex w of degree 2. If the signed graph $\hat{G}-w$ has circular chromatic number strictly less than 4, then $\hat{G}$ also has circular chromatic number strictly less than 4.*

Proof. Let $\hat{G}$ be a minimum counterexample to the theorem. Then it follows immediately that G is connected and has no vertex of degree 1. Let u and v be the two neighbors of w. Since circular chromatic number is invariant under switching, and without loss of generality, we may assume both uw and vw are positive edges in $\hat{G}$.

Let $\hat{G}' = \hat{G}-w$ and let ϵ be a positive real number smaller than 2, such that $\hat{G}'$ admits a circular $(4-\epsilon)$-coloring. Let C be the circle of circumference $4-\epsilon$.

By rotational symmetries of the circle we can assume that $\varphi(u) = 0$. Then considering symmetries along the diameters of the circle, in particular the one that contains 0, we may assume $\varphi(v) \geqslant 2-\frac{\epsilon}{2}$. Furthermore, we may assume $\varphi(v) < 2$ as otherwise we can complete φ to a coloring of $\hat{G}$ simply by setting $\varphi(w) = 1$.

Our aim is to present a circular $(4-\frac{\epsilon}{4})$-coloring ψ of $\hat{G}$. To this end, first we do a uniform scaling of the circle C to a circle C' to get a circular $(4-\frac{\epsilon}{2})$-coloring φ' of $\hat{G}'$. More precisely $\varphi' : V(\hat{G}') \to [0, 4-\frac{\epsilon}{2})$ is defined as follows:

$$\varphi'(x) = \frac{4-\frac{\epsilon}{2}}{4-\epsilon}\varphi(x).$$

The mapping φ' has the property that for a positive edge xy the points $\varphi'(x)$ and $\varphi'(y)$ are at distance (on C') at least $1+\frac{\epsilon}{8-2\epsilon}$ and that the same holds for the distance between $\varphi'(x)$ and the antipodal of $\varphi'(y)$ whenever xy is a negative edge. Observe that $\varphi'(u) = 0$ and $\varphi'(v) \geqslant 2-\frac{\epsilon}{4}$.

Next we introduce a circular $(4-\frac{\epsilon}{4})$-coloring of $\hat{G}'$ by inserting an interval of length $\frac{\epsilon}{4}$ inside C' to obtain a circle C'' of circumference $4-\frac{\epsilon}{4}$. Assuming this interval is inserted at point $1-\frac{\epsilon}{8}$ of C', the new coloring ψ of $\hat{G}'$ is defined as follows:

$$\psi(x) = \begin{cases} \varphi'(x), & \text{if } \varphi(x) < 1-\frac{\epsilon}{8}, \\ \varphi'(x)+\frac{\epsilon}{4}, & \text{if } \varphi(x) \geqslant 1-\frac{\epsilon}{8}. \end{cases}$$

We need to verify that ψ is a circular coloring of $\hat{G}'$. For a positive edge xy, it's immediate to see that the distance of $\psi(x)$ and $\psi(y)$ is at least 1, because in changing C' to C'' the distance between two points does not decrease. For a negative edge xy, we note that since the diameter of the circle is changed, the antipodal of each point is shifted by $\frac{\epsilon}{8}$. To be more precise, if a is a point of circle C' with a_1 as its antipodal, and a' and a'_1 are the images of these points at C'' after inserting an interval of length $\frac{\epsilon}{4}$, the antipodal of a' on C'' is at distance $\frac{\epsilon}{8}$ from a'_1 (see Figures 4.1 and 4.2). Since in C' the distance between $\varphi'(x)$ and the antipodal of $\varphi'(y)$ is at least $1+\frac{\epsilon}{8-2\epsilon}$, even after this shift of $\frac{\epsilon}{8}$ the distance between $\psi(x)$ and the antipodal of $\psi(y)$ is at least 1 and, therefore, ψ is a circular $(4-\frac{\epsilon}{4})$-coloring.

Finally, as $\psi(u) = 0$ and $\psi(v) \geqslant 2$, we may complete the circular $(4-\frac{\epsilon}{4})$-coloring ψ of $\hat{G}'$ to $\hat{G}$ simply by setting $\psi(w) = 1$. □

We observe that in this proof for two vertices x and y of $\hat{G}-w$ if we have $\varphi(x) = \varphi(y)$, then we have $\psi(x) = \psi(y)$.

From the statement of this theorem, it follows immediately that every signed 2-degenerate simple graph admits a $(4-\epsilon)$-coloring for some positive real number ϵ. Next we use the notion of tight cycle to give a precise upper bound in terms of the number of vertices.

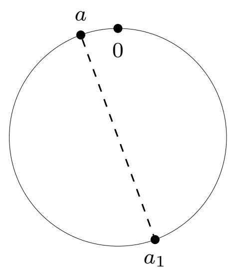


Figure 4.1. Circle C' with $r' = 4 - \frac{\epsilon}{2}$

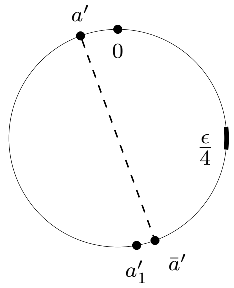


Figure 4.2. Circle C'' with $r'' = 4 - \frac{\epsilon}{4}$

Theorem 4.1.3. *For any signed 2-degenerate simple graph (G, σ) on n vertices, $n \geqslant 2$, we have:*

- *For each odd value of n, $\chi_c(G, \sigma) \leqslant 4 - \frac{4}{n+1}$,*
- *For each even value of n, $\chi_c(G, \sigma) \leqslant 4 - \frac{4}{n}$.*

Proof. As stated in Proposition 3.1.11, we know that $\chi_c(G, \sigma) = \frac{p}{q}$ where p is twice the length of a cycle in G. Thus p is an even integer satisfying $p \leqslant 2n$. Since $\chi_c(G, \sigma) < 4$ we have $\frac{p}{q} < 4$, in other words, $p < 4q$. As p and q are integers, and moreover p is an even integer, we have $p \leqslant 4q - 2$. Therefore, $\chi_c(G, \sigma) \leqslant \frac{4q-2}{q} = 4 - \frac{2}{q}$. On the other hand $\chi_c(G, \sigma) \leqslant \frac{2n}{q}$.

For a fixed n, the sequence $(\frac{2n}{q})_{q \in \mathbb{N}}$ is decreasing, whereas the sequence $(4 - \frac{2}{q})_{q \in \mathbb{N}}$ is increasing. It is easy to check that

$$\max_{q \in \mathbb{N}} \min \left\{ \frac{2n}{q}, 4 - \frac{2}{q} \right\} = \begin{cases} 4 - \frac{4}{n+1} & \text{for } q = \frac{n+1}{2} \text{ if } n \text{ is odd,} \\ 4 - \frac{4}{n} & \text{for } q = \frac{n}{2} \text{ if } n \text{ is even.} \end{cases}$$

This completes the proof. □

In Theorem 4.1.3, we have proved the inequality of Theorem 4.1.1. For the moreover part, we construct a sequence of signed 2-degenerate simple graphs Ω_i reaching the bound for $n = 2i + 1$. For even values of n, it would be enough to add an isolated vertex to Ω_i.

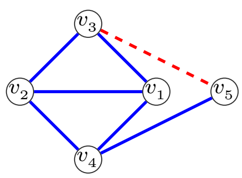


Figure 4.3. Ω_2

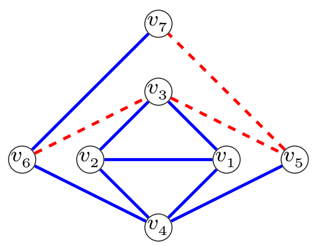


Figure 4.4. Ω_3

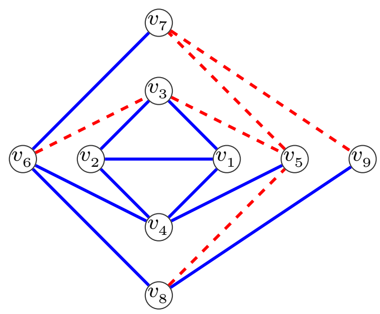


Figure 4.5. Ω_4

Let $\Omega_1 = (K_3, +)$, that is the complete graph on three vertices v_1, v_2, v_3 with all edges being positive. Starting with Ω_1, we define the sequence Ω_i of signed graphs as follows. Given Ω_i on vertices $v_1, v_2, \ldots, v_{2i+1}$, we first add a vertex v_{2i+2} which a copy of v_{2i+1}, i.e., it sees each of the two neighbors of v_{2i+1} with edges of the same sign. Then we add a new vertex v_{2i+3} which is joined to v_{2i+1} and v_{2i+2}, to one with a negative edge and to the other with a positive edge. Observe that Ω_i has $2i+1$ vertices and is 2-degenerate. The elements Ω_2, Ω_3 and Ω_4 of the sequence are illustrated in Figures 4.3, 4.4, 4.5 respectively.

Proposition 4.1.4. *Given a signed graph Ω_i as defined above, we have that*

$$\chi_c(\Omega_i) = 4 - \frac{4}{|V(\Omega_i)|+1}.$$

Proof. We prove by induction a slightly stronger claim. Let $r_i = 4 - \frac{2}{i+1}$. We claim that $\chi_c(\Omega_i) = r_i$ and, moreover, in any circular r_i-coloring of Ω_i, any tight cycle is a Hamiltonian cycle.

The case $i = 1$ of this claim is immediate. That $\chi_c(\Omega_i) \leqslant r_i$ follows from Theorem 4.1.3. To show that $\chi_c(\Omega_i) \geqslant r_i$, it is enough to show that Ω_i is not r_{i-1}-colorable, because there are no rational numbers between r_{i-1} and r_i with a numerator at most $2(2i+1)$. To this end, and toward a contradiction, assume ψ is a circular r_{i-1}-coloring of Ω_i. We claim that $\psi(v_{2i-1}) = \psi(v_{2i})$. That is because ψ is also a circular r_{i-1}-coloring of Ω_{i-1}, and in any such a coloring, any tight cycle (of Ω_{i-1}) is a Hamiltonian cycle. As v_{2i-1} is of degree 2 in Ω_{i-1} and v_{2i} is a copy of v_{2i-1}, we must have $\psi(v_{2i-1}) = \psi(v_{2i})$. But then to complete the circular r_{i-1}-coloring to v_{2i+1} we must have a point on the circle which is at distance at least 1 from both $\psi(v_{2i-1})$ and its antipodal. But that is only possible if the circumference of circle used for coloring is at least 4, a contradiction. Thus $\chi_c(\Omega_i) = \frac{4i+2}{i+1}$. For the moreover part, we observe that $gcd(4i+2, i+1) = 1$ when i is even and $gcd(4i+2, i+1) = 2$ when i is odd. Recall that $\chi_c(G,\sigma) = \frac{2p}{q}$ where p is the length of a tight cycle (with respect to a circular $\frac{2p}{q}$-coloring). Since $|V(\Omega_i)| = 2i+1$, such a tight cycle of Ω_i in a circular $\frac{4i+2}{i+1}$-coloring is a Hamiltonian cycle. □

4.1.2 Signed d-degenerate simple graphs for $d \geqslant 3$

From the previous section, we can observe that the supremum of the circular chromatic number of signed 2-degenerate simple graphs is 4 but it is never achieved. Now, we will consider all the other cases for $k \geqslant 3$.

Theorem 4.1.5. *For any positive integer $d \geqslant 2$, $\chi_c(\mathcal{SD}_d) = 2\lfloor \frac{d}{2} \rfloor + 2$.*

Proof. First we show that every $(G,\sigma) \in \mathcal{SD}_d$ admits a circular $(2\lfloor \frac{d}{2} \rfloor + 2)$-coloring. Equivalently, (G,σ) admits an edge-sign preserving homomorphism to $K^s_{2\lfloor \frac{d}{2} \rfloor+2}$ whose vertices are labelled $0, 1, \ldots, 2\lfloor \frac{d}{2} \rfloor + 1$ in a cyclic order. Recall that in $K^s_{2\lfloor \frac{d}{2} \rfloor+2}$ between any pair of vertices i, j there are both positive and negative edges, unless $i = j$ or $i = j + \lfloor \frac{d}{2} \rfloor + 1$. When $i = j$, there is a negative loop but no positive loop; when $i = j + \lfloor \frac{d}{2} \rfloor + 1$, i and j are connected by a positive edge but not a negative edge. Thus, given a vertex u of (G,σ) and a partial mapping ϕ of (G,σ) to $K^s_{2\lfloor \frac{d}{2} \rfloor+2}$, if at most d neighbors of u are already colored, then ϕ can be extended to u. This now can be applied on the ordering of vertices of G which is a witness of G being d-degenerate.

To prove that the upper bound is tight, we consider three cases.

- For $d = 2$, the limit of the circular chromatic numbers of those signed graphs in Proposition 4.1.4 is 4.
- For odd integer $d \geqslant 3$, this bound is tight by considering the signed complete graphs $(K_{d+1}, +)$.
- For even integer $d \geqslant 4$, we now construct a d-degenerate graph G together with a signature σ such that $\chi_c(G,\sigma) = d + 2$.

Define a signed graph Ω_d as follows. Take $(K_d, +)$ whose vertices are labelled $x_1, x_2, \dots, x_d$. For each pair $i, j \in [d]$ $(i \neq j)$, we add a vertex $y_{i,j}$ and join it to each of x_i and x_j with a negative edge, and to all the other x_k's with positive edges. Since each $y_{i,j}$ is of degree d and after removing all of them we are left with a K_d, we have $\Omega_d \in \mathcal{SD}_d$. We claim that $\chi_c(\Omega_d) = d + 2$.

Assume this is not true and φ is a circular r-coloring of Ω_d and $r < d + 2$. Without loss of generality, we may assume that $\varphi(x_1), \varphi(x_2), \dots, \varphi(x_d)$ are cyclicly ordered on C^r in a clockwise orientation. Furthermore, we may also assume that $\varphi(x_1), \varphi(x_2)$ has the maximum distance among all the pairs $\varphi(x_i), \varphi(x_{i+1})$ where the addition of the index is taken (mod d). As the distance between each consecutive pair $\varphi(x_i), \varphi(x_{i+1})$ is at least 1, it follows that, except for x_1, x_2, $d_{(\text{mod } r)}(\varphi(x_i), \varphi(x_{i+1})) < 2$. We will now show that there is no possible choice for $y_{1,1+\frac{d}{2}}$. A point between $\varphi(x_i)$ and $\varphi(x_{i+1})$ for $i \in \{2, 3, \dots, \frac{d}{2} - 1\} \cup \{\frac{d}{2} + 2, \dots, d - 1\}$ is at distance less than 1 from one of the two and cannot be the color of $y_{1,1+\frac{d}{2}}$ because $x_i y_{1,1+\frac{d}{2}}$, $x_{i+1} y_{1,1+\frac{d}{2}}$ are both positive edges. If $\varphi(y_{1,1+\frac{d}{2}}) \in [\varphi(x_1), \varphi(x_2)]$, then we show that $d_{(\text{mod } r)}(\varphi(y_{1,1+\frac{d}{2}}), \varphi(x_{1+\frac{d}{2}}) \geqslant \frac{d}{2}$, which is a contradiction because $y_{1,1+\frac{d}{2}} x_{1+\frac{d}{2}}$ is a negative edge. To see this, we consider clockwise and anti-clockwise distances of $\varphi(y_{1,1+\frac{d}{2}})$ and $\varphi(x_{1+\frac{d}{2}})$. On the anti-clockwise direction, $(\varphi(y_{1,1+\frac{d}{2}}), \varphi(x_{1+\frac{d}{2}}))$ contains $\frac{d}{2}$ intervals of the form (x_i, x_{i+1}), each of which is of length at least 1. On the clockwise direction, first of all, $y_{1,1+\frac{d}{2}} x_2$ is a positive edge which means $d_{(\text{mod } r)}(\varphi(y_{1,1+\frac{d}{2}}), \varphi(x_2)) \geqslant 1$, and, furthermore, $(\varphi(y_{1,1+\frac{d}{2}}), \varphi(x_{1+\frac{d}{2}}))$ contains $\frac{d}{2} - 1$ intervals of form (x_i, x_{i+1}) (for $i \in \{2, 3, \dots, \frac{d}{2}\}$). If $\varphi(y_{1,1+\frac{d}{2}}) \in [\varphi(x_d), \varphi(x_1)]$, then the same argument shows that $d_{(\text{mod } r)}(\varphi(y_{1,1+\frac{d}{2}}), \varphi(x_{1+\frac{d}{2}})) \geqslant \frac{d}{2}$. If either $\varphi(y_{1,1+\frac{d}{2}}) \in [\varphi(x_{\frac{d}{2}}), \varphi(x_{\frac{d}{2}+1})]$ or $\varphi(y_{1,1+\frac{d}{2}}) \in [\varphi(x_{\frac{d}{2}+1}), \varphi(x_{\frac{d}{2}+2})]$, then $d_{(\text{mod } r)}(\varphi(y_{1,1+\frac{d}{2}}), \varphi(x_1)) \geqslant \frac{d}{2}$, which is a contradiction as $y_{1,1+\frac{d}{2}} x_1$ is a negative edge. □

It follows directly from Proposition 4.1.5 that $\chi_c(G, \sigma) \leqslant 2\lfloor \frac{\Delta(G)}{2} \rfloor + 2$. It's clear that when $\Delta(G)$ is odd, this bound is attained by the complete graph $K_{\Delta(G)+1}$. But we don't know whether this bound of $2\lfloor \frac{\Delta(G)}{2} \rfloor + 2$ is tight when $\Delta(G)$ is even. We are also curious about the Brooks Theorem for signed graphs in terms of circular coloring.

4.2 Signed series-parallel simple graphs

In this section, we consider the class $\mathcal{SSP}$ of signed series-parallel simple graphs and its subclass $\mathcal{SO}$ of signed outerplanar simple graphs. It was proved in [NRS15] that every signed K_4-minor-free simple graph admits a (switching) homomorphism to the signed Paley graph $SPal_5$, depicted in Figure 4.6. It is easy to check that $SPal_5$ is a signed subgraph of $K^s_{10;3}$ (as it labeled). Hence, we have

$$\chi_c(\mathcal{SO}) \leqslant \chi_c(\mathcal{SSP}) \leqslant \frac{10}{3}.$$

We then show that the upper bound is tight.

Theorem 4.2.1. $\chi_c(\mathcal{SSP}) = \chi_c(\mathcal{SO}) = \frac{10}{3}$.

Proof. Let (F, σ) be the signed graph of Figure 4.7. It is easily observed that it is simple, outerplanar, and, thus, series-parallel. We will show that $\chi_c(F, \sigma) = \frac{10}{3}$ to obtain the equality in the theorem.

Since (F, σ) contains a positive triangle as a subgraph, its circular chromatic number is at least 3. By Proposition 3.1.11, the only possible values for $\chi_c(F, \sigma)$ are 3 and $\frac{10}{3}$. It remains to show that this graph does not admit a circular 3-coloring, equivalently, (F, σ) does not admit a (switching)

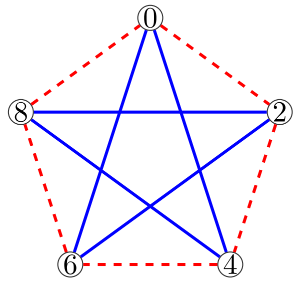


Figure 4.6. $SPal_5$

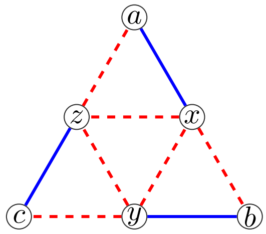


Figure 4.7. (F, σ)

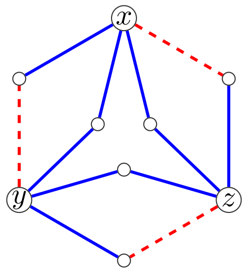


Figure 4.8. $S(K_3)$

homomorphism to $\hat{K}^s_{6;2}$. Recall that $\hat{K}^s_{6;2}$ is built from a positive triangle by adding to each vertex a negative loop. Suppose to the contrary that $(F, \sigma) \to \hat{K}^s_{6;2}$. Then at least one edge of the negative triangle xyz is mapped to a negative loop of $\hat{K}^s_{6;2}$, as in $\hat{K}^s_{6;2}$ every negative closed walk must contain a negative loop. Whichever edge of xyz is mapped to a negative loop, its two end vertices are identified and the resulting signed graph has a digon. But $\hat{K}^s_{6;2}$ contains no negative even closed walk of length 2, a contradiction. So $\chi_c(F, \sigma) > 3$ and thus $\chi_c(F, \sigma) = \frac{10}{3}$. □

Providing a slightly improved girth condition for this class, we prove that every signed triangle-free series-parallel graph (the class denoted by $\mathcal{SSP}_4$), in particular, every signed bipartite series-parallel graph is circular 3-colorable.

Theorem 4.2.2. $\chi_c(\mathcal{SSP}_4) = 3$.

Proof. Let G_{xy} be a triangle-free series-parallel graph with two terminals x and y of length ℓ. Assume that σ is a signature on G_{xy}. We shall show that $(G_{xy}, \sigma) \xrightarrow{s.p.} K^s_{6;2}$.

Given a positive integer ℓ and $\omega \in \{+, -\}$, we define

$$D^{\omega}_{\ell} = \begin{cases} \{0,1\}, & \ell = 1,\ \omega = -. \\ \{2,3\}, & \ell = 1,\ \omega = +. \\ \{1,2\}, & \ell = 2,\ \omega \in \{+,-\}. \\ \{0,1,2,3\}, & \text{otherwise.} \end{cases}$$

Given a signed triangle-free series-parallel graph (G_{xy}, σ), let P^*_{xy} denote the shortest signed path between x and y and let $\ell(x, y)$ denote the length of P^*_{xy}. It suffices to show the following claim.

Claim. *Given a signed triangle-free series-parallel graph (G_{xy}, σ), let $\ell = \ell(x, y)$ and $\omega = \sigma(P^*_{xy})$. For any $i, j \in V(K^s_{6;2})$ satisfying that $d_{(\mathrm{mod}\ 6)}(i, j) \in D^{\omega}_{\ell}$, there exists an edge-sign preserving homomorphism φ of (G_{xy}, σ) to $K^s_{6;2}$ such that $\varphi(x) = i$ and $\varphi(y) = j$.*

We show this by introduction based on the construction of (G_{xy}, σ). For $(G_{xy}, \sigma) = (K_2, \pi)$ where $K_2 = xy$(so $\ell(x, y) = 1$), it is trivial. Now assume that the statement holds for two signed triangle-free series-parallel graphs $(G^1_{x_1y_1}, \sigma_1)$ and $(G^2_{x_2y_2}, \sigma_2)$. Assume that $\ell(x_i, y_i) = \ell_i$ for $i \in \{1, 2\}$, without loss of generality, say $\ell_1 \leqslant \ell_2$. Suppose that $\sigma_1(P^*_{x_1y_1}) = \alpha$ and $\sigma_2(P^*_{x_2y_2}) = \beta$.

Let (G_{xy}, σ) be a signed triangle-free series-parallel graph obtained from $(G^1_{x_1y_1}, \sigma_1)$ and $(G^2_{x_2y_2}, \sigma_2)$ by a parallel operation with identifying the vertices x_1 with x_2 and y_1 with y_2. So $P^*_{xy} = P^*_{x_1y_1}$ by the definition of the parallel operation and thus $\ell(x, y) = \ell_1$ and $\sigma(P^*_{xy}) = \alpha$. Also, as (G_{xy}, σ) is triangle-free, $\ell_1 + \ell_2 \geqslant 4$. First we claim that for any ℓ_1, ℓ_2 satisfying that $\ell_1 + \ell_2 \geqslant 4$ and $\ell_1 \leqslant \ell_2$, $D^{\alpha}_{\ell_1} \subset D^{\beta}_{\ell_2}$ for any $\alpha, \beta \in \{+, -\}$. It could be easily verified for the following cases: (1) $\ell_1 = 1$,

$\ell_2 \geqslant 3$, (2) $\ell_1 = \ell_2 = 2$, (3) $\ell_1 = 2$, $\ell_2 \geqslant 3$, (4) $\ell_2 \geqslant \ell_1 \geqslant 3$. Since $D^{\alpha}_{\ell_1} \subset D^{\beta}_{\ell_2}$, for any i, j satisfying that $d_{(\text{mod } 6)}(i, j) \in D^{\alpha}_{\ell_1}$, there exists $\varphi_1 : (G^1_{x_1y_1}, \sigma_1) \to K^s_{6;2}$ and $\varphi_2 : (G^2_{x_2y_2}, \sigma_2) \to K^s_{6;2}$ such that $\varphi_1(x_1) = \varphi_2(x_2) = i$ and $\varphi_1(y_1) = \varphi_2(y_2) = j$. We define $\varphi : V(G_{xy}) \to K^s_{6;2}$ such that

$$\varphi(v) = \begin{cases} \varphi_1(v), & \text{for } v \in V(G^1_{x_1y_1}). \\ \varphi_2(v), & \text{for } v \in V(G^2_{x_2y_2}). \end{cases}$$

Such φ is an edge-sign preserving homomorphism of (G_{xy}, σ) to $K^s_{6;2}$ satisfying that $\varphi(x) = i$ and $\varphi(y) = j$.

Let (G_{xy}, σ) be a signed triangle-free series-parallel graph obtained from $(G^1_{x_1y_1}, \sigma_1)$ and $(G^2_{x_2y_2}, \sigma_2)$ by a series operation, without loss of generality, with identifying the vertices y_1 and x_2. So $\ell(x, y) = \ell_1 + \ell_2 := \ell$ and $\ell(x, y)$ is attained by the signed path $P^*_{xy} = P^*_{x_1y_1} \cup P^*_{x_2y_2}$ with $\sigma(P^*_{xy}) = \alpha\beta$. First we prove the following claim:

Claim. *For any i, j satisfying that $d_{(\text{mod } 6)}(i, j) \in D^{\alpha\beta}_{\ell_1+\ell_2}$, there exists a $k \in V(K^s_{6;2})$ such that $d_{(\text{mod } 6)}(i, k) \in D^{\alpha}_{\ell_1}$ and $d_{(\text{mod } 6)}(k, j) \in D^{\beta}_{\ell_2}$.*

Proof of the claim: As $\ell_1 \leqslant \ell_2$, we have six cases to discuss: (1) $\ell_1 = \ell_2 = 1$, (2) $\ell_1 = 1, \ell_2 = 2$, (3) $\ell_1 = 1$, $\ell_2 \geqslant 3$, (4) $\ell_1 = \ell_2 = 2$, (5) $\ell_1 = 2$, $\ell_2 \geqslant 3$, (6) $\ell_2 \geqslant \ell_1 \geqslant 3$. Suppose that i, j satisfy that $d_{(\text{mod } 6)}(i, j) \in D^{\alpha\beta}_{\ell_1+\ell_2}$ and without loss of generality, $i + d_{(\text{mod } 6)}(i, j) \equiv j (\text{mod } 6)$. We consider all the possibilities and choose proper k satisfying $d_{(\text{mod } 6)}(i, k) \in D^{\alpha}_{\ell_1}$ and $d_{(\text{mod } 6)}(k, j) \in D^{\beta}_{\ell_2}$ that as follows:

(1) Suppose that $\ell_1 = \ell_2 = 1$ and $d_{(\text{mod } 6)}(i, j) \in D^{\alpha\beta}_2 = \{1, 2\}$. If $\alpha = \beta = +$, then $k = i - 2$; if $\alpha = \beta = -$, then $k = i + 1$; if $\alpha = +, \beta = -$, then $k = i + 2$; if $\alpha = -, \beta = +$, then $k = i - 1$.

(2) Suppose that $\ell_1 = 1, \ell_2 = 2$ and $d_{(\text{mod } 6)}(i, j) \in D^{\alpha\beta}_3 = \{0, 1, 2, 3\}$. If $\alpha = \beta = +$, then $k = i+2$ when $d_{(\text{mod } 6)}(i, j) \in \{0, 1\}$ and $k = i - 2$ when $d_{(\text{mod } 6)}(i, j) \in \{2, 3\}$; if $\alpha = \beta = -$, then $k = i+1$ when $d_{(\text{mod } 6)}(i, j) \in \{0, 2\}$ and $k = i-1$ when $d_{(\text{mod } 6)}(i, j) \in \{1, 3\}$; if $\alpha = +, \beta = -$, then $k = i + 2$ when $d_{(\text{mod } 6)}(i, j) \in \{0, 1\}$ and $k = i - 2$ when $d_{(\text{mod } 6)}(i, j) \in \{2, 3\}$; if $\alpha = -, \beta = +$, then $k = i + 1$.

(3) Suppose that $\ell_1 = 1$, $\ell_2 \geqslant 3$ and $d_{(\text{mod } 6)}(i, j) \in D^{\alpha\beta}_{\ell} = \{0, 1, 2, 3\}$. If $\alpha = +$, then $k = i + 2$; if $\alpha = -$, then $k = i + 1$.

(4) Suppose that $\ell_1 = \ell_2 = 2$ and $d_{(\text{mod } 6)}(i, j) \in D^{\alpha\beta}_{\ell} = \{0, 1, 2, 3\}$. Since $D^+_2 = D^-_2 = \{1, 2\}$, for any $\alpha, \beta \in \{+, -\}$, if $d_{(\text{mod } 6)}(i, j) \in \{0, 1\}$, then $k = i + 2$ and if $d_{(\text{mod } 6)}(i, j) \in \{2, 3\}$, then $k = i - 2$.

(5) Suppose that $\ell_1 = 2$, $\ell_2 \geqslant 3$ and $d_{(\text{mod } 6)}(i, j) \in D^{\alpha\beta}_{\ell} = \{0, 1, 2, 3\}$. Since $D^+_2 = D^-_2 = \{1, 2\}$, for any $\alpha, \beta \in \{+, -\}$, $k = i + 2$.

(6) Suppose that $\ell_2 \geqslant \ell_1 \geqslant 3$ and $d_{(\text{mod } 6)}(i, j) \in D^{\alpha\beta}_{\ell} = \{0, 1, 2, 3\}$. Since $D^+_{\ell} = D^-_{\ell} = \{0, 1, 2, 3\}$ with $\ell \geqslant 3$, for any $\alpha, \beta \in \{+, -\}$, $k = i + 2$.

This completes the proof of the claim. ♢

For any $i, j \in V(K^s_{6;2})$ satisfying that $d_{(\text{mod } 6)}(i, j) \in D^{\alpha\beta}_{\ell_1+\ell_2}$, by the above claim, we could find a vertex k of $K^s_{6;2}$ such that $d_{(\text{mod } 6)}(i, k) \in D^{\alpha}_{\ell_1}$ and $d_{(\text{mod } 6)}(k, j) \in D^{\beta}_{\ell_2}$. By hypothesis of $(G^i_{x_iy_i}, \sigma_i)$ for $i \in \{1, 2\}$, there exists $\varphi_1 : (G^1_{x_1y_1}, \sigma_1) \xrightarrow{s.p.} K^s_{6;2}$ such that $\varphi_1(x_1) = i$ and $\varphi_1(y_1) = k$, and φ_2 :

$(G^2_{x_2y_2}, \sigma_2) \xrightarrow{s.p.} K^s_{6;2}$ such that $\varphi_2(x_2) = k$ and $\varphi_2(y_2) = j$. Similarly, we define $\varphi : V(G_{xy}) \to K^s_{6;2}$ such that

$$\varphi(v) = \begin{cases} \varphi_1(v), & \text{for } v \in V(G^1_{x_1y_1}). \\ \varphi_2(v), & \text{for } v \in V(G^2_{x_2y_2}). \end{cases}$$

Such φ is an edge-sign preserving homomorphism of (G_{xy}, σ) to $K^s_{6;2}$ satisfying that $\varphi(x) = i$ and $\varphi(y) = j$.

Moreover, note that $S(K_3)$, shown in Figure 4.8, is a signed triangle-free series-parallel graph satisfying that $\chi_c(S(K_3)) = 3$ by Theorem 3.3.12. This bound is tight. □

4.3 Signed k-chromatic graphs

Following Corollary 3.3.3, we know that given a graph G and an arbitrary signature σ, $\chi_c(G, \sigma) \leqslant 2\chi_c(G)$, noting that the tightness is obtained by signed graphs with multiple-edges (thus, girth 2). In this section, we strengthen this claim by showing that the tightness stands even for signed simple graphs of arbitrary large girth.

To prove the result, we will use the concept of *augmented tree* introduced in [AKR+16]. It has been introduced to prove, in a constructive way, Erdös' theorem that given integers k and g, there exists a graph of girth at least g whose chromatic number is larger than k.

Some basic notations about rooted trees are needed. Given a rooted tree T, the *level* of a vertex v is defined to be the distance from v to the root vertex in T. For a leaf u of T, there is a unique path in T from the root to u, denoted by P_u. Moreover, vertices in $P_u - \{u\}$ are *ancestors* of u.

A *complete k-ary tree* is a rooted tree, in which each non-leaf vertex has k children and all the leaves are in the same level. Given a complete k-ary tree T, a *standard labeling* of the edges of T is a labeling $\phi : E(T) \to \{1, 2, \ldots, k\}$ such that for each non-leaf vertex v, v has each of its k children labeled from 1 to k correspondingly. Furthermore, given a standard labeling ϕ of $E(T)$ and a k-mapping $f : V(T) \to \{1, 2, \ldots, k\}$ of the vertices of T, the *f-path* $P_f = (v_1, v_2, \ldots, v_m)$ of T is the path from the root vertex v_1 to a leaf v_m of T such that for each $i = 1, 2, \ldots, m - 1$, $f(v_i) = \phi(v_iv_{i+1})$.

A *q-augmented k-ary tree* is obtained from a complete k-ary tree by adding, for each leaf u, q edges connecting u to q of its ancestors. These q edges are called the *augmenting edges* from u. Note that each cycle in the q-augmented k-ary tree contains at least one augmenting edge. For positive integers k, q, g, a *(k, q, g)-graph* is a q-augmented k-ary tree which is bipartite and has girth at least g. The existence of such a (k, q, g)-graph for any positive integers k, p, g has been proved in [AKR+16].

Lemma 4.3.1. *For any positive integers $k, q, g \geqslant 2$, there exists a (k, q, g)-graph.*

Now we shall prove the next theorem which supports the tightness of the bound in Corollary 3.3.3 and is also of independent interest.

Theorem 4.3.2. *For any integers $k, g \geqslant 2$ and any $\epsilon > 0$, there is a graph G of girth at least g satisfying that $\chi(G) = k$ and a signature σ such that $\chi_c(G, \sigma) > 2k - \epsilon$.*

Proof. Let k and g be two integers which are both larger or equal to 2. It suffices to prove that for any integer p, there is a graph G of girth at least g whose chromatic number is at most k and there is a signature σ such that (G, σ) is not $(2kp, p + 1)$-colorable.

Let H be a $(2kp, k, 2kg)$-graph, which is a bipartite k-augmented $2kp$-ary tree T of girth at least $2kg$. Let ϕ be a standard $2kp$-labeling of $E(T)$. For each $v \in V(T)$, we denote the level of v by $\ell(v)$.

Starting from the root, we partition the vertices in every consecutive k levels into a group and let $\theta(v) = \ell(v) \bmod k$.

For each leaf v of T, we denote the vertices on P_v that are connected to v by augmenting edges by $u_{v,1}, u_{v,2}, \dots, u_{v,k}$. For each i, let $u'_{v,i} \in P_v$ be the closest descendant of $u_{v,i}$ with $\theta(u'_{v,i}) = i$ and let $e_{v,i}$ be the edge connecting $u'_{v,i}$ to its child on P_v. Given a leaf v and an index $i \in \{1, \dots, k\}$, we consider the following three sets:

$$A_{v,i} = \{\phi(e_{v,i}), \phi(e_{v,i}) + 1, \dots, \phi(e_{v,i}) + p\},\ B_{v,i} = \{x + kp \mid x \in A_{v,i}\},\ C_{v,i} = A_{v,i} \cup B_{v,i}.$$

The addition above are all carried out modulo $2kp$. Note that $B_{v,i}$ is a kp-shift of $A_{v,i}$. Also we can regard the elements in $B_{v,i}$ as the antipodals of those in $A_{v,i}$ with respect to the set $[2kp]$. Since $|C_{v,i}| = |A_{v,i} \cup B_{v,i}| = 2(p+1)$ for each i but $\bigcup_{i=1}^{k} C_{v,i} \subseteq [2kp]$, by Pigeonhole Principle, there exists a pair of distinct indices, say i, j, such that $C_{v,i} \cap C_{v,j} \neq \varnothing$.

There are two possibilities. If $A_{v,i} \cap A_{v,j} \neq \varnothing$ (thus $B_{v,i} \cap B_{v,j} \neq \varnothing$), then

$$d_{(\text{mod } 2kp)}(\phi(e_{v,i}), \phi(e_{v,j})) \leqslant p.$$

If $A_{v,i} \cap B_{v,j} \neq \varnothing$ (and hence $B_{v,i} \cap A_{v,j} \neq \varnothing$), then

$$d_{(\text{mod } 2kp)}(\phi(e_{v,i}), \overline{\phi(e_{v,j})}) \leqslant p.$$

Let L be the set of leaves of T. For each $v \in L$, we define one edge e_v on $V(T)$ as follows:

- If $d_{(\text{mod } 2kp)}(\phi(e_{v,i}), \phi(e_{v,j})) \leqslant p$, then let e_v be a positive edge connecting $u'_{v,i}$ and $u'_{v,j}$.
- If $d_{(\text{mod } 2kp)}(\phi(e_{v,i}), \overline{\phi(e_{v,j})}) \leqslant p$, then let e_v be a negative edge connecting $u'_{v,i}$ and $u'_{v,j}$.

Let (G, σ) be the signed graph with vertex set $V(T)$ and with edge set $\{e_v : v \in L\}$, where the signs of the edges are defined as above. We shall show that (G, σ) has the desired properties.

First, observe that θ is a proper k-coloring of G. So G has chromatic number at most k.

Next, we show that G has girth at least g. For each edge $e_v = u'_{v,i}u'_{v,j}$ of G, let B_v be the path which is the union of the subpath of P_v from $u'_{v,i}$ to $u_{v,i}$ and the path $u_{v,i}vu_{v,j}$ and the subpath of P_v from $u_{v,j}$ to $u'_{v,j}$. Then B_v has length at most $2k$. If C is a cycle in G, then replace each edge e_v of C by the path B_v, we obtain a closed walk in H. As H has girth at least $2kg$, we conclude that C has length at least g and hence G has girth at least g.

Finally, we show that (G, σ) is not $(2kp, p+1)$-colorable. Assume f is a $(2kp, p+1)$-coloring of (G, σ). As f could also be regarded as a $2kp$-coloring of the vertices of T, there is a unique f-path, say P_v. Assume $e_v = u'_{v,i}u'_{v,j}$. It follows from the definition of f-path that $f(u'_{v,i}) = \phi(e_{v,i})$ and $f(u'_{v,j}) = \phi(e_{v,j})$. By the definition of e_v, the following claims hold.

- If e_v is a positive edge, then $d_{(\text{mod } 2kp)}(\phi(e_{v,i}), \phi(e_{v,j})) \leqslant p$;
- If e_v is a negative edge, then $d_{(\text{mod } 2kp)}(\phi(e_{v,i}), \overline{\phi(e_{v,j})}) \leqslant p$.

This is in contrary to the assumption that f is a $(2kp, p+1)$-coloring of (G, σ). □

Remark: The graph constructed above is shown to have chromatic number at most k. However, since $\frac{2kp}{p+1} < \chi_c(G, \sigma) \leqslant 2\chi(G)$, we conclude that $\chi(G) = k$ when $p+1 \geqslant 2k$. It is not known whether there is a finite k-chromatic graph of girth at least g and a signature σ such that $\chi_c(G, \sigma) = 2k$. Also, it is unknown whether for every rational $\frac{p}{q}$, any integer g, and any $\epsilon > 0$, there is a graph G with $\chi_c(G) \leqslant \frac{p}{q}$ and a signature σ such that $\chi_c(G, \sigma) > \frac{2p}{q} - \epsilon$.

Replacing k-chromatic graphs by k-critical graphs, we have a similar result. The following theorem about circular chromatic number of critical graphs of large girth was proved in [Zhu01b].

Theorem 4.3.3. *For any integer $k \geqslant 3$ and $\epsilon > 0$, there is an integer g such that any k-critical graph of girth at least g has circular chromatic number at most $k - 1 + \epsilon$.*

As a consequence of Theorem 4.3.3 and Corollary 3.3.3, we know that for any integer $k \geqslant 3$ and $\epsilon > 0$, there is an integer g such that any k-critical graph G of girth at least g has signed circular chromatic number at most $2k - 2 + \epsilon$. However, this bound is not tight. The following proposition follows from Theorem 3.3.4.

Proposition 4.3.4. *If G is a k-critical graph, then for any signature σ, $\chi_c(G, \sigma) \leqslant 2k - 2$.*

Proof. Let σ be a signature on G. If $(G, \sigma) = (G, +)$, then $\chi_c(G, \sigma) \leqslant \chi(G, \sigma) = \chi(G) = k$. If $\sigma(e) = -$ for some edge e, then the subgraph of G induced by positive edges has chromatic number at most $k - 1$. Hence, $\chi_c(G, \sigma) \leqslant 2(k - 1)$. □

4.4 Signed planar simple graphs

The last but not the least family that we are interested in is the class of signed planar simple graphs. In this section, we work on bounding the circular chromatic number of this family.

Since planar simple graphs are 5-degenerate, by Proposition 4.1.5, $\chi_c(\mathcal{SP}) \leqslant 6$. It was conjectured in [MRŠ16] that every planar simple graph admits a 0-free 4-coloring. If the conjecture was true, it would have implied the best possible bound of 4 for the circular chromatic number of signed planar simple graphs. However, this conjecture was disproved in [KN21] using a dual notion. Later a direct proof of a counterexample is given in [NP22]. Extending this construction, we build a signed planar simple graph whose circular chromatic number is $4 + \frac{2}{3}$, which is the best lower bound we found till now.

Theorem 4.4.1. $\frac{14}{3} \leqslant \chi_c(\mathcal{SP}) \leqslant 6$.

The construction is broken down into several constructions of certain gadgets. Similar to the gadget of [KN21], we start with a mini-gadget depicted in Figure 4.9 and state its circular coloring property in Lemma 4.4.2.

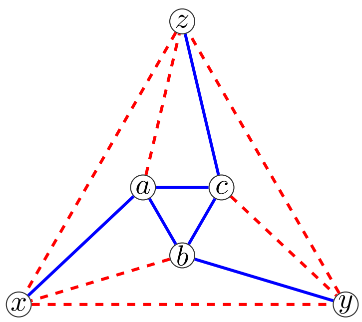


Figure 4.9. Mini-gadget (T, π)

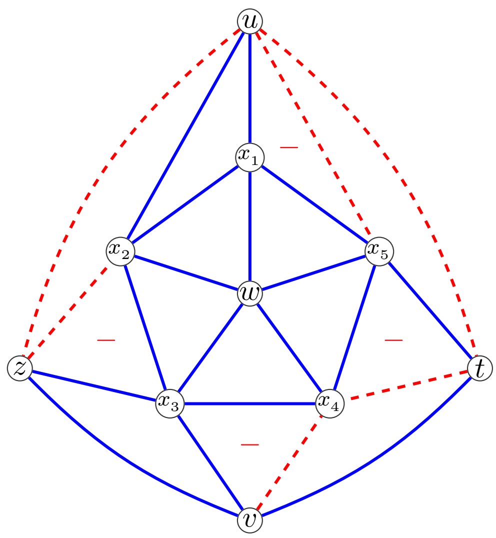


Figure 4.10. A signed Wenger Graph

We first introduce some notation to state the properties. Let r be a positive real number. Let ϕ be a mapping of a set $\{v_1, \ldots, v_k\}$ of points (or vertices of a graph) to C^r. We denote by

$I_{\phi;v_1,\ldots,v_q,\bar{v}_{q+1},\ldots,\bar{v}_k}$ an interval of minimum length which contains $\{\phi(v_i) : i = 1,\ldots,q\} \cup \{\overline{\phi(v_i)} : i = q+1,\ldots,k\}$ and by $\ell_{\phi;v_1,\ldots,v_q,\bar{v}_{q+1},\ldots,\bar{v}_k}$ the length of this interval. Note that the minimality of the length implies that the two end points of the interval $I_{\phi;v_1,\ldots,v_q,\bar{v}_{q+1},\ldots,\bar{v}_k}$ are in $\{\phi(v_i) : i = 1,\ldots,q\} \cup \{\overline{\phi(v_i)} : i = q+1,\ldots,k\}$

Lemma 4.4.2. *For $0 \leqslant \alpha < 2$, assume ϕ is a circular $(4+\alpha)$-coloring of the signed graph (T,π) of Figure 4.9. Then $\ell_{\phi;x,y,z} \in [1-\frac{\alpha}{2}, 1+\frac{\alpha}{2}]$. Moreover, for any t_1,t_2,t_3 satisfying that $\max\{d_{(\mathrm{mod}\, r)}(t_i,t_j) \mid i,j \in \{1,2,3\}\} \in [1-\frac{\alpha}{2}, 1+\frac{\alpha}{2}]$, there exists a circular r-coloring ϕ of (T,π) such that $\phi(x)=t_1, \phi(y)=t_2, \phi(z)=t_3$.*

Proof. Let $r = 4+\alpha$ and let ϕ be a circular r-coloring of (T,π). Without loss of generality we may assume that $\phi(x)$, $\phi(y)$ and $\phi(z)$ are on C^r in the clockwise order, and assume that the interval $[\phi(z),\phi(x)]$ is a longest interval among $[\phi(x),\phi(y)]$, $[\phi(y),\phi(z)]$ and $[\phi(z),\phi(x)]$. Thus $I_{\phi;x,y,z} = [\phi(x),\phi(z)]$. We first claim that $[\phi(z),\phi(x)]$ contains $\overline{\phi(y)}$. Otherwise, either $[\overline{\phi(y)},\phi(y)]$ or $[\phi(y),\overline{\phi(y)}]$ which is of length $\frac{r}{2}$, is included in either $(\phi(x),\phi(y)]$ or $[\phi(y),\phi(z))$. This is a contradiction as $[\phi(z),\phi(x)]$ is longest among the three. As $\overline{\phi(y)}$ is contained in $[\phi(z),\phi(x)]$, and as y is adjacent to both z and x with a negative edge, we conclude that $[\phi(z),\phi(x)]$ is of length at least 2. On the other hand, since z and x are adjacent with a negative edge, one of the two intervals, $[\phi(z),\phi(x)]$ or $[\phi(x),\phi(z)]$ is of length at most $\frac{r}{2}-1 = 1+\frac{\alpha}{2}$. As $\alpha<2$, the only option is that $[\phi(x),\phi(z)]$ is of length at most $1+\frac{\alpha}{2}$.

For the other direction, assume $\ell_{\phi;x,y,z} < 1-\frac{\alpha}{2}$, say $I_{\phi;x,y,z} = [0,\beta]$ for some $\beta < 1-\frac{\alpha}{2}$. Each of a,b,c is joined by a positive edge and a negative edge to vertices in x,y,z. This implies that $\phi(a),\phi(b),\phi(c) \in [1, 1+\beta+\frac{\alpha}{2}] \cup [3+\frac{\alpha}{2}, 3+\alpha+\beta]$. As each of the intervals $[1,1+\beta+\frac{\alpha}{2}]$ and $[3+\frac{\alpha}{2}, 3+\alpha+\beta]$ has length strictly smaller than 1, two of the vertices a,b,c are colored by colors of distance less than 1 in C^r. But abc is a triangle with three positive edges, a contradiction.

For the "moreover" part, without loss of generality, we assume that $t_3 = 0, t_1 \in [1-\frac{\alpha}{2}, 1+\frac{\alpha}{2}], t_2 \in [0,t_1]$. If $t_1 \in [1-\frac{\alpha}{2}, 1]$, then let $\phi(a) = 3+\frac{\alpha}{2}$, $\phi(b) = 2$ and $\phi(c) = 1$; if $t_1 \in [1, 1+\frac{\alpha}{2}]$, then let $\phi(a) = 3+\frac{\alpha}{2}$, $\phi(b) = 2+\frac{\alpha}{2}$ and $\phi(c) = 1$. It is straightforward to verify that ϕ is a circular r-coloring of (T,π). □

By taking $\alpha = \frac{2}{3}-\epsilon$ and a switching at the vertex z, we have the following formulation of Lemma 4.4.2 which we will use frequently.

Corollary 4.4.3. *Let (T,π') be a signed graph obtained from (T,π) by a switching at the vertex z, and let ϕ be a circular $(\frac{14}{3}-\epsilon)$-coloring of (T,π') where $0<\epsilon<\frac{2}{3}$. Then $\ell_{\phi;x,y,\bar{z}} \in [\frac{2}{3}+\frac{\epsilon}{2}, \frac{4}{3}-\frac{\epsilon}{2}]$.*

We define $\tilde{W}$ to be the signed graph obtained from signed Wenger graph of Figure 4.10 by completing each of the four negative facial triangles to a switching of the mini-gadget of Figure 4.9. Next we show that $\tilde{W}$ has a property similar to signed indicators, more precisely:

Lemma 4.4.4. *Let $r = \frac{14}{3}-\epsilon$ with $0<\epsilon\leqslant\frac{2}{3}$. For any circular r-coloring ϕ of $\tilde{W}$, $\ell_{\phi;u,v} \geqslant \frac{4}{9}$.*

The proof of Lemma 4.4.4 is long and detailed case-analysis, so we leave it to the end of this section. Let Γ be obtained from $\tilde{W}$ by adding a negative edge uv. Let $\mathcal{I} = (\Gamma,u,v)$ be a signed indicator. It follows from Lemma 4.4.4 that for $4 \leqslant r < \frac{14}{3}$, $(\mathcal{I},r) \subseteq [\frac{4}{9}, \frac{r}{2}-1]$.

Theorem 4.4.5. *Let $\Omega = K_4(\mathcal{I})$. Then Ω is a signed planar simple graph with $\chi_c(\Omega) = \frac{14}{3}$.*

Proof. First we show that Ω admits a circular $\frac{14}{3}$-coloring.

For $r = \frac{14}{3}$, there is a circular r-coloring ϕ of Γ with $\phi(u) = \phi(v)$, defined as $\phi(u) = \phi(v) = 0$, $\phi(w) = 3$, $\phi(x_1) = 2$, $\phi(x_2) = 1$, $\phi(x_3) = 2$, $\phi(x_4) = \frac{1}{3}$, $\phi(x_5) = 4$ and $\phi(z) = \phi(t) = 1$. We observe that each of the four negative triangles satisfies the conditions of Lemma 4.4.2, and that the coloring of its vertices can be extended to the inner part of the mini-gadget. Let v_1, v_2, v_3, v_4 be the 4 vertices of K_4. Then there is a circular $\frac{14}{3}$-coloring ϕ of $K_4(\mathcal{I})$ with $\phi(v_i) = 0$ for $i = 1, 2, 3, 4$. So $\chi_c(\Omega) \leqslant \frac{14}{3}$.

It remains to show that $\chi_c(\Omega) \geqslant \frac{14}{3}$. Assume to the contrary that $\chi_c(\Omega) < \frac{14}{3}$ and let ϕ be a circular r-coloring of Ω for some $4 \leqslant r < \frac{14}{3}$. Note that for the purpose of applying Lemma 4.4.4, we assume $r \geqslant 4$. Without loss of generality, assume that $\phi(v_1)$, $\phi(v_2)$, $\phi(v_3)$ and $\phi(v_4)$ are on C^r in this cyclic order.

As $(\mathcal{I}, r) \subseteq [\frac{4}{9}, \frac{r}{2} - 1]$, we know that for any $1 \leqslant i < j \leqslant 4$,

$$\frac{4}{9} \leqslant d_{(\text{mod } r)}(\phi(v_i), \phi(v_j)) \leqslant \frac{r}{2} - 1.$$

By symmetry, we may assume $d_{(\text{mod } r)}(\phi(v_1), \phi(v_3)) = \ell([\phi(v_1), \phi(v_3)])$ and $d_{(\text{mod } r)}(\phi(v_2), \phi(v_4)) = \ell([\phi(v_2), \phi(v_4)])$.

Hence

$$\ell([\phi(v_1), \phi(v_4)]) = \ell([\phi(v_1), \phi(v_2)]) + \ell([\phi(v_2), \phi(v_3)]) + \ell([\phi(v_3), \phi(v_4)]) \geqslant 3 \times \frac{4}{9} = \frac{4}{3} > \frac{r}{2} - 1,$$

and

$$\ell([\phi(v_4), \phi(v_1)]) \geqslant r - (\ell([\phi(v_1), \phi(v_3)]) + \ell([\phi(v_2), \phi(v_4)])) \geqslant 2 > \frac{r}{2} - 1.$$

This implies that $d_{(\text{mod } r)}(\phi(v_1), \phi(v_4)) > \frac{r}{2} - 1$, a contradiction. □

4.4.1 Proof of Lemma 4.4.4

Assume to the contrary that ϕ is a circular r-coloring of $\tilde{W}$ with $\ell_{\phi;u,v} = \eta < \frac{4}{9}$. Without loss of generality, we assume that $\phi(u) = 0$ and $\phi(v) = \eta$. Since each of $\phi(z)$ and $\phi(t)$ is of distance at least 1 from both $\overline{\phi(u)}$ and $\phi(v)$, we have:

$$\phi(z), \phi(t) \in \begin{cases} [1+\eta, \frac{4}{3} - \frac{\epsilon}{2}] \cup [\frac{10}{3} - \frac{\epsilon}{2}, \frac{11}{3} + \eta - \epsilon], & \text{if } \eta \leqslant \frac{1}{3} - \frac{\epsilon}{2}, \\ [\frac{10}{3} - \frac{\epsilon}{2}, \frac{11}{3} + \eta - \epsilon], & \text{otherwise.} \end{cases} \tag{4.1}$$

Lemma 4.4.6. $\phi(w) \notin (\frac{5}{3} - \epsilon, 3 + \eta) \cup (4 - \frac{3\epsilon}{2}, \frac{2}{3} + \eta + \frac{\epsilon}{2})$.

Proof. Let $\phi(w) = \delta$. First we show that $\delta \notin (\frac{5}{3} - \epsilon, 3 + \eta)$. Assume to the contrary that $\delta \in (\frac{5}{3} - \epsilon, 3 + \eta)$. As x_2 is joined to u and w by positive edges,

$$\phi(x_2) \in \begin{cases} [1, \delta - 1], & \text{if } \delta > \frac{8}{3} - \epsilon, \\ [\delta + 1, \frac{11}{3} - \epsilon], & \text{if } \delta < 2, \\ [1, \delta - 1] \cup [\delta + 1, \frac{11}{3} - \epsilon], & \text{if } 2 \leqslant \delta \leqslant \frac{8}{3} - \epsilon. \end{cases} \tag{4.2}$$

$$\phi(x_3) \in \begin{cases} [1+\eta, \delta - 1], & \text{if } \delta > \frac{8}{3} + \eta - \epsilon, \\ [\delta + 1, \frac{11}{3} + \eta - \epsilon], & \text{if } \delta < 2 + \eta, \\ [1+\eta, \delta - 1] \cup [\delta + 1, \frac{11}{3} + \eta - \epsilon], & \text{if } 2 + \eta \leqslant \delta \leqslant \frac{8}{3} + \eta - \epsilon. \end{cases} \tag{4.3}$$

For a depiction of these cases, see Figure 4.11.

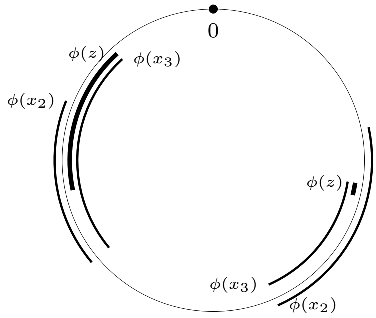


Figure 4.11. A sketch of locating $\phi(z)$, $\phi(x_2)$ and $\phi(x_3)$ on C^r

Claim. *The following restrictions on the value of $\phi(z)$ hold:*

[I]. *If $\delta < 3$ and $\phi(x_2) \in [1, \delta - 1]$, then $\eta \leqslant \frac{1}{3} - \frac{\epsilon}{2}$ and $\phi(z) \in [1 + \eta, \frac{4}{3} - \frac{\epsilon}{2}]$.*

[II]. *If $\phi(x_2) \in [\delta + 1, \frac{11}{3} - \epsilon]$, then $\phi(z) \in [\frac{10}{3} - \frac{\epsilon}{2}, \frac{11}{3} + \eta - \epsilon]$.*

[III]. *If $\phi(x_3) \in [1 + \eta, \delta - 1]$, then $\phi(z) \in [\frac{10}{3} - \frac{\epsilon}{2}, \frac{11}{3} + \eta - \epsilon]$.*

[IV]. *If $\delta > \frac{5}{3} + \eta - \epsilon$ and $\phi(x_3) \in [\delta + 1, \frac{11}{3} + \eta - \epsilon]$, then $\eta \leqslant \frac{1}{3} - \frac{\epsilon}{2}$ and $\phi(z) \in [1 + \eta, \frac{4}{3} - \frac{\epsilon}{2}]$.*

Proof of the claim: We prove the claim case by case.

[I]. Assume to the contrary (by Condition (4.1)) that $\phi(z) \in [\frac{10}{3} - \frac{\epsilon}{2}, \frac{11}{3} + \eta - \epsilon]$ and $\phi(x_2) \in [1, \delta - 1]$. Then $d_{(\text{mod } r)}(\phi(x_2), \phi(z)) \geqslant \min\{\frac{10}{3} - \frac{\epsilon}{2} - (\delta - 1), \frac{14}{3} - \epsilon + 1 - (\frac{11}{3} + \eta - \epsilon)\} > \frac{4}{3} - \frac{\epsilon}{2}$, contradicting the fact that x_2z is a negative edge.

[II]. Assume to the contrary (by Condition (4.1)) that $\phi(z) \in [1 + \eta, \frac{4}{3} - \frac{\epsilon}{2}]$ and $\eta \leqslant \frac{1}{3} - \frac{\epsilon}{2}$. Then $d_{(\text{mod } r)}(\phi(x_2), \phi(z)) \geqslant \min\{2 + \eta, \delta - \frac{1}{3} + \frac{\epsilon}{2}\} > \frac{4}{3} - \frac{\epsilon}{2}$, contradicting the fact that x_2z is a negative edge.

[III]. Assume to the contrary (by Condition (4.1)) that $\phi(z) \in [1 + \eta, \frac{4}{3} - \frac{\epsilon}{2}]$, $\phi(x_3) \in [1 + \eta, \delta - 1]$ and hence $\delta \geqslant 2 + \eta$. As $\delta \in (\frac{5}{3} - \epsilon, 3 + \eta)$, $d_{(\text{mod } r)}(\phi(x_3), \phi(z)) \leqslant \delta - 1 - (1 + \eta) < 1$, contradicting the fact that x_3z is a positive edge.

[IV]. Assume to the contrary (by Condition (4.1)) that $\phi(z) \in [\frac{10}{3} - \frac{\epsilon}{2}, \frac{11}{3} + \eta - \epsilon]$. As $\delta > \frac{5}{3} + \eta - \epsilon$, $d_{(\text{mod } r)}(\phi(x_3), \phi(z)) \leqslant \frac{11}{3} + \eta - \epsilon - (\delta + 1) < 1$, contradicting the fact that x_3z is a positive edge.

This completes the proof of the claim. ◊

To complete the proof of this lemma, we partition the interval $(\frac{5}{3} - \epsilon, 3 + \eta)$ into three parts and consider three cases depending on to which part δ belongs.

Case (i) $\delta \in (\dfrac{5}{3} - \epsilon, 2 + \eta)$.

As $\delta < 2 + \eta$, by Condition (4.3), $\phi(x_3) \in [\delta + 1, \frac{11}{3} + \eta - \epsilon]$. Thus $\overline{\phi(x_3)} \in [\delta - \frac{4}{3} + \frac{\epsilon}{2}, \frac{4}{3} + \eta - \frac{\epsilon}{2}]$. By Condition (4.2), $\phi(x_2) \in [1, \delta - 1] \cup [\delta + 1, \frac{11}{3} - \epsilon]$.

Subcase (i-1) $\phi(x_2) \in [1, \delta - 1]$ and hence, by Condition (4.2), $\delta \geqslant 2$.

As $\delta < 2 + \eta < 3$, by [I], $\phi(z) \in [1 + \eta, \frac{4}{3} - \frac{\epsilon}{2}]$ and $\eta \leqslant \frac{1}{3} - \frac{\epsilon}{2}$. Hence $\delta < 2 + \eta \leqslant \frac{7}{3} - \frac{\epsilon}{2}$. Consider the interval $I_{\phi;\bar{x}_3,x_2,z}$, see Figure 4.12. If $\overline{\phi(x_3)}$ is the starting point of this interval, then since $\delta - 1 < \frac{4}{3} - \frac{\epsilon}{2}$, we have $[\overline{\phi(x_3)}, \phi(z)] \subseteq [\delta - \frac{4}{3} + \frac{\epsilon}{2}, \frac{4}{3} - \frac{\epsilon}{2}]$. If the starting of $I_{\phi;\bar{x}_3,x_2,z}$ is $\phi(x_2)$ or $\phi(z)$, then since $\delta - 1 < \frac{4}{3} - \frac{\epsilon}{2}$, we have $[\phi(x_2), \overline{\phi(x_3)}] \subseteq [1, \frac{4}{3} + \eta - \frac{\epsilon}{2}]$. In either case, $I_{\phi;\bar{x}_3,x_2,z}$ has length at most $\frac{2}{3} - \epsilon$, contrary to Corollary 4.4.3.

Subcase (i-2) $\phi(x_2) \in [\delta + 1, \frac{11}{3} - \epsilon]$.

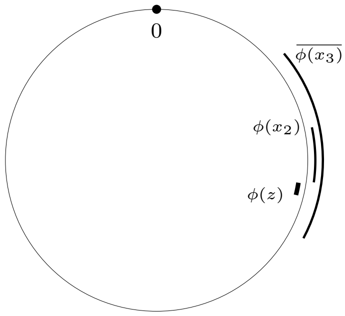


Figure 4.12. Subcase (i-1): Restrictions on the negative triangle x_3x_2z.

By [II], $\phi(z) \in [\frac{10}{3} - \frac{\epsilon}{2}, \frac{11}{3} + \eta - \epsilon]$. Note that $\ell([\frac{10}{3} - \frac{\epsilon}{2}, \frac{11}{3} + \eta - \epsilon]) = \frac{1}{3} + \eta - \frac{\epsilon}{2} < 1$. Since $d_{(\text{mod } r)}(\phi(x_3), \phi(z)) \geqslant 1$ (as x_3z is a positive edge) and $\phi(x_3) \in [\delta + 1, \frac{11}{3} + \eta - \epsilon]$, we conclude that $\delta \leqslant \frac{5}{3} + \eta - \epsilon$ and $\phi(x_3) \in [\delta + 1, \frac{8}{3} + \eta - \epsilon]$. This implies that $I_{\phi;x_3,x_2} \subseteq [\delta + 1, \frac{11}{3} - \epsilon]$. As $\delta > \frac{5}{3} - \epsilon$, $\ell([\delta + 1, \frac{11}{3} - \epsilon]) < 1$, contrary to the fact that x_2x_3 is a positive edge.

Case (ii) $\delta \in [2 + \eta, \dfrac{8}{3} + \eta - \epsilon]$.

Depending on the ranges of $\phi(x_2)$ and $\phi(x_3)$, we consider four subcases.

Subcase (ii-1) $\phi(x_2) \in [1, \delta - 1]$ and $\phi(x_3) \in [1 + \eta, \delta - 1]$.

By [III], $\phi(z) \in [\frac{10}{3} - \frac{\epsilon}{2}, \frac{11}{3} + \eta - \epsilon]$. As $\phi(x_2), \phi(x_3) \in [1, \delta - 1]$, $\ell([1 + \eta, \delta - 1]) < 1$ and x_2x_3 is a positive edge, we have $\delta \geqslant 3$ and $\phi(x_2) \in [1, \delta - 2]$. However, the distance of points in $[\frac{10}{3} - \frac{\epsilon}{2}, \frac{11}{3} + \eta - \epsilon]$ and $[1, \delta - 2]$ is at least $2 - \eta$ which is strictly larger than $\frac{4}{3} - \frac{\epsilon}{2}$, contradicting that x_2z is a negative edge.

Subcase (ii-2) $\phi(x_2) \in [1, \delta - 1]$ and $\phi(x_3) \in [\delta + 1, \frac{11}{3} + \eta - \epsilon]$. $(\overline{\phi(x_3)} \in [\delta - \frac{4}{3} + \frac{\epsilon}{2}, \frac{4}{3} + \eta - \frac{\epsilon}{2}])$

By [IV], $\phi(z) \in [1 + \eta, \frac{4}{3} - \frac{\epsilon}{2}]$ and by Condition (4.1), $\eta \leqslant \frac{1}{3} - \frac{\epsilon}{2}$. Note that the interval $I_{\phi;\bar{x}_3,x_2,z}$ is one of the following intervals:

$$[\overline{\phi(x_3)}, \phi(z)] \subseteq [\delta - \frac{4}{3} + \frac{\epsilon}{2}, \frac{4}{3} - \frac{\epsilon}{2}],\ [\overline{\phi(x_3)}, \phi(x_2)] \subseteq [\delta - \frac{4}{3} + \frac{\epsilon}{2}, \delta - 1],\ [\phi(z), \phi(x_2)] \subseteq [1 + \eta, \delta - 1],$$

$$[\phi(z), \overline{\phi(x_3)}] \subseteq [1 + \eta, \frac{4}{3} + \eta - \frac{\epsilon}{2}], \text{ and } [\phi(x_2), \overline{\phi(x_3)}], [\phi(x_2), \phi(z)] \subseteq [1, \frac{4}{3} + \eta - \frac{\epsilon}{2}].$$

All the above intervals have lengths at most $\frac{2}{3} - \epsilon$, implying that $\ell_{\phi;\bar{x}_3,x_2,z} < \frac{2}{3} + \frac{\epsilon}{2}$, this contradicts Corollary 4.4.3.

Subcase (ii-3) $\phi(x_2) \in [\delta + 1, \frac{11}{3} - \epsilon]$ and $\phi(x_3) \in [1 + \eta, \delta - 1]$.

By Condition (4.2), $\delta \leqslant \frac{8}{3} - \epsilon$, and by [III], $\phi(z) \in [\frac{10}{3} - \frac{\epsilon}{2}, \frac{11}{3} + \eta - \epsilon]$. Observe that $\overline{\phi(x_3)} \in [\frac{10}{3} + \eta - \frac{\epsilon}{2}, \frac{4}{3} + \delta - \frac{\epsilon}{2}]$. So $I_{\phi;\bar{x}_3,x_2,z}$ is one of the following intervals:

$$[\overline{\phi(x_3)}, \phi(z)], [\overline{\phi(x_3)}, \phi(x_2)] \subseteq [\frac{10}{3} + \eta - \frac{\epsilon}{2}, \frac{11}{3} + \eta - \epsilon],\ [\phi(x_2), \overline{\phi(x_3)}] \subseteq [\delta + 1, \frac{4}{3} + \delta - \frac{\epsilon}{2}],$$

$$[\phi(x_2), \phi(z)] \subseteq [\delta{+}1, \frac{11}{3}{+}\eta{-}\epsilon],\ [\phi(z), \phi(x_2)] \subseteq [\frac{10}{3}{-}\frac{\epsilon}{2}, \frac{11}{3}{-}\epsilon], \text{ and } [\phi(z), \overline{\phi(x_3)}] \subseteq [\frac{10}{3}{-}\frac{\epsilon}{2}, \frac{4}{3}{+}\delta{-}\frac{\epsilon}{2}].$$

Thus the $\ell_{\phi;\bar{x}_3,x_2,z} < \frac{2}{3} - \epsilon$, contradicting Corollary 4.4.3.

Subcase (ii-4) $\phi(x_2) \in [\delta + 1, \frac{11}{3} - \epsilon]$ and $\phi(x_3) \in [\delta + 1, \frac{11}{3} + \eta - \epsilon]$.

The interval $[\delta + 1, \frac{11}{3} + \eta - \epsilon]$ has length at most $\frac{2}{3} - \epsilon < 1$. This contradicts the fact that x_2x_3 is a positive edge.

Case (iii) $\delta \in (\frac{8}{3} + \eta - \epsilon, 3 + \eta)$.

As $\delta > \frac{8}{3} + \eta - \epsilon \geqslant \frac{8}{3} - \epsilon$, by Conditions (4.2) and (4.3), $\phi(x_2) \in [1, \delta - 1]$ and $\phi(x_3) \in [1 + \eta, \delta - 1]$.

As $\ell([1 + \eta, \delta - 1]) < 1$ and $d_{(\text{mod } r)}(\phi(x_2), \phi(x_3)) \geqslant 1$, we conclude that $\delta \geqslant 3$ and $\phi(x_2) \in [1, \delta - 2]$. As x_2z is a negative edge, and the distance between the intervals $[\frac{10}{3} - \frac{\epsilon}{2}, \frac{11}{3} + \eta - \epsilon]$ and $[1, \delta - 2]$ is strictly larger than $\frac{4}{3} - \frac{\epsilon}{2}$, we know that $\phi(z) \notin [\frac{10}{3} - \frac{\epsilon}{2}, \frac{11}{3} + \eta - \epsilon]$. By Condition (4.1), $\phi(z) \in [1 + \eta, \frac{4}{3} - \frac{\epsilon}{2}]$ and $\eta \leqslant \frac{1}{3} - \frac{\epsilon}{2}$. This implies that $\phi(z)$ and $\phi(x_3)$ are both in $[1 + \eta, \delta - 1]$. However, $\delta < 3 + \eta$, so $\ell([1 + \eta, \delta - 1]) < 1$, contradicting the fact that x_3z is a positive edge.

This completes the proof that $\phi(w) \notin (\frac{5}{3} - \epsilon, 3 + \eta)$.

We observe that in this proof, the vertex x_1 plays no role. In other words, the conclusion holds for the signed subgraph induced on $G \backslash x_1$. In this subgraph a switching at $U = \{w, x_2, x_3, x_4, x_5\}$ results in an isomorphic copy where x_4 and x_5 play the role of x_2 and x_3. Thus for the mapping ϕ' defined as $\phi'(v) = \phi(v)$ for $v \in V(\tilde{W}) - U$ and $\phi'(v) = \overline{\phi(v)}$ for $v \in U$, we have $\phi'(w) \notin (\frac{5}{3} - \epsilon, 3 + \eta)$. Hence, $\phi(w) \notin (4 - \frac{3\epsilon}{2}, \frac{2}{3} + \eta + \frac{\epsilon}{2})$. □

If $\eta > 1 - \frac{3\epsilon}{2}$, then by the Lemma 4.4.6, we have no choice for $\phi(w)$. Thus we assume in the rest of the proof that $\eta \leqslant 1 - \frac{3\epsilon}{2}$ and

$$\phi(w) \in [3 + \eta, 4 - \frac{3\epsilon}{2}] \cup [\frac{2}{3} + \eta + \frac{\epsilon}{2}, \frac{5}{3} - \epsilon].$$

The two cases will be considered separately.

Case A. $\phi(w) \in [3 + \eta, 4 - \frac{3\epsilon}{2}]$.

As ux_5 is a negative edge and $\phi(u) = 0$, we have $\phi(x_5) \in [\frac{10}{3} - \frac{\epsilon}{2}, \frac{4}{3} - \frac{\epsilon}{2}]$. As $\ell([3+\eta, 4-\frac{3\epsilon}{2}]) < 1$, and x_5w is a positive edge, we conclude that $3 + \eta, \phi(w), \phi(x_5), \frac{4}{3} - \frac{\epsilon}{2}$ occur in this cyclic order. This implies that

$$\phi(x_5) \in [4 + \eta, \frac{4}{3} - \frac{\epsilon}{2}].$$

For $i = 1, 2, 3, 4$, by considering the edges between x_i and u, v, w, similar arguments as above lead to the following restrictions on the value of $\phi(x_i)$:

$$\phi(x_1) \in [1, 3 - \frac{3\epsilon}{2}],\ \phi(x_2) \in [1, 3 - \frac{3\epsilon}{2}],\ \phi(x_3) \in [1 + \eta, 3 - \frac{3\epsilon}{2}],\ \phi(x_4) \in [4 + \eta, \frac{4}{3} + \eta - \frac{\epsilon}{2}].$$

By Condition (4.1), based on the choices of $\phi(z)$ and $\phi(t)$, we consider four cases.

Case A-1 $\eta \leqslant \frac{1}{3} - \frac{\epsilon}{2}$ and $\phi(z), \phi(t) \in [1 + \eta, \frac{4}{3} - \frac{\epsilon}{2}]$.

We will update the ranges of $\phi(x_i)$'s as depicted in Figure 4.13. In this figure the range of $\phi(x_i)$'s are shown as an interval partitioned to two parts. The full interval represent the restriction we have started with. We then show that the dotted part of the interval is not available for $\phi(x_i)$, thus updating the range to the solid part of the interval.

As $\ell([1 + \eta, \frac{4}{3} - \frac{\epsilon}{2}]) < 1$, $\phi(x_3) \in [1 + \eta, 3 - \frac{3\epsilon}{2}]$ and zx_3 is a positive edge, the points $1 + \eta, \phi(z), \phi(x_3), 3 - \frac{3\epsilon}{2}$ occur in C^r in this cyclic order. This implies that

$$\phi(x_3) \in [2 + \eta, 3 - \frac{3\epsilon}{2}].$$

As $\ell([2+\eta, 3-\frac{3\epsilon}{2}]) < 1$, $\phi(x_2) \in [1, 3-\frac{3\epsilon}{2}]$ and x_2x_3 is a positive edge, the points $1, \phi(x_2), \phi(x_3), 3-\frac{3\epsilon}{2}$ occurs in C^r in this cyclic order. This implies that

$$\phi(x_2) \in [1, 2 - \frac{3\epsilon}{2}].$$

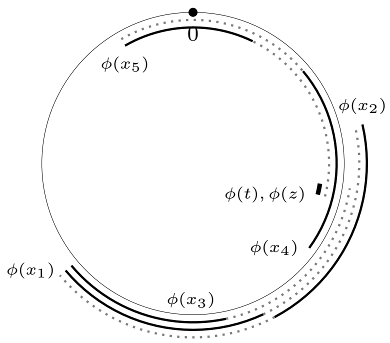


Figure 4.13. Case A-1: Updating ranges of $\phi(x_i)$'s

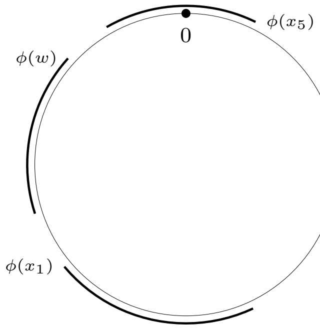


Figure 4.14. Case A-1: Restrictions on x_1x_5u

By considering the positive edges x_5t and then x_4x_5, similar arguments show that

$$\phi(x_5) \in [4+\eta, \frac{1}{3}-\frac{\epsilon}{2}] \text{ and } \phi(x_4) \in [\frac{1}{3}+\eta+\epsilon, \frac{4}{3}+\eta-\frac{\epsilon}{2}].$$

Considering the positive edge x_1x_2 and the range of $\phi(x_2)$ given above, a similar argument shows that

$$\phi(x_1) \in [2, 3-\frac{3\epsilon}{2}] \text{ and hence } \overline{\phi(x_1)} \in [\frac{13}{3}-\frac{\epsilon}{2}, \frac{2}{3}-\epsilon].$$

Now consider the negative triangle x_1x_5u. If $I_{\phi;\bar{x}_1,x_5,u} = [\overline{\phi(x_1)}, \phi(x_5)]$, then $I_{\phi;\bar{x}_1,x_5,u} \subseteq [\frac{13}{3}-\frac{\epsilon}{2}, \frac{1}{3}-\frac{\epsilon}{2}]$ but $\ell([\frac{13}{3}-\frac{\epsilon}{2}, \frac{1}{3}-\frac{\epsilon}{2}]) = \frac{2}{3}-\epsilon < \frac{2}{3}+\frac{\epsilon}{2}$, contrary to Corollary 4.4.3. Also $\phi(u)=0$ cannot be an end point of the interval $I_{\phi;\bar{x}_1,x_5,u}$, as 0 is at distance less than $\frac{2}{3}+\frac{\epsilon}{2}$ from each of the four end points of the intervals that are the ranges of $\overline{\phi(x_1)}$ and $\phi(x_5)$. Thus $I_{\phi;\bar{x}_1,x_5,u} = [\phi(x_5), \overline{\phi(x_1)}]$. By Corollary 4.4.3, $\ell([\phi(x_5), \overline{\phi(x_1)}]) \geqslant \frac{2}{3}+\frac{\epsilon}{2}$. Thus

$$\ell([\phi(x_1), \phi(x_5)]) = \frac{r}{2} - \ell([\phi(x_5), \overline{\phi(x_1)}]) \leqslant \frac{5}{3}-\epsilon.$$

As $\phi(x_1) \in [2, 3-\frac{3\epsilon}{2}]$, $\phi(w) \in [3+\eta, 4-\frac{3\epsilon}{2}]$ and $\phi(x_5) \in [4+\eta, \frac{1}{3}-\frac{\epsilon}{2}]$, we conclude that $\phi(w) \in [\phi(x_1), \phi(x_5)]$ (see Figure 4.14). Since wx_1 and wx_5 are positive edges, we have

$$2 \leqslant \ell([\phi(x_1), \phi(w)]) + \ell([\phi(w), \phi(x_5)]) = \ell([\phi(x_1), \phi(x_5)]) \leqslant \frac{5}{3}-\epsilon,$$

a contradiction.

Case A-2 $\eta \leqslant \frac{1}{3}-\frac{\epsilon}{2}$, $\phi(z) \in [1+\eta, \frac{4}{3}-\frac{\epsilon}{2}]$, and $\phi(t) \in [\frac{10}{3}-\frac{\epsilon}{2}, \frac{11}{3}+\eta-\epsilon]$.

The proof is similar to the previous case. The positive edge zx_3 and the negative edge tx_4 further restrict the ranges of $\phi(x_3)$, $\phi(x_4)$. Then, the new ranges of $\phi(x_3)$ and $\phi(x_5)$, together with the positive edges x_3x_2 and x_4x_5 further restrict the range of $\phi(x_2)$, $\phi(x_5)$. As the computations are very similar to the previous case, we just list the conclusion of this argument:

$$\phi(x_3) \in [2+\eta, 3-\frac{3\epsilon}{2}],\ \phi(x_2) \in [1, 2-\frac{3\epsilon}{2}],\ \phi(x_5) \in [\frac{1}{3}+\eta+\epsilon, \frac{4}{3}-\frac{\epsilon}{2}] \text{ and } \phi(x_4) \in [4+\eta, \frac{1}{3}-\frac{\epsilon}{2}].$$

Next we consider the negative triangle vx_3x_4. As

$$\overline{\phi(x_3)} \in [\frac{13}{3}+\eta-\frac{\epsilon}{2}, \frac{2}{3}-\epsilon],\ \phi(x_4) \in [4+\eta, \frac{1}{3}-\frac{\epsilon}{2}], \text{ and } \phi(v) = \eta,$$

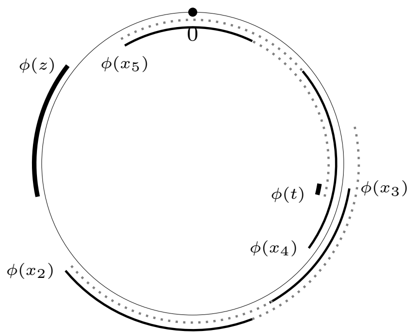


Figure 4.15. Case A-3: Updating ranges of $\phi(x_i)$'s

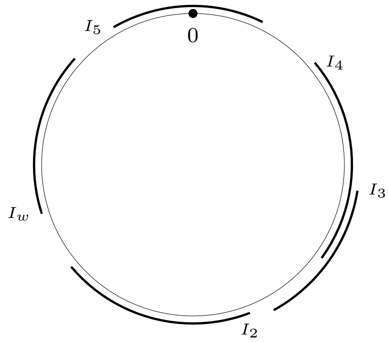


Figure 4.16. Case A-3: Restrictions on $wx_5x_4x_3x_2$

similar analysis as in the previous case shows that $I_{\phi;\bar{x}_3,x_4,v} = [\phi(x_4), \overline{\phi(x_3)}]$ and $\ell(\phi(x_4), \overline{\phi(x_3)}) \geqslant \frac{2}{3} + \frac{\epsilon}{2}$. This means that $\ell([\phi(x_3), \phi(x_4)]) < 2$. A similar argument shows that $\phi(w) \in [\phi(x_3), \phi(x_4)]$. As x_3w, x_4w are positive edges, we have

$$2 \leqslant \ell([\phi(x_3), \phi(w)]) + \ell([\phi(w), \phi(x_4)]) = \ell([\phi(x_3), \phi(x_4)]) < 2,$$

a contradiction.

Case A-3 $\eta \leqslant \frac{1}{3} - \frac{\epsilon}{2}$, $\phi(z) \in [\frac{10}{3} - \frac{\epsilon}{2}, \frac{11}{3} + \eta - \epsilon]$, and $\phi(t) \in [1 + \eta, \frac{4}{3} - \frac{\epsilon}{2}]$.

We will update the ranges of $\phi(x_2), \ldots, \phi(x_5)$ as depicted in Figure 4.15.

We first have that

$$\phi(w) \in [3 + \eta, 4 - \frac{3\epsilon}{2}] := I_w.$$

Recall that $\phi(x_2) \in [1, 3 - \frac{3\epsilon}{2}]$ and $\phi(x_5) \in [4+\eta, \frac{4}{3} - \frac{\epsilon}{2}]$. If $\phi(x_2) \in [1, 2)$, then $d_{(\mathrm{mod}\, r)}(\phi(x_2), \phi(z)) > \frac{4}{3} - \frac{\epsilon}{2}$, contrary to the fact that x_2z is a negative edge. Thus

$$\phi(x_2) \in [2, 3 - \frac{3\epsilon}{2}] := I_2.$$

If $\phi(x_5) \in (\frac{1}{3} - \frac{\epsilon}{2}, \frac{4}{3} - \frac{\epsilon}{2}]$, then $d_{(\mathrm{mod}\, r)}(\phi(x_5), \phi(t)) < 1$, contrary to the fact that x_5t is a positive edge. Therefore,

$$\phi(x_5) \in [4 + \eta, \frac{1}{3} - \frac{\epsilon}{2}] := I_5.$$

Note that $\ell(I_5) < 1$. As $\phi(x_4) \in [4 + \eta, \frac{4}{3} + \eta - \frac{\epsilon}{2}]$ and $d_{(\mathrm{mod}\, r)}(\phi(x_5), \phi(x_4)) \geqslant 1$ (as x_4x_5 is a positive edge), we conclude that the four points $4 + \eta, \phi(x_5), \phi(x_4), \frac{4}{3} + \eta - \frac{\epsilon}{2}$ occurs in C^r in this cyclic order and

$$\phi(x_4) \in [\frac{1}{3} + \eta + \epsilon, \frac{4}{3} + \eta - \frac{\epsilon}{2}] := I_4.$$

Note that $\ell(I_2) < 1$, x_2x_3 is a positive edge and $\phi(x_3) \in [1+\eta, 3 - \frac{3\epsilon}{2}]$. Thus the points $2, \phi(x_2), \phi(x_3)$ occurs in C^r in this cyclic order. Hence,

$$\phi(x_3) \in [1 + \eta, 2 - \frac{3\epsilon}{2}] := I_3.$$

The intervals I_w, I_5, I_4, I_3, I_2 are each of length less than 1, and except for I_3 and I_4 there is no intersection among them (see Figure 4.16). Since $\ell(I_3) < 1$ and x_3x_4 is a positive edge, we have that $\phi(x_4) \notin I_3$. Thus the points $\phi(w), \phi(x_5), \phi(x_4), \phi(x_3), \phi(x_2)$ occur in C^r in this cyclic order.

As C^r is of length $\frac{14}{3} - \epsilon$, the colors of some two consecutive vertices of the 5-cycle $wx_5x_4x_3x_2$ is less than 1, but all the edges of this cycle are positive. This is a contradiction.

Case A-4 $\phi(z), \phi(t) \in [\frac{10}{3} - \frac{\epsilon}{2}, \frac{11}{3} + \eta - \epsilon]$.

Similarly, we obtain that

$$\phi(w) \in [3 + \eta, 4 - \frac{3\epsilon}{2}], \quad \phi(x_2) \in [2, 3 - \frac{3\epsilon}{2}], \quad \phi(x_4) \in [4 + \eta, \frac{1}{3} + \eta - \frac{\epsilon}{2}],$$

$$\phi(x_5) \in [\frac{1}{3} + \eta + \epsilon, \frac{4}{3} - \frac{\epsilon}{2}], \text{ and } \phi(x_1) \in [\frac{4}{3} + \eta + \epsilon, 2 - \frac{3\epsilon}{2}].$$

The points $\phi(w)$, $\phi(x_4)$, $\phi(x_5)$, $\phi(x_1)$ and $\phi(x_2)$ occur in C^r in this cyclic order. As all the edges of the 5-cycle $wx_4x_5x_1x_2$ are positive, this is a contradiction.

Case B. $\phi(w) \in [\frac{2}{3} + \eta + \frac{\epsilon}{2}, \frac{5}{3} - \epsilon]$.

Similarly, by considering the edges between each of x_i's and vertices u, v, w, we have that

$$\phi(x_1), \phi(x_2) \in [\frac{5}{3} + \eta + \frac{\epsilon}{2}, \frac{11}{3} - \epsilon], \quad \phi(x_3) \in [\frac{5}{3} + \eta + \frac{\epsilon}{2}, \frac{11}{3} + \eta - \epsilon],$$

$$\phi(x_4) \in [\frac{10}{3} + \eta - \frac{\epsilon}{2}, \frac{2}{3} - \epsilon], \text{ and } \phi(x_5) \in [\frac{10}{3} - \frac{\epsilon}{2}, \frac{2}{3} - \epsilon].$$

Based on the choices of $\phi(z)$ and $\phi(t)$, we have four sub-cases to discuss.

Case B-1 $\phi(z), \phi(t) \in [\frac{10}{3} - \frac{\epsilon}{2}, \frac{11}{3} + \eta - \epsilon]$.

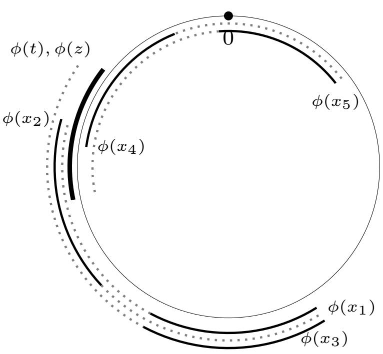


Figure 4.17. Case B-1: Updating ranges of $\phi(x_i)$'s

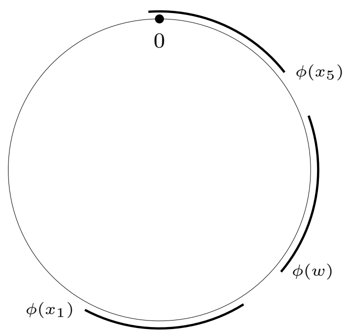


Figure 4.18. Case B-1: Restrictions on x_1x_5u

We will update the ranges of $\phi(x_i)$'s as depicted in Figure 4.17.

The positive edges zx_3 and tx_5 further restrict the ranges of $\phi(x_3)$ and $\phi(x_5)$. Then the new ranges of $\phi(x_3)$ and $\phi(x_5)$, through the positive edges x_3x_2 and x_5x_4, further restrict the ranges of $\phi(x_2)$ and $\phi(x_4)$. By similar computation as previous cases, we have

$$\phi(x_3) \in [\frac{5}{3} + \eta + \frac{\epsilon}{2}, \frac{8}{3} - \epsilon], \quad \phi(x_2) \in [\frac{8}{3} + \eta + \frac{\epsilon}{2}, \frac{11}{3} - \epsilon],$$

$$\phi(x_5) \in [\frac{13}{3} + \eta - \frac{\epsilon}{2}, \frac{2}{3} - \epsilon], \text{ and } \phi(x_4) \in [\frac{10}{3} + \eta - \frac{\epsilon}{2}, \frac{13}{3} - 2\epsilon].$$

Considering the positive edge x_1x_2 and the range of $\phi(x_2)$ given above, we obtain that

$$\phi(x_1) \in [\frac{5}{3} + \eta + \frac{\epsilon}{2}, \frac{8}{3} - \epsilon].$$

Next we consider the negative triangle x_1x_5u. As $\overline{\phi(x_1)} \in [4+\eta, \frac{1}{3}-\frac{\epsilon}{2}]$, $\phi(x_5) \in [\frac{13}{3}+\eta-\frac{\epsilon}{2}, \frac{2}{3}-\epsilon]$ and $\phi(u) = 0$, similar analysis shows that $I_{\phi;\bar{x}_1,x_5,u} = [\overline{\phi(x_1)}, \phi(x_5)]$ and $\ell([\overline{\phi(x_1)}, \phi(x_5)]) \geqslant \frac{2}{3} + \frac{\epsilon}{2}$. It implies that $\ell([\phi(x_5), \phi(x_1)]) = \frac{5}{3} - \frac{\epsilon}{2} < 2$. We observe that $\phi(w) \in [\phi(x_5), \phi(x_1)]$ (see Figure 4.18) and since x_5w, x_1w are both positive edges, we have that

$$2 \leqslant \ell([\phi(x_5, w)]) + \ell(\phi(w), \phi(x_1)) = \ell([\phi(x_5), \phi(x_1)]) < 2,$$

a contradiction.

Case B-2 $\eta \leqslant \dfrac{1}{3} - \dfrac{\epsilon}{2}$, $\phi(z) \in [\dfrac{10}{3} - \dfrac{\epsilon}{2}, \dfrac{11}{3} + \eta - \epsilon]$, and $\phi(t) \in [1+\eta, \dfrac{4}{3} - \dfrac{\epsilon}{2}]$.

The positive edge zx_3 and the negative edge tx_4 further restrict the ranges of $\phi(x_3)$ and $\phi(x_4)$ respectively. Then the new ranges of $\phi(x_3)$ and $\phi(x_4)$, through the positive edges x_3x_2 and x_4x_5, further restrict the ranges of $\phi(x_2)$ and $\phi(x_5)$. By similar computation as previous cases, we have

$$\phi(x_3) \in [\frac{5}{3} + \eta + \frac{\epsilon}{2}, \ \frac{8}{3} - \epsilon], \phi(x_2) \in [\frac{8}{3} + \eta + \frac{\epsilon}{2}, \frac{11}{3} - \epsilon],$$

$$\phi(x_4) \in [\frac{13}{3} + \eta - \frac{\epsilon}{2}, \frac{2}{3} - \epsilon], \text{ and } \phi(x_5) \in [\frac{10}{3} - \frac{\epsilon}{2}, \frac{13}{3} - 2\epsilon].$$

Next we consider the negative triangle x_3x_4v. As $\overline{\phi(x_3)} \in [4+\eta, \frac{1}{3} - \frac{\epsilon}{2}]$, $\phi(x_4) \in [\frac{13}{3} + \eta - \frac{\epsilon}{2}, \frac{2}{3} - \epsilon]$ and $\phi(v) = \eta$, similar analysis shows that $I_{\phi;\bar{x}_3,x_4,v} = [\overline{\phi(x_3)}, \phi(x_4)]$ and $\ell([\overline{\phi(x_3)}, \phi(x_4)]) \geqslant \frac{2}{3} + \frac{\epsilon}{2}$. This means that $\ell([\phi(x_4), \phi(x_3)]) < 2$. We observe that $\phi(w) \in [\phi(x_4), \phi(x_3)]$ and as x_4w, x_3w are both positive edges, we have that

$$2 \leqslant \ell([\phi(x_4, w)]) + \ell(\phi(w), \phi(x_3)) = \ell([\phi(x_4), \phi(x_3)]) < 2,$$

a contradiction.

Case B-3 $\eta \leqslant \dfrac{1}{3} - \dfrac{\epsilon}{2}$ and $\phi(z), \phi(t) \in [1+\eta, \dfrac{4}{3} - \dfrac{\epsilon}{2}]$.

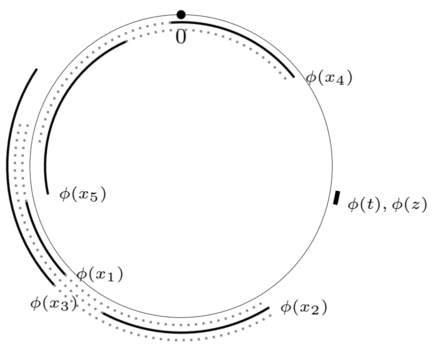


Figure 4.19. Case B-3: Updating ranges of $\phi(x_i)$'s

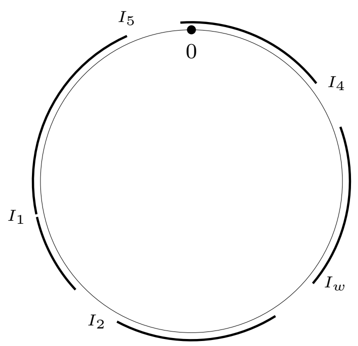


Figure 4.20. Case B-3: Restrictions on $wx_5x_4x_3x_2$

We will update the ranges of $\phi(x_i)$'s as depicted in Figure 4.19.

We know that

$$\phi(w) \in [\frac{2}{3} + \eta + \frac{\epsilon}{2}, \frac{5}{3} - \epsilon] := I_w.$$

Recall that $\phi(x_2) \in [\frac{5}{3} + \eta + \frac{\epsilon}{2}, \frac{11}{3} - \epsilon]$ and $\phi(x_4) \in [\frac{10}{3} + \eta - \frac{\epsilon}{2}, \frac{2}{3} - \epsilon]$. If $\phi(x_2) \in (\frac{8}{3} - \epsilon, \frac{11}{3} - \epsilon]$, then $d_{(\bmod r)}(\phi(x_2), \phi(z)) > \frac{4}{3} - \frac{\epsilon}{2}$, contrary to the fact that x_2z is a negative edge. Thus

$$\phi(x_2) \in [\frac{5}{3} + \eta + \frac{\epsilon}{2}, \frac{8}{3} - \epsilon] := I_2.$$

If $\phi(x_4) \in [\frac{10}{3} + \eta - \frac{\epsilon}{2}, \frac{13}{3} + \eta - \frac{\epsilon}{2})$, then $d_{(\bmod r)}(\phi(x_4), \phi(t)) > \frac{4}{3} - \frac{\epsilon}{2}$, contrary to the fact that x_4t is a negative edge. Therefore,

$$\phi(x_4) \in [\frac{13}{3} + \eta - \frac{\epsilon}{2}, \frac{2}{3} - \epsilon] := I_4.$$

Note that $\ell(I_4) < 1$. As $\phi(x_5) \in [\frac{10}{3} - \frac{\epsilon}{2}, \frac{2}{3} - \epsilon]$ and $d_{(\bmod r)}(\phi(x_5), \phi(x_4)) \geqslant 1$ (as x_4x_5 is a positive edge), we conclude that

$$\phi(x_5) \in [\frac{10}{3} - \frac{\epsilon}{2}, \frac{13}{3} - 2\epsilon] := I_5.$$

Recall that $\phi(x_3) \in [\frac{5}{3} + \eta + \frac{\epsilon}{2}, \frac{11}{3} + \eta - \epsilon]$. Similarly, $\ell(I_2) < 1$, and x_2x_3 is a positive edge. Thus $\phi(x_3) \in [\frac{8}{3} + \eta + \frac{\epsilon}{2}, \frac{11}{3} + \eta - \epsilon]$. By the restriction from the positive edges x_1x_2 and x_1x_5 and the new ranges I_2 and I_5, we have

$$\phi(x_1) \in [\frac{8}{3} + \eta + \frac{\epsilon}{2}, \frac{10}{3} - 2\epsilon] := I_1.$$

The intervals I_w, I_2, I_1, I_5, I_4 are each of length less than 1, and there is no intersection among them (see Figure 4.20). As C^r is of circumference $\frac{14}{3} - \epsilon$, the colors of some two consecutive vertices of the 5-cycle $wx_2x_1x_5x_4$ is less than 1, but all the edges of this cycle are positive. It is a contradiction.

Case B-4 $\eta \leqslant \frac{1}{3} - \frac{\epsilon}{2}$, $\phi(z) \in [1 + \eta, \frac{4}{3} - \frac{\epsilon}{2}]$, and $\phi(t) \in [\frac{10}{3} - \frac{\epsilon}{2}, \frac{11}{3} + \eta - \epsilon]$.

Similarly, we could obtain that

$$\phi(w) \in [\frac{2}{3}+\eta+\frac{\epsilon}{2}, \frac{5}{3}-\epsilon] := I_w,\ \ \phi(x_2) \in [\frac{5}{3}+\eta+\frac{\epsilon}{2}, \frac{8}{3}-\epsilon] := I_2,\ \ \phi(x_3) \in [\frac{8}{3}+\eta+\frac{\epsilon}{2}, \frac{11}{3}+\eta-\epsilon] := I_3,$$

$$\phi(x_4) \in [\frac{10}{3} + \eta - \frac{\epsilon}{2}, \frac{13}{3} - 2\epsilon] := I_4,\ \text{and}\ \phi(x_5) \in [\frac{13}{3} + \eta - \frac{\epsilon}{2}, \frac{2}{3} - \epsilon] := I_5.$$

The intervals I_w, I_2, I_3, I_4, I_5 are each of length less than 1, and except for I_3 and I_4 there is no intersection among them. Since $\ell(I_4) < 1$ and x_3x_4 is a positive edge, $\phi(x_3) \notin I_4$. That is again a contradiction because of the 5-cycle $wx_2x_3x_4x_5$ all whose edges are positive.

This completes the proof of Lemma 4.4.4.

4.5 Further questions

The circular coloring of graphs has been studied extensively in the literature. Many of the results and problems on the circular coloring of graphs would also be interesting in the framework of signed graphs. In the previous sections, we have studied the circular chromatic number of several classes of signed graphs. Here we present a collection of interesting problems for future work.

4.5.1 Signed graph minors

As mentioned in Section 1.1, one of the most intriguing conjectures in graph theory is the Hadwiger's conjecture and its strengthening, the odd Hadwiger's conjecture. Using the development in this work, the famous odd Hadwiger's conjecture can be restated as follows.

Conjecture 4.5.1. [Odd Hadwiger's conjecture restated] *If $(G, -)$ has no $(K_{k+1}, -)$-minor, then $\chi_c(G, +) \leqslant k$.*

To generalize this, one may ask:

Question 4.5.2. *Assuming (G, σ) has no $(K_{k+1}, -)$-minor, what is the best upper bound on $\chi_c(G, -\sigma)$?*

Let $f(k)$ be the answer to Question 4.5.2. By Theorem 3.3.4, one can observe that if Conjecture 4.5.1 holds, then $f(k) \leqslant 2k$. The best-known result so far for the odd Hadwiger's conjecture is in [Pos20], where the upper bound of $O(k \cdot (\log \log k)^6)$ is proved. Considering Theorem 3.3.4, we have $f(k) = O(k \cdot (\log \log k)^6)$.

4.5.2 Signed planar graphs

For the class of signed planar simple graphs, the upper bound of 6 follows from the fact that these graphs are 5-degenerate. With our definition of circular chromatic number and development in this work, one may restate a conjecture of [MRŠ16] as to "the circular chromatic number of the class of signed planar simple graphs is 4". However, this conjecture is recently disproved in [KN21]. The first counterexample provided in [KN21] is essentially the subgraph $K_3(\mathcal{I})$ of the signed graph of Theorem 4.4.5. The work of [KN21] is based on the dual interpretation of the circular four-coloring of signed planar graphs. The examples build there then are based on non-Hamiltonian cubic bridgeless planar graphs. The underlying graph of the signed graph of Figure 4.10 is the dual of Tutte fragment used to build the first example of a non-Hamiltonian cubic bridgeless planar graph and referred to as Wenger graph in some literature. This graph itself is used as a building block in a number of coloring results. We note, furthermore, that since in Theorem 4.4.5 we give the exact value of the circular chromatic number of $K_4(\mathcal{I})$, one does not expect to improve the lower bound using this particular gadget.

It remains an open problem to decide the exact value of the circular chromatic number of the class of signed planar simple graphs or to improve the bounds (of $\frac{14}{3}$ and 6) from either direction.

4.5.3 Spectrum question

In the previous question one may also ask for the full possible range of circular chromatic number of a given family of signed graphs. For example, it is known [HZ00] that a rational number r is the circular chromatic number of a non-trivial K_4-minor-free graph if and only if $r \in [2, \frac{8}{3}] \cup \{3\}$. As for signed K_4-minor-free simple graphs, we extended the upper bound to $\frac{10}{3}$. In a recent paper [PZ22], it has been proved that every rational number between $\frac{8}{3}$ and $\frac{10}{3}$ is the circular chromatic number of a K_4-minor-free signed simple graph. Spectrum of the circular chromatic number of series-parallel graphs of given girth and circular chromatic number of planar graphs were studied in [Mos97; PZ02; PZ04; Zhu99b; Zhu99a]. Similar questions are interesting for signed planar graphs and other families of signed graphs.

4.5.4 Jaeger-Zhang conjecture and extensions

Note that $-C_{-\ell}$ is the signed cycle of length ℓ where the number of positive edges is odd. Then $\chi_c(-C_{-\ell}) = \frac{2\ell}{\ell-1}$. Recall that given a positive integer ℓ and a signed graph (G, σ) satisfying $g_{ij}(G, \sigma) \geqslant g_{ij}(-C_{-\ell})$ for $ij \in \mathbb{Z}_2^2$, $\chi_c(G, -\sigma) \leqslant \frac{2\ell}{\ell-1}$ if and only if $(G, \sigma) \to -C_{-\ell}$.

Note that when $\ell = 2k+1$, $-C_{-(2k+1)}$ is switching equivalent to C_{2k+1} and a signed graph (G, σ) satisfying $g_{ij}(G, \sigma) \geqslant g_{ij}(C_{2k+1})$ is equivalent to $(G, +)$; when $\ell = 2k$, $-C_{-2k}$ is switching equivalent to C_{-2k} and a signed graph (G, σ) satisfying $g_{ij}(G, \sigma) \geqslant g_{ij}(C_{-2k})$ is bipartite. We recall the following general question.

Question 1.4.4. *Given a positive integer $\ell \geqslant 2$, what is the smallest value $f(\ell)$ such that every signed planar graph (G, σ) of girth at least $f(\ell)$ has $\chi_c(G, \sigma) \leqslant \frac{2\ell}{\ell-1}$?*

We restrict our study to the class of signed planar graphs with all positive edges when $\ell = 2k+1$ and to the class of signed bipartite planar graphs when $\ell = 2k$.

Question 4.5.3. *For $\ell = 2k+1$, what is the smallest value $f_+(\ell)$ such that every signed planar graph $(G,+)$ of odd-girth at least $f_+(\ell)$ has $\chi_c(G,+) \leqslant \frac{2k+1}{k}$?*

As mentioned in Section 1.4, Jaeger-Zhang conjecture (Conjecture 1.4.3) claims that $f_+(\ell) = 4k+1$. That $f_+(3) = 5$ is a restatement of the well-known Grötzsch theorem. Also, $f_+(5) \leqslant 11$ [DP17; CL20] and $f_+(7) \leqslant 17$ [PS22; CL20]. A general lower bound $4k+1$ of $f_+(2k+1)$ is given in [Zha02] and the best upper bound till now follows from a flow result of [LTWZ13] that $f_+(2k+1) \leqslant 6k+1$.

Question 4.5.4. *For $\ell = 2k$, what is the smallest value $f_b(\ell)$ such that every signed bipartite planar graph (G,σ) of negative-girth at least $f_b(\ell)$ has $\chi_c(G,\sigma) \leqslant \frac{4k}{2k-1}$?*

The above question is a restatement of Question 1.5.3 and this is also the main question we will discuss in Part IV. It follows from a result of [CNS20] that $4k-2 \leqslant f_b(2k) \leqslant 8k-2$. It is obvious that $f_b(2) = 2$ since every signed bipartite graph admits a homomorphism to a digon. Furthermore, in Chapter 7, we prove that for every signed bipartite planar graph (G,σ) of negative-girth at least 4, $\chi_c(G,\sigma) < 4$. In Chapter 8, based on an edge-density result of C_{-4}-critical signed graphs using the potential technique and discharging method, we have that $f_b(4) = 8$. In Chapter 9, our result implies that $f_b(3) = 6$. The proof is based on an edge-coloring result of [DKK16], which in turn relied on an earlier work of B. Guenin [Gue03] and the 4-color theorem, and Grötzsch's theorem. Furthermore, recently we have proved that $f_b(6) \leqslant 14$ and $f_b(8) \leqslant 20$ in [LSWW22].

For any integer $\ell \geqslant 2$, it is known that the general answer $f(\ell)$ to Question 1.4.4 exists and is finite. As an example, a result of [CNS20] implies that every signed planar graph of girth at least 10 admits a homomorphism to (K_4, e) which is a signed graph on K_4 with only one negative edge. Their result implies that $f(3) \leqslant 10$ as $\chi_c(K_4, e) = 3$. In Chapter 10, we study the homomorphism of sparse signed graphs to (K_6, M) where the condition of $mad(G) < \frac{14}{5}$ is proved to be sufficient. Noting that $\chi_c(K_6, M) = 3$, as a corollary, we improve the upper bound of $f(3)$ to be 7. For other values, this question is still open.

Note that similar questions might be asked also for the class of signed K_5-minor-free graphs.

Part III

Circular Flows in Signed Graphs

5 | Circular flows in mono-directed signed graphs

This chapter is based on the following paper:

[LNWZ22] J. Li, R. Naserasr, Z. Wang, and X. Zhu. "Circular flow in mono-directed signed graphs". In: *In preparation* (2022)

5.1 Introduction

Following Tait's reformulation of the 4-color theorem, W.T. Tutte [Tut54] introduced the concept of nowhere-zero integer flows, which is dual to the concept of proper vertex coloring when restricted to planar graphs. In 1988, A. Vince [Vin88] introduced a natural refinement of proper vertex coloring of graphs, which is now called the *circular coloring*. The dual notion, *circular flows* in graphs, was introduced by L.A. Goddyn, M. Tarsi, and C.Q. Zhang [GTZ98] in 1998.

In Part II, we have extended the notion of the circular coloring of graphs to signed graphs and studied the circular chromatic number of some families of signed graphs. In this chapter, based on the coloring-flow duality in signed planar graphs, we introduce the dual concept: circular flows in mono-directed signed graphs.

Definition 5.1.1. Given a signed graph (G, σ) and a real number r, a *circular r-flow* in (G, σ) is a pair (D, f) where D is an orientation on G and $f : E(G) \to (-r, r)$ satisfies the followings.

- For each positive edge e of (G, σ), $|f(e)| \in [1, r - 1]$.
- For each negative edge e of (G, σ), $|f(e)| \in [0, \frac{r}{2} - 1] \cup [\frac{r}{2} + 1, r)$.
- For each vertex v of (G, σ), its total out-flow equals the total in-flow in the orientation D, i.e.,

$$\sum_{(v,w)\in D} f(vw) = \sum_{(u,v)\in D} f(uv).$$

This definition naturally extends the concept of circular flows from graphs to signed graphs. Considering a graph as a signed graph $(G, +)$, this is indeed the concept of the circular flow in graphs defined in [GTZ98]. Since the orientation will not influence of the existence of the circular r-flow in a signed graph, we may take $f(e) \geqslant 0$ for the simplicity.

We need to mention that there is another classical and commonly-accepted notion of flows that has been largely studied on signed graphs, using bidirected edges [Bou83; RZ11]. The bi-direction idea was motivated by the dual of tension on graphs embedded on non-orientable surfaces. In such an approach, each edge of a signed graph is assigned with two arrows, one for each end, in such a way that the two arrows on a positive edge are in the same direction, and the two arrows on a negative edge are in opposite directions. The condition at each vertex is the same: the total in-flow must be equal to the total out-flow. However, one should note that in this line of study, negative edges behave like a "black hole", meaning that a negative edge either creates a flow out of nothing or absorbs the in-flow.

In our definition, each edge of a signed graph is given a single direction (i.e., a single arrow), just as in the classic sense of flows in graphs. That is the reason why we call it circular flow in mono-directed signed graphs. Our definition of the circular flow in signed graphs is the dual notion to circular coloring and it generalizes the flow-coloring duality of Tutte [Tut49] and Goddyn-Tarsi-Zhang [GTZ98] on planar graphs. Given a signed plane graph (G, σ), the *dual signed graph* of (G, σ) is the signed plane graph (G^*, σ^*) defined as follows: G^* is the dual graph of the underlying graph G and $\sigma^*(e^*) = \sigma(e)$ for each edge $e^* \in E(G^*)$ where e^* is the dual edge of e.

Theorem 5.1.2. *For a signed planar graph (G, σ) and its dual signed graph (G^*, σ^*), (G, σ) admits a circular r-coloring if and only if (G^*, σ^*) admits a circular r-flow.*

The organization of this chapter is as follows. In Section 5.2, we introduce an operation called "inversing" on signed graphs, as a dual notion of the switching operation and prove some basic properties of signed graphs under inversing. In Section 5.3, with transforming the circular coloring of signed graphs to the tension on signed graphs, we show the duality between the concept of circular flow and circular coloring of signed graphs. Similar to circular flows in graphs, in Section 5.4, we introduce several equivalent definitions of circular flows in signed graphs. Especially, by Tutte's lemma, in the circular $\frac{p}{q}$-flow, we may use values from the group $\mathbb{Z}_p$ instead of $\mathbb{Z}$. In Section 5.4.4, we discuss how several graph operations influence the circular flow index of the resulting signed graphs. In Section 5.5, we introduce the modulo ℓ-orientation on signed graphs and characterize the existence of the modulo ℓ-orientations on two classes of signed graphs. Furthermore, on those two special classes, we indicate the relation between the modulo ℓ-orientation and homomorphism to cycles.

5.2 Switching and inversing

A key concept that separates a signed graph from a 2-edge-colored graph is the switching operation. The circular chromatic number (also, the existence of the circular coloring) in Definition 3.1.1, as well as many other concepts in signed graphs, are invariant under switching. Recall that two signed graphs are said to be switching equivalent if one can be obtained from the other by switching on some vertices, in other words, when the symmetric difference of the negative edges of two signed graphs is an edge-cut. In planar graphs, the dual of an edge-cut is a cycle, or more generally, an even-degree subgraph.

To capture the duality between colorings and flows, we define a new operation on the cycles of signed graphs. Given a signed graph (G, σ) and a cycle C of (G, σ), an *inversing on the cycle C* is to multiply the signs of all the edges of C by $-$. Repeating this operation on a number of cycles then is to multiply the signs of all edges of an even-degree subgraph by $-$. Two signed graphs (G, σ) and (G, σ') are said to be *inversing equivalent* if one can be obtained from the other by inversing on some cycles, in other words, when the symmetric difference of the negative edges of two signed graphs is an even-degree subgraph. We observe that the inversing-equivalent relation is an equivalence relation on the set of all signatures on G.

Noting that the sign of a cycle is invariant under switching, similarly, as inversing only applies on the edges of a cycle, the sign of a cut (defined to be the product of the signs of its edges) remains invariant under inversing. Thus, for a given cut, there are only two inversing equivalent signatures on it. Thus, depending on the parity of the number of negative edges of a cut and the sign of a cut, we have four types of cuts: *type* 00 is a positive cut that has an even number of edges, *type* 01 is a positive cut that has an odd number of edges, *type* 10 is a negative cut that has an even number of edges, and *type* 11 is a negative cut that has an odd number of edges. For a given signed graph (G, σ), we define $C_{ij}(G, \sigma)$ to be the size (i.e., the number of edges) of a smallest cut of (G, σ) of type ij where $ij \in \mathbb{Z}_2^2$. When there is no such a cut, we write $C_{ij}(G, \sigma) = \infty$. A first fact, easy to observe, is the following.

Lemma 5.2.1. *Given a signed graph (G, σ), and cuts (X, X^c) of type ij and (X', X'^c) of type $i'j'$ for $ij, i'j' \in \mathbb{Z}_2^2$, the symmetric difference of (X, X^c) and (X', X'^c) is a cut of type $ij + i'j'$.*

In the next lemma, similar to Lemma 2.2.1, we show that the signs of cuts uniquely determine the inversing-equivalence class to which a signed graph belongs.

Lemma 5.2.2. *Two signed graphs (G, σ) and (G, σ') are inversing equivalent if and only if they have the same set of negative cuts.*

Proof. We have already observed that if (G, σ) and (G, σ') are inversing equivalent, then each cut has the same sign in these two signed graphs. For the inverse, assume that the sign of each cut in (G, σ) is the same as its sign in (G, σ'). In particular, it is the case for the cuts of the form $(\{v\}, V(G)\backslash\{v\})$ for every $v \in V(G)$. Hence, if we take the symmetric difference S of the sets of negative edges in (G, σ) and (G, σ'), then the subgraph induced by these edges will be of even degree on each vertex. Therefore, (G, σ') is obtained from (G, σ) by inversing on the even-degree subgraph induced by S. □

We note that in applying this lemma, one does not need to verify the condition for all the cuts. It is rather enough to verify it for a basis of the cut-space, that is the binary vector space consisting of all the cuts with the operation being the symmetric difference. A particular case would be to consider all but one of the cuts $(\{v\}, V(G)\backslash\{v\})$ from each connected component. We have the following result as a corollary.

Corollary 5.2.3. *Given a graph G on n vertices with c connected components, there are 2^{n-c} distinct inversing-equivalent classes of signed graphs on G.*

Next we show that given a signed graph (G, σ), one can apply an inversing so that the set of negative edges are all among edges of a spanning tree of G.

Lemma 5.2.4. *Let G be a connected graph and let T be a spanning tree of G. Given a signed graph (G, σ), there exists an inversing-equivalent signed graph (G, σ') of (G, σ) such that all the negative edges are in $E(T)$.*

Proof. For each edge $e \notin T$, let C_e be the unique cycle in the graph obtained from adding e to T. If e is a negative edge in (G, σ), then apply a switching on C_e. After applying this process on all the edges of $E(G\backslash T)$, we obtain a signed graph (G, σ') in which each edge in $E(G\backslash T)$ is positive, proving this lemma. □

A key example in the study of signed graphs equipped with the inversing operation is the dual of C_{-k} and we denote it by C^*_{-k}. That is a signed graph on kK_2 with the product of the signs of the edges being negative. Observe that $C_{00}(C^*_{-k}) = 0$, $C_{01}(C^*_{-k}) = \infty$ and that of the two values of

$C_{11}(C^*_{-k})$ and $C_{10}(C^*_{-k})$, depending on the parity of k, one is k and the other is ∞. There are two sub-classes of signed graphs then proved to be of special importance in the study of circular flow in signed graphs.

- A signed graph (G, σ) satisfying that $C_{ij}(G, \sigma) \geqslant C_{ij}(C^*_{-(2\ell+1)})$, for any $ij \in \mathbb{Z}_2^2$, is inversing equivalent to $(G, -)$.
- A signed graph (G, σ) satisfying that $C_{ij}(G, \sigma) \geqslant C_{ij}(C^*_{-2\ell})$, for any $ij \in \mathbb{Z}_2^2$, has all of its vertices to be of even degrees.

Note that the restriction on the first class captures the classic graphs and the second class is the class of even-degree (Eulerian if connected) signed graphs.

5.3 Flow-coloring duality

In this section, we use the notion of tension on signed graphs to build the connection between the circular coloring of signed graphs and circular flow in signed graphs.

5.3.1 Tension

Given a cycle C and a direction (clockwise or anti-clockwise) of C, we denote the set of forward edges by C^F and the set of backward edges by C^B. Extending the notion of r-tensions from graphs to signed graphs, we have the following.

Definition 5.3.1. Given a signed graph (G, σ) and a real number r, an *r-tension* of (G, σ) is a pair (D, ϕ) where D is an orientation on G and $\phi : E(D) \to [0, r)$ satisfies the following conditions.

- For each positive edge e of (G, σ), $\phi(e) \in [1, r-1]$.
- For each negative edge e of (G, σ), $\phi(e) \in [0, \frac{r}{2} - 1] \cup [\frac{r}{2} + 1, r)$.
- For any cycle C of (G, σ), $\sum_{e \in C^F} \phi(e) = \sum_{e \in C^B} \phi(e)$.

It is clear that there is a natural one-to-one correspondence between the circular r-coloring of (G, σ) and the r-tension on (G, σ). This is formally proved in the next lemma.

Lemma 5.3.2. *A signed graph (G, σ) admits a circular r-coloring if and only if (G, σ) admits an r-tension.*

Proof. Let (G, σ) be a signed graph and let φ be a circular r-coloring of (G, σ). An r-tension, based on φ, is defined as follows: For each edge xy of (G, σ), if $\varphi(x) < \varphi(y)$, we orient it as (x, y) and if $\varphi(x) = \varphi(y)$, we orient it arbitrarily. Let D be the resulting orientation on G. Define $\phi : E(G) \to [0, r)$ such that for each $e = (x, y) \in D$, $\phi(e) = \varphi(y) - \varphi(x)$. Thus such a pair (D, ϕ) is an r-tension on (G, σ).

It remains to show that if (G, σ) has an r-tension, then it admits a circular r-coloring. Without loss of generality, we assume that G is connected. Assume that (G, σ) has an r-tension (D, ϕ). We define a vertex mapping $\varphi : V(G) \to \mathbb{R}$ as follows: choose an arbitrary vertex v^* and define $\varphi(v^*) = 0$. For each other vertex u, let P be a walk from v^* to u and let

$$\varphi(u) = \sum_{e \in P^F} \phi(e) - \sum_{e \in P^B} \phi(e) \pmod{r}.$$

where P^F and P^B are the sets of forward and backward edges in P, respectively. Since $\phi(C^F) - \phi(C^B) = 0$ holds for every cycle, and thus every closed walk, φ is well-defined. That is to say, it does not depend on the choice of the v^*u-walk. It is easy to verify that φ is a circular r-coloring of (G, σ). □

Note that the third cycle condition of Definition 5.3.1 is crucial. Interchanging the roles of cycles and cuts (especially, $(\{v\}, V(G)\backslash\{v\})$), we obtain the concept of circular flows in mono-directed signed graphs. We have seen this in Theorem 5.1.2.

5.3.2 Application of Hoffman's circulation theorem

A natural question that follows the definition of an r-tension is that: Does one need to verify the conditions on all the cycles of G? That is indeed not necessary for most signed graphs. The research for providing a minimal set of cycles that is sufficient to check is ongoing. Here we present an extension from one of the basic results from graphs to signed graphs. Recall that in this work, the role of the signature is to give different limits on the values assigned to positive and negative edges. A more general setting would be to have a certain limit on each of edges. A necessary and sufficient condition for the existence of a flow satisfying such limits is given by A.J. Hoffman in 1960.

Theorem 5.3.3. [Hoffman's circulation theorem] [Hof60] *Let G be a digraph and let $s, t : A(G) \to \mathbb{R}_{\geqslant 0}$ satisfy $s(e) \leqslant t(e)$ for each $e \in A(G)$. Then there exists a function $f : E \to \mathbb{R}_{\geqslant 0}$ satisfying that $\sum\limits_{(u,v)\in A(G)} f(uv) = \sum\limits_{(v,w)\in A(G)} f(wv)$ with $s(e) \leqslant f(e) \leqslant t(e)$ for each $e \in A(G)$ if and only if*

$$\sum_{(u,v)\in A(G), u\in U, v\notin U} s(uv) \leqslant \sum_{(x,y)\in A(G), x\notin U, y\in U} t(xy) \qquad \forall U \subset V(G).$$

When considering the circular r-flows in signed graphs, we have the following corollary. Given a cut C of (G, σ), we denote the set of forward edges of C by C^F and the set of backward edges of C by C^B.

Lemma 5.3.4. *A signed graph $\hat{G}$ admits a circular r-flow if and only if there is an orientation D and a partition $\pi = (E_1^-, E_2^-)$ of $E_{\hat{G}}^-$ such that for any cut (X, X^c) of $\hat{G}$,*

$$\sum_{u\in X, v\in X^c, (u,v)\in D} s(uv) \leqslant \sum_{x\in X^c, y\in X, (x,y)\in D} t(xy)$$

where $$s(e) = \begin{cases} 1, & \text{if } e \in E_{\hat{G}}^+, \\ 0, & \text{if } e \in E_1^-, \\ \frac{r}{2}+1, & \text{if } e \in E_2^-, \end{cases} \quad \text{and} \quad t(e) = \begin{cases} r-1, & \text{if } e \in E_{\hat{G}}^+, \\ \frac{r}{2}-1, & \text{if } e \in E_1^-, \\ r, & \text{if } e \in E_2^-. \end{cases}$$

Applying the same technique to the tension rather than the flow, with the same function $s(e)$ and $t(e)$ defined above, we have the following conclusion. We leave the details to curious readers.

Lemma 5.3.5. *A signed graph $\hat{G}$ has an r-tension if and only if there is an orientation D and a partition $\pi = (E_1^-, E_2^-)$ of $E_{\hat{G}}^-$ such that for any cycle C of $\hat{G}$, $\sum\limits_{e\in C^F} s(e) \leqslant \sum\limits_{e\in C^B} t(e)$.*

5.4 Basic properties and equivalent definitions

In this section, we give basic properties of circular r-flow in signed graphs. Most of these properties are direct extensions from graphs. Thus we may omit some proofs.

Recall that the third condition in Definition 5.1.1 implies that given a signed graph (G, σ), if there is a circular r-flow (D, f) in (G, σ), then for any cut (U, U^c) of G,

$$\sum_{(v,w)\in D, \forall v\in U, w\in U^c} f(vw) = \sum_{(u,v)\in D, \forall v\in U, u\in U^c} f(uv). \tag{5.1}$$

The first observation is about characterizing the class of signed graphs that admits a circular r-flow for some finite r, following from the above observation. This is dual to the fact that the circular chromatic number is finite for the class of signed graphs without positive loops.

Proposition 5.4.1. *A signed graph (G, σ) admits a circular r-flow for some finite r if and only if it has no positive bridge.*

Note that the inversing operation never changes the sign of a bridge, that is dual to the fact that a switching never changes the sign of a loop. Another basic property is the following.

Proposition 5.4.2. *If a signed graph (G, σ) admits a circular r-flow, then (G, σ) admits a circular r'-flow for every $r' \geqslant r$.*

It is because that given real values r' and r where $r' \geqslant r$, if (D, f) is a circular r-flow in (G, σ), then naturally $(D, \frac{r'}{r} f)$ is a circular r'-flow in (G, σ).

Given a signed graph (G, σ) without a positive bridge, the *circular flow index* of (G, σ), denoted by $\Phi_c(G, \sigma)$, is the smallest rational number r such that (G, σ) admits a circular r-flow, i.e.,

$$\Phi_c(G, \sigma) = \inf\{r \mid (G, \sigma) \text{ admits a circular } r\text{-flow}\}.$$

Observation 5.4.3. *Given a graph G, a signature σ on G and a real number r, if G admits a circular r-flow, then (G, σ) admits a circular $2r$-flow.*

5.4.1 Circular modulo r-flow

We have another geometrical interpretation of circular coloring. Given a circular r-coloring φ of a signed graph (G, σ), for each edge uv of (G, σ), $\varphi(u)$ and $\varphi(v)$ partition the circle C^r into two segments and the shorter one is of length $d_{(\text{mod } r)}(\varphi(u), \varphi(v))$. This leads to the following equivalent and mainstream definition of *circular modulo r-flow*. The equivalence between this notion and the circular r-flow would be indicated later by using Tutte's lemma.

Definition 5.4.4. Given a signed graph (G, σ) and a real number $r \geqslant 2$, a *circular modulo r-flow* in (G, σ) is a pair (D, f) where D is an orientation on G and $f : E(G) \to [0, r)$ satisfies the followings.

- For each positive edge e of (G, σ), $f(e) \in [1, r-1]$.
- For each negative edge e of (G, σ), $f(e) \in [0, \dfrac{r}{2} - 1] \cup [\dfrac{r}{2} + 1, r)$.
- For each vertex v of (G, σ), there is an integer a such that $\sum_{(v,w)\in D} f(vw) - \sum_{(u,v)\in D} f(uv) = ar$.

In this view, edges of (G, σ) are oriented and each is given a value from the group $\mathbb{R}/[0, r)$. To observe that the existence of a circular modulo r-flow is independent of the choice of the orientation D, one notes that if the orientation of an edge e is reversed, then it is enough to change the value of $f(e)$ to its inverse value in the group $\mathbb{R}/[0, r)$, that is $r - f(e)$ when $[0, r)$ is used to represent the elements of this group.

Note that if (G, σ) admits a circular modulo r-flow (D, f), then it also admits a circular modulo r-flow (D', f') such that $f'(e) \in [1, \frac{r}{2}]$ for each positive edge e and $f'(e) \in [0, \frac{r}{2} - 1]$ for each negative edge e. We simply define f' as follows: $f'(e) = r - f(e)$ if for positive edge e, $f(e) \in (\frac{r}{2}, r - 1]$ and if for negative edge e, $f(e) \in [\frac{r}{2} + 1, r)$, and $f'(e) = f(e)$ otherwise. Flip those oriented positive edges satisfying that $f(e) \in (\frac{r}{2}, r - 1]$ and those oriented negative edges satisfying that $f(e) \in [\frac{r}{2} + 1, r)$ and obtain D'.

We show that the existence of a circular modulo r-flow in a signed graph (G, σ), and thus the value of circular flow index of (G, σ), is independent of the inversing operation.

Proposition 5.4.5. *Let (G, σ) and (G, σ') be two inversing-equivalent signed graphs. Then every circular modulo r-flow in (G, σ) corresponds to a circular modulo r-flow in (G, σ').*

Proof. To see this, let (D, f) be a circular modulo r-flow in (G, σ) and let $\hat{C}$ be a signed cycle of (G, σ). Let σ' be the signature of G obtained from σ after an inversing on $\hat{C}$. We define f' as follows:

$$f'(e) = \begin{cases} f(e), & \text{if } e \notin C, \\ |\frac{r}{2} - f(e)|, & \text{if } e \in C. \end{cases}$$

We define an orientation D' as follows: If $f(e) \geqslant \frac{r}{2}$, then e is oriented in D' as same in D and otherwise, e is oriented in D' as opposite in D. It is easily verified that (D', f') satisfies all the conditions of being a circular modulo r-flow in (G, σ'). □

Later we will show the equivalence between the circular r-flow and the circular modulo r-flow for rational r. Thus we note that the circular flow index being independent of the inversing operation is dual to the circular chromatic number being independent of the switching operation.

5.4.2 Tight cut and rationality of the circular flow index

Note that the existence of the circular r-flow (or circular modulo r-flow) (D, f) is independent from the choice of the orientation, sometimes, for convenience, we may choose an orientation such that all the values $f(e)$ are all non-negative. We call such a flow a *non-negative circular r-flow* (or a *non-negative circular modulo r-flow*).

Now we introduce the concept of *tight cut*. Given a signed graph (G, σ) and a non-negative circular r-flow (D, f) of (G, σ), a positive edge e is said to be *tight* if either $f(e) = 1$ or $f(e) = r - 1$, and a negative edge e is said to be *tight* if either $f(e) = \frac{r}{2} - 1$ or $f(e) = \frac{r}{2} + 1$. Furthermore, a cut (X, X^c) of (G, σ) is said to be *tight* with respect to (D, f) if for each edge $e = uv$ where $u \in X, v \in X^c$, it satisfies that

$$f(e) = \begin{cases} 1, & \text{if } e \text{ is a positive edge and } (u, v) \in D, \\ r - 1, & \text{if } e \text{ is a positive edge and } (v, u) \in D, \\ \frac{r}{2} + 1, & \text{if } e \text{ is a negative edge and } (u, v) \in D, \\ \frac{r}{2} - 1, & \text{if } e \text{ is a negative edge and } (v, u) \in D. \end{cases}$$

The concept of tight cut in the circular flow is the dual notion of tight cycle in the circular coloring. An analogue of the ideas from circular coloring then can be employed to prove that the circular flow index of a (finite) signed graph is a rational number and that the infimum in the definition is attained. The main idea is stated in the next lemma for which we only provide a sketch proof.

Lemma 5.4.6. *Given a signed graph (G, σ), $\Phi_c(G, \sigma) = r$ if and only if (G, σ) admits a circular r-flow and for any circular r-flow (D, f), (G, σ) has a tight cut with respect to (D, f).*

Proof. We first prove the "only if" part. Assume that $\Phi_c(G, \sigma) = r$ but (G, σ) has no tight cut with respect to some non-negative circular r-flows. We choose such a circular r-flow (D, f) with minimum number of tight edges.

First we claim that there is no tight edge in (G, σ) with respect to (D, f). Otherwise, let uv be a tight edge. As (G, σ) has no tight cut with respect to (D, f), we can find a path P from u to v consisting of non-tight edges. Then by shifting the values along all edges of cycle $C = P + uv$ by a sufficiently small amount, we get a new circular r-flow in which e is no longer a tight edge and no new tight edge is created. It is a contradiction to the choice of (D, f), proving this claim. Since there is no tight edge in (G, σ) with respect to (D, f), for some positive ϵ, $(D, \frac{1}{1+\epsilon} f)$ is also a circular r-flow in (G, σ), a contradiction.

For the "if" part, we need to show that if $\Phi_c(G, \sigma) < r$, then there is a circular r-flow in (G, σ) such that there is no tight cut. Assume $\Phi_c(G, \sigma) = r' < r$ and let (D, f') be a non-negative circular r'-flow in (G, σ). Let $f = \frac{r}{r'} f'$. Then (D, f) is a non-negative circular r-flow in (G, σ) and (G, σ) contains no tight edge with respect to (D, f). □

Assume that $\Phi_c(G, \sigma) = r$ and (D, f) is a circular r-flow in (G, σ). By Lemma 5.4.6, there exists a tight cut (X, X^c) with respect to (D, f). Assume that in $E(X, X^c)$, there are s_1 positive edges e having $f(e) = r - 1$, s_2 positive edges e having $f(e) = 1$, t_1 negative edges having $f(e) = \frac{r}{2} - 1$, and t_2 negative edges having $f(e) = \frac{r}{2} + 1$. According to Condition (5.1), we have that

$$s_1(r - 1) + t_1(\frac{r}{2} - 1) = s_2 + t_2(\frac{r}{2} + 1). \tag{5.2}$$

Thus, we can determine the circular flow index r of (G, σ):

$$r = \frac{2(s_1 + s_2 + t_1 + t_2)}{2s_1 + t_1 - t_2} = \frac{2|(X, X^c)|}{2s_1 + t_1 - t_2}. \tag{5.3}$$

As s_1, s_2, t_1 and t_2 are all non-negative integers, we have the following result.

Theorem 5.4.7. *For every finite signed graph (G, σ) with no positive bridge, the circular flow index $\Phi_c(G, \sigma)$ is a rational number and is the minimum over all r such that (G, σ) admits a circular r-flow, i.e.,*

$$\Phi_c(G, \sigma) = \min\{\frac{p}{q} \mid (G, \sigma) \text{ admits a circular } \frac{p}{q}\text{-flow}\}.$$

Computing $\Phi_c(G, \sigma)$

Observe that in Formula (5.2), the values of s_1, s_2, t_1 and t_2 are all bounded by the number of edges of G. Therefore, Formula (5.3) limits the possible choices of $\Phi_c(G, \sigma)$ to a rational number whose numerator and denominator each is bounded by $2|E(G)|$.

From this discussion and also from the fact that given an orientation D of a signed graph (G, σ), verifying if (D, f) is a circular r-flow in (G, σ) can be done in a polynomial time in size of the inputs, it follows that:

Theorem 5.4.8. *Given a signed graph (G, σ) and a rational number r, the problem of determining whether $\Phi_c(G, \sigma) \leqslant r$ is in the class of decidable problems and is in fact in the class NP.*

However, the problem in an NP-hard problem and thus one would not expect an algorithm for the general class of signed graphs which runs in a sub-exponential time. That the problem is NP-hard follows from duality to the circular coloring problem. In particular, the problem includes the question of 3-colorability of planar graphs which is among the well-known NP-hard problems.

5.4.3 Real-valued $\frac{p}{q}$-flow and modulo $\frac{p}{q}$-flow

In the previous discussion, we have observed that the circular flow index of any finite signed graph with no positive bridge is a rational number. With a further attention to the details, we observe that in fact, given $r = \frac{p}{q}$, in defining a circular r-flow in a signed graph, we may restrict ourselves to the values of the form $\frac{i}{q}$, for $i \in \{0, 1, \dots, p-1\}$. Then multiplying all values by q, we may work with integer values. Thus we have the following definition, where the set of solutions is restricted, but the existence of a solution is equivalent to the existence of one in the previous definitions and thus it can be equivalently employed to obtain the circular flow index of a given signed graph.

Definition 5.4.9. Given an even integer p and an integer q where $q \leqslant \frac{p}{2}$, a $\frac{p}{q}$*-flow* of a signed graph (G, σ) is a pair (D, f) where D is an orientation on G and $f : E(G) \to \mathbb{Z}$ satisfies the followings.

- For each positive edge e of (G, σ), $|f(e)| \in \{q, \dots, p-q\}$.
- For each negative edge e of (G, σ), $|f(e)| \in \{0, \dots, \dfrac{p}{2} - q\} \cup \{\dfrac{p}{2} + q, \dots, p-1\}$.
- For each vertex v of (G, σ), $\sum\limits_{(v,w) \in D} f(vw) = \sum\limits_{(u,v) \in D} f(uv)$.

Our definitions so far are restricted to the real values. We next define the *modulo* $\frac{p}{q}$*-flow*, which is equivalent to the $\frac{p}{q}$-flow in Definition 5.4.9. The equivalence follows directly from Tutte's lemma (Lemma 2.1.1). We note that in the statement of Tutte's lemma there is no restriction on the values $f(e)$, in particular, $f(e) = 0$ is a possibility even though it is normally applied to nowhere-zero flows.

Definition 5.4.10. Given an even integer p and an integer q where $q \leqslant \frac{p}{2}$, a *modulo* $\frac{p}{q}$*-flow* of (G, σ) is a pair (D, f) where D is an orientation on G and $f : E(G) \to \mathbb{Z}_p$ satisfies the followings.

- For each positive edge e of (G, σ), $|f(e)| \in \{q, \dots, p-q\}$.
- For each negative edge e of (G, σ), $|f(e)| \in \{0, \dots, \dfrac{p}{2} - q\} \cup \{\dfrac{p}{2} + q, \dots, p-1\}$.
- For each vertex v of (G, σ), $\sum\limits_{(v,w) \in D} f(vw) \equiv \sum\limits_{(u,v) \in D} f(uv) \pmod{p}$.

It follows from the discussion above that: Given a signed graph (G, σ) and a rational number $r = \frac{p}{q}$ where p is even and $p \geqslant 2q$, the following claims are equivalent:

(1) (G, σ) admits a circular r-flow.

(2) (G, σ) admits a $\frac{p}{q}$-flow.

(3) (G, σ) admits a modulo $\frac{p}{q}$-flow.

(4) (G, σ) admits a circular modulo r-flow.

Note that we shall use the modulo $\frac{p}{q}$-flow concept most frequently when we determine the circular flow index of a given signed graph.

5.4.4 Operations on signed graphs

Assume $\mathcal{I} = (\Gamma, u, v)$ is a signed indicator and $r \geqslant 2$ is a real number. Let $\mathcal{I}_{uv}$ be obtained from $\mathcal{I}$ by adding an edge (without any signature) between u, v. Note that it might be a multi-edge but with a slight abuse of the notation, we use uv to denote this edge.

Definition 5.4.11. Given a signed indicator $\mathcal{I} = (\Gamma, u, v)$ and a real number r, for $x \in [0, r)$, we say the value x is *feasible for* $\mathcal{I}$ with respect to r if there is a pair (D, ϕ) where D is an orientation on $\mathcal{I}_{uv}$ and $\phi : E(\mathcal{I}_{uv}) \to [0, r)$ satisfies the following conditions:

- For each positive edge e of Γ, $\phi(e) \in [1, r-1]$;
- For each negative edge e of Γ, either $\phi(e) \in [0, \frac{r}{2} - 1]$ or $\phi(e) \in [\frac{r}{2} + 1, r)$;
- For the additional edge uv, $\phi(uv) = x$;
- For each vertex w of Γ, there is an integer a such that $\sum\limits_{(z,w)\in D} \phi(zw) - \sum\limits_{(w,y)\in D} \phi(wy) = ar$.

Note that since the calculation is taken modulo r, if x is feasible for $\mathcal{I}_{uv}$, then $r - x$ is also feasible for $\mathcal{I}_{uv}$. Thus we only consider $x \in [0, \frac{r}{2}]$.

Definition 5.4.12. Given a signed indicator $\mathcal{I} = (\Gamma, u, v)$ and a real number $r \geqslant 2$, let

$$Z^*(\mathcal{I}, r) = \{x \in [0, \frac{r}{2}] \mid x \text{ is feasible for } \mathcal{I} \text{ with respect to } r\ \}.$$

For $\mathcal{I} = (\Gamma, u, v)$, as $\mathcal{I}_{uv}$ is not a signed graph, with a slight abuse of the notation, we define $\Phi_c(\mathcal{I}_{uv}) := \min\{r \mid Z^*(\mathcal{I}, r) \neq \varnothing\}$. Note that $\Phi_c(\mathcal{I}_{uv})$ is not necessary to be the same as $\Phi_c(\Gamma)$.

Assume that $\mathcal{I} = (\Gamma, u, v)$ is a signed indicator.

- $\mathcal{I}$ is said to be a *plus flow indicator* if for each $r \geqslant \Phi_c(\mathcal{I}_{uv})$, there is a value $f(r)$ such that $Z^*(\mathcal{I}, r) = [f(r), \frac{r}{2}]$.
- $\mathcal{I}$ is said to be a *minus flow indicator* if for each $r \geqslant \Phi_c(\mathcal{I}_{uv})$, there is a value $g(r)$ such that $Z^*(\mathcal{I}, r) = [0, \frac{r}{2} - g(r)]$.
- $\mathcal{I}$ is said to be a *plus-minus flow indicator* if for each $r \geqslant \Phi_c(\mathcal{I}_{uv})$, there is a value $h(r)$ such that $Z^*(\mathcal{I}, r) = [h(r), \frac{r}{2} - h(r)]$.

Proposition 5.4.13. *Given a plus flow, minus flow, or plus-minus flow indicator $\mathcal{I} = (\Gamma, u, v)$, the corresponding function, f, g, or h, defined on $[\Phi_c(\mathcal{I}_{uv}), +\infty)$, is a continuous non-increasing function. In particular, for r large enough, the value of each function at r will be* 0.

The following three lemmas are dual to Lemmas 3.3.9, 3.3.10 and 3.3.11. So we omit the details of the proofs.

Lemma 5.4.14. *Assume that $\mathcal{I} = (\Gamma, u, v)$ is a plus flow indicator and r, r' are real numbers satisfying that $r > \Phi_c(\mathcal{I}_{uv})$ and $r' > 2$. If $Z^*(\mathcal{I}, r) = [\frac{r}{r'}, \frac{r}{2}]$, then given a graph G,*

$$\Phi_c(G(\mathcal{I})) = r \quad \textit{if and only if} \quad \Phi_c(G) = r'.$$

Lemma 5.4.15. *Assume that $\mathcal{I} = (\Gamma, u, v)$ is a plus-minus flow indicator and r, r' are real numbers satisfying that $r > \Phi_c(\mathcal{I}_{uv})$ and $r' > 2$. If $Z^*(\mathcal{I}, r) = [\frac{r}{2r'}, \frac{r}{2} - \frac{r}{2r'}]$, then given a graph G,*

$$\Phi_c(G(\mathcal{I})) = r \quad \textit{if and only if} \quad \Phi_c(G) = r'.$$

Lemma 5.4.16. *Assume $\mathcal{I}^+ = (\Gamma_+, u_1, v_1)$ is a plus flow indicator and $\mathcal{I}^- = (\Gamma_-, u_2, v_2)$ is a minus flow indicator. Assume that $r > \max\{\Phi_c(\mathcal{I}^+_{u_1v_1}), \Phi_c(\mathcal{I}^-_{u_2v_2})\}$ and $r' > 2$ are two real numbers and let $t = \frac{r}{r'}$. If $Z^*(\mathcal{I}_+, r) = [t, \frac{r}{2}]$ and $Z^*(\mathcal{I}_-, r) = [0, \frac{r}{2} - t]$, then given a signed graph Ω,*

$$\Phi_c(\Omega(\mathcal{I}_+, \mathcal{I}_-)) = r \text{ if and only if } \Phi_c(\Omega) = r'.$$

Let $\hat{P}_2^2$ be a signed multi-path on three vertices u, w, v, and between u, w there are two parallel edges of the same sign, between w, v there are two parallel edges of different signs.

Example 5.4.17. Assume that $\mathcal{I} = (\Gamma, u, v)$ is a signed indicator.

(1) If Γ is a signed $2K_2$ with one positive edge and one negative edge connecting two vertices u and v, then $\mathcal{I}$ is a plus flow indicator and $Z^*(\mathcal{I}, r) = [2 - \frac{r}{2}, \frac{r}{2}]$ for $2 \leqslant r < 4$ and $Z^*(\mathcal{I}, r) = [0, \frac{r}{2}]$ for $r \geqslant 4$.

(2) If Γ is a signed $2K_2$ with both positive edges connecting two vertices u and v, then $\mathcal{I}$ is a minus flow indicator and $Z^*(\mathcal{I}, r) = [0, r-2]$ for $2 \leqslant r < 4$ and $Z^*(\mathcal{I}, r) = [0, \frac{r}{2}]$ for $r \geqslant 4$.

(3) If Γ is a signed multi-path $\hat{P}_2^2$ connecting two vertices u and v, then $\mathcal{I}$ is a plus-minus flow indicator and $Z^*(\mathcal{I}, r) = [2 - \frac{r}{2}, r-2]$ for $\frac{8}{3} \leqslant r < 4$ and $Z^*(\mathcal{I}, r) = [0, \frac{r}{2}]$ for $r \geqslant 4$.

(4) If Γ is a signed 2-path connecting two vertices u and v with one positive edge and one negative edge, then $\mathcal{I}$ is a plus-minus flow indicator and $Z^*(\mathcal{I}, r) = [1, \frac{r}{2} - 1]$ for $r \geqslant 4$.

Recall that $T_\ell(G, \sigma)$ is a signed graph obtained from (G, σ) by replacing each edge e with an ℓ-path of signature $-\sigma(e)$. Since here we work with inversing-equivalent signed graphs, we need to indicate which signature we choose for the T_ℓ-construction. Especially, for $T_2(G, +)$, we choose the signed 2-path to consist of one positive edge and one negative edge. Considering the signed indicator of Example 5.4.17 (4) and applying Lemma 5.4.15, we obtain the following result.

Lemma 5.4.18. *Given two integers p and q satisfying that $\frac{p}{q} \geqslant 2$, a graph G admits a circular $\frac{p}{q}$-flow if and only if $T_2(G, +)$ admits a circular $\frac{2p}{q}$-flow.*

We denote by ℓK_2^o (or ℓK_2^e) a signed ℓK_2 which contains an odd (or, respectively, even) number of positive edges. Considering signed indicators $\mathcal{I}_+ = \ell K_2^o$ and $\mathcal{I}_- = \ell K_2^e$, we obtain the next lemma, which is dual to Lemma 3.3.17.

Lemma 5.4.19. *Given an integer $\ell \geqslant 1$ and a real number $r < \frac{2\ell}{\ell-1}$,*

$$Z^*(\ell K_2^o, r) = [\ell - (\ell-1)\frac{r}{2}, \frac{r}{2}] \text{ and } Z^*(\ell K_2^e, r) = [0, \ell\frac{r}{2} - \ell].$$

We combine Lemmas 5.4.16 and 5.4.19, where we take $\Gamma_+ = \ell K_2^o, \Gamma_- = \ell K_2^e$ and $t = \ell - (\ell-1)\frac{r}{2}$, and then we have the following result.

Lemma 5.4.20. *For any signed graph Ω,*

$$\chi_c(\Omega(\ell K_2^o, \ell K_2^e)) = \frac{2\ell\chi_c(\Omega)}{(\ell-1)\chi_c(\Omega) + 2}.$$

In particular, when $\ell = 2$, $\Omega(2K_2^o, 2K_2^e) = 2\Omega$ (recall $\ell\hat{G}$ in Section 2.3.2) and then we have $\chi_c(2\Omega) = \frac{4\chi_c(\Omega)}{\chi_c(\Omega)+2}$.

Lemma 5.4.21. *Given a real number r, $r \geqslant 2$, a signed graph $\hat{G}$ admits a circular r-flow if and only if $2\hat{G}$ admits a circular $\frac{4r}{r+2}$-flow.*

For a graph G, we define $S^*(G)$ as follows: replace each edge xy with P_2^2 by identifying x and y with u and v, respectively. One may observe that $S^*(G)$ is the dual to the notion of $S(G)$ defined before (see Section 2.3.4) when restricted to planar graphs. Observing that $S^*(G)$ is the same as $2T_2(G,+)$, the next lemma follows from Lemmas 5.4.18 and 5.4.21.

Lemma 5.4.22. *A graph G admits a circular 4-flow if and only if $S^*(G)$ admits a circular $\frac{16}{5}$-flow.*

Hence, we have another restatement of the 4-color theorem as follows.

Theorem 5.4.23. [4-color theorem restated] *Given a planar graph G, $S^*(G)$ admits a circular $\frac{16}{5}$-flow.*

5.5 Modulo ℓ-orientations and homomorphisms to cycles

One of the main motivations for the study of circular flow in signed graphs is to provide settings in which inductive approach to a conjecture of Jaeger (Question 1.4.1) becomes more feasible. It remains an open problem then to provide the best edge-connectivity conditions that would work for Jaeger's circular flow conjecture. For a given integer k, assume that $f(k)$ is the smallest value such that every $f(k)$-edge-connected graph admits a circular $(2+\frac{1}{k})$-flow and it has already been shown $4k+1 < f(k) \leqslant 6k$ (see Section 1.4 for details and references). To prove tighter upper bounds on the value of $f(k)$, one plausible approach is by induction on k. However, moving from $\mathbb{Z}_{2k+1}$ to $\mathbb{Z}_{2k+3}$ counts as two steps. To introduce an intermediary step based on $\mathbb{Z}_{2k}$, we use the settings in signed graphs. We refer to [NY21] and references therein for a successful example of such an approach. Here, we provide a similar packing property which would be the first step towards such an approach.

As in this thesis, we will work with signed graphs, in which we need to distinguish positive and negative edges, so we need the following notation. We use $\overleftarrow{d_D^+}(v)$ to denote the *positive-out-degree* of a vertex v with respect to an orientation D, and similarly, $\overrightarrow{d_D^+}(v)$ *positive-in-degree*, $\overleftarrow{d_D^-}(v)$ *negative-out-degree* and $\overrightarrow{d_D^-}(v)$ *negative-in-degree*.

Definition 5.5.1. A signed graph (G,σ) is said to be *modulo ℓ-orientable* if there exists an inversing-equivalent signature σ' and an orientation D on G such that, with respect to (G,σ'), we have that

$$(\ell-1)(\overleftarrow{d_D^+}(v) - \overrightarrow{d_D^+}(v)) = \overleftarrow{d_D^-}(v) - \overrightarrow{d_D^-}(v). \tag{5.4}$$

Such an orientation D is called a *modulo ℓ-orientation* of (G,σ).

We observe that the conditions on this definition can only happen on two special classes of signed graphs.

- If ℓ is an odd number, then the left side, being multiplied by $\ell-1$ is an even number. Thus in the subgraph induced by the set of negative edges with respect to σ', the difference of in-degree and out-degree is an even number. Hence, in this subgraph, the degree of each vertex is even. So we can apply some inversing on cycles to inverse all edges into positive signs. In this case, we have a signed graph (G,σ) where all cuts are positive and thus (G,σ) is inversing-equivalent to $(G,+)$.

- If ℓ is an even number, then we conclude that at each vertex the difference of in-degree and out-degree for positive edges and negative edges, with respect to σ', is of the same parity. That means the total degree of a vertex is even. In this case, we have an even-degree graph.

To present the potential induction idea to attack Jaeger's conjecture and its extension to signed Eulerian graphs (formalized in Question 6.4.8), we provide a necessary and sufficient condition of the existence of modulo ℓ-orientations in two special classes of signed graphs. Note that it generalizes Theorem 2.1.3. Let $-C^*_{-\ell}$ denote the dual signed graph of $-C_{-\ell}$.

Theorem 5.5.2. *A signed graph (G, σ) satisfying that $C_{ij}(G, \sigma) \geqslant C_{ij}(-C^*_{-\ell})$ admits a modulo ℓ-orientation if and only if the following two conditions are satisfied:*

1. *there is a partition $\{E_1, E_2, \ldots, E_\ell\}$ of $E(G)$ such that each E_i is the set of positive edges of a signature which is inversing equivalent to σ, and*
2. *there is an orientation on G such that for each vertex v and any pair $i, j \in \{1, \ldots, \ell\}$, we have that*
$$\overleftarrow{d_{E_i}}(v) - \overrightarrow{d_{E_i}}(v) = \overleftarrow{d_{E_j}}(v) - \overrightarrow{d_{E_j}}(v). \tag{5.5}$$

Proof. One direction of the problem is easy. Suppose that $E(G)$ is partitioned into $E_1, E_2, \ldots, E_\ell$, each E_i being the set of positive edges of the signature σ_i where each σ_i is inversing equivalent to σ. Assume that D is an orientation satisfying Condition 2 of the theorem. We then add up Equations (5.5) for the pairs $(1, j)$, $j = 2, 3 \ldots, \ell$, and we obtain the main equality (Equation (5.4)) for a modulo ℓ-orientation on (G, σ), with respect to the inversing-equivalent signed graph (G, σ_1). And we know that (G, σ) must belong to one of the two special classes that satisfy $C_{ij}(G, \sigma) \geqslant C_{ij}(-C^*_{-\ell})$.

For the converse, suppose that, with respect to an inversing-equivalent signature σ', the signed graph (G, σ') admits a modulo ℓ-orientation D. We will apply some operations on (G, σ') to get a new signed graph (G'', σ''). To construct G'' from G, we may split vertices, thus we will have more vertices than those in G, and hence the names of edges might change, but there will be a clear correspondence between the edges of G and G''. Moreover, with respect to this natural correspondence between edges, the orientation on the edges will remain the same. Thus, with a minor abuse of notation, we will use D to denote the orientation on the edges in G'' and all the intermediary constructions. Similarly, we will use σ to denote the sign of the edges in all these intermediary graphs.

To construct G'', if at a vertex v of G there are both incoming and outgoing edges of the same sign with respect to the signature σ', then taking one of each at random, say uv and vw of sign η, we delete these two edges and add the oriented edge uw with the sign η. We repeat this operation until there are no such pairs of edges. Let (G_1, σ') be the resulting signed graph. We first observe that each vertex of G_1 is either a source or a sink with respect to the orientation D. That at a given vertex v all the edges of the same sign are in the same direction is the consequence of our operation. To observe that positive and negative edges of a vertex of G_1 are also in the same direction, we first note that G_1 has the same set of vertices as G. Next we note that the operation mentioned above does not change the difference of in-degree and out-degree in the set of positive (or negative) edges of a vertex v. In other words,

$$\overleftarrow{d^+_G}(v) - \overrightarrow{d^+_G}(v) = \overleftarrow{d^+_{G_1}}(v) - \overrightarrow{d^+_{G_1}} \text{ and } \overleftarrow{d^-_G}(v) - \overrightarrow{d^-_G}(v) = \overleftarrow{d^-_{G_1}}(v) - \overrightarrow{d^-_{G_1}}(v).$$

Thus D is a modulo ℓ-orientation on (G_1, σ') as well. We conclude that at each vertex v, $(\ell - 1)d^+_{G_1}(v) = d^-_{G_1}(v)$. We then split each vertex v into $d^-(v)$ copies $v_1, v_2, \ldots v_{d^-(v)}$ where each v_i takes one positive neighbor of v and $\ell - 1$ negative neighbors of v. Let (G'', σ'') be the final signed graph. Recall that (G'', σ'') is equipped with an orientation induced by D. We then have an ℓ-regular graph oriented such that each vertex is either a source or a sink and where each vertex has exactly one negative edge incident to it. Thus the set of negative edges form a perfect matching and that the underlying graph is a bipartite graph with source vertices forming one part and sink vertices forming the other part. Let E_1 be the perfect matching corresponding to the positive edges and consider the $(\ell-1)$-regular subgraph of G'' obtained from removing edges of E_1. Any partition of the edges on this $(\ell-1)$-regular graph into perfect matching then works as $E_2, E_3, \ldots, E_\ell$. $\square$

We claim that for $\ell = 2k+1$, Definition 5.5.1 is equivalent to the classic definition of modulo $(2k+1)$-orientation on graphs (see Section 2.1.2). When $\ell = 2k+1$, we only consider signed graphs $(G,+)$. For any signed graph $(G,+)$, the conditions of Definition 5.5.1 implies that there is an inversing-equivalent signature such that $\overleftarrow{d_D}(v) - \overrightarrow{d_D}(v) = (2k+1)(\overleftarrow{d_D^+}(v) - \overrightarrow{d_D^+}(v)) \equiv 0 \pmod{2k+1}$. Thus G admits a modulo $(2k+1)$-orientation. The other direction needs to use Theorem 2.1.3. Assume that G admits a modulo $(2k+1)$-orientation. Thus there is a partition $\{E_1, \ldots, E_{2k+1}\}$ of $E(G)$ and an orientation D on G satisfying the conditions of Theorem 2.1.3. Without loss of generality, we may consider a signature, say σ_1, where the set of positive edges is E_1. Note that in (G,σ_1), $E_2 \cup \cdots \cup E_{2k+1}$ is the set of negative edges and moreover, (G,σ_1) is inversing-equivalent to $(G,+)$. Hence, with respect to (G,σ_1) and the orientation D on G, Equality (5.4) is satisfied.

Lemma 5.5.3. [Jae88] *A graph admits a circular $\frac{2k+1}{k}$-flow if and only if it admits a modulo $(2k+1)$-orientation.*

Our definition then helps to fill the gap in the parity by the case of modulo $2k$-orientation on signed Eulerian graphs in Lemma 5.5.5. To see this lemma, we first give an equivalent definition of a circular $\frac{4p}{2p-1}$-flow in a signed Eulerian graph.

Lemma 5.5.4. *A signed Eulerian graph $\hat{G}$ admits a circular $\frac{4p}{2p-1}$-flow if and only if it admits a $4p$-flow (D,f) such that for each positive edge e, $f(e) \in \{2p-1, 2p+1\}$ and for each negative edge e, $f(e) \in \{-1,1\}$.*

Proof. Let $\hat{G}$ be a signed Eulerian graph. One direction is easy that any $4p$-flow satisfying the above conditions is also a circular $\frac{4p}{2p-1}$-flow. We show that if $\hat{G}$ admits a circular $\frac{4p}{2p-1}$-flow (d,φ), then we can modify it and obtain a $4p$-flow as required. Note that by the equivalence of definitions, we may assume that $\varphi(e) \in \{2k-1, 2k, 2k+1\}$ for each positive edge e and $\varphi(e) \in \{-1,0,1\}$ for each negative edge e. Let $E' = \{e \in E(G) \mid \varphi(e) \in \{0, 2k\}$. Since G is Eulerian, the set E' induces an even-degree subgraph. Then we partition E' into edge-disjoint cycles. For each cycle C in this partition, we modify φ on C as follows: If C is a directed cycle, then let $f(e) = \varphi(e)+1$ for each edge $e \in C$. Otherwise, the cycle C is divided into segments $P_1, P_2, \ldots, P_{2t}$, where each P_i is a directed path and for each i, one of P_i, P_{i+1} is a forward directed path and the other is a backward directed path. Let $f(e) = \varphi(e)+1$ for $e \in P_1, P_3, \ldots, P_{2t-1}$ and $f(e) = \varphi(e)-1$ for $e \in P_2, P_4, \ldots, P_{2t}$. For edges $e \notin E'$, let $f(e) = \varphi(e)$. It is easy to verify that (D,f) is a $4p$-flow as required. $\square$

Lemma 5.5.5. *A signed Eulerian graph admits a circular $\frac{4k}{2k-1}$-flow if and only if it admits a modulo $2k$-orientation.*

Proof. Let (D,f) be a $4k$-flow in (G,σ) satisfying that $f(e) \in \{2k-1, 2k+1\}$ for each positive edge e, and $f(e) \in \{-1,1\}$ for each negative edge e. We choose a mapping $\varphi : E(G) \to \{1,2,\ldots,2k\}$ such that $\varphi(e) \equiv f(e) \pmod{2k}$ for each edge e. Note that we have $\varphi(e) \in \{1, 2k-1\}$ for each $e \in E(G)$. Next we define a pair (D', f') as follows: For each edge e with $\varphi(e) = 2k-1$, we reverse the direction of e in D (to obtain D') and set $f'(e) = 1$; for any other edge, both the direction and the value remain the same as in D and φ (respectively). As f' accepts only 1 as output, it is easily observed that D' could be viewed as a modulo $2k$-orientation on the underlying graph G, i.e., $\overleftarrow{d_{D'}}(v) - \overrightarrow{d_{D'}}(v) \equiv 0 \pmod{2k}$.

Next, based on the directed graph D' and following the methods of the proof of Theorem 5.5.2, we build a new digraph as follows: As long as there is a vertex, say v, with both indegree and outdegree nonzero, taking a pair of arcs, say (u, v) and (v, w), we delete both of them and add a new arc (u, w). At the end of this process, we obtain a digraph D_1 where each vertex is either a source or a sink. We note that D_1 might not be unique and depends on how we apply the process. However, we have $\overleftarrow{d_{D_1}}(v) - \overrightarrow{d_{D_1}}(v) = \overleftarrow{d_{D'}}(v) - \overrightarrow{d_{D'}}(v)$ for any vertex $v \in V(D_1)$. Thus in the underlying graph of D_1, we conclude that the degree of each vertex is a multiple of $2k$. We further note that in the process of constructing D_1 from D', each edge of D' has a unique corresponding edge in D_1. More precisely, edges of D' correspond to a partition of edges of D' into directed paths.

For the final step, we split vertices of D_1 so to have a $2k$-regular oriented graph D_2. As D_2 inherits the property that each vertex is either a source or a sink, its underlying graph is a $2k$-regular bipartite graphs with sources forming one part and sinks forming the other. Being a bipartite regular graph, D_2 admits a perfect matching. Let M be one such perfect matching. We define a signature σ' on G as follows: $\sigma'(e)$ is positive if the image of e is an edge of M, otherwise it is negative. It follows from our construction that with respect to D' and the signature σ' we have $(2k-1)(\overleftarrow{d^+}(v) - \overrightarrow{d^+}(v)) = \overleftarrow{d^-}(v) - \overrightarrow{d^-}(v)$. To complete the proof, what remains is to show that σ and σ' are inversing equivalent.

We shall prove that for each vertex $v \in V(G)$, $\prod_{e\sim v} \sigma(e) = \prod_{e\sim v} \sigma'(e)$. Let $d_1^-(v)$ (or $d_2^-(v)$) represent the number of negative edges e with $f(e) = 1$ (respectively, $f(e) = -1$), and $d_1^+(v)$ (or $d_2^+(v)$) represent the number of positive edges e with $f(e) = 2k+1$ (respectively, $f(e) = 2k-1$). Observe that $\prod_{e\sim v} \sigma(e)$ depends on the parity of $\overleftarrow{d_1^-}(v)+\overrightarrow{d_1^-}(v)+\overleftarrow{d_2^-}(v)+\overrightarrow{d_2^-}(v) := \alpha$ with respect to the orientation D. Following the procedure to build D_1, we know that with respect to the orientation D, there is a β such that $(\overleftarrow{d_1^-}(v)+\overleftarrow{d_1^+}(v))+(\overrightarrow{d_2^-}(v)+\overrightarrow{d_2^+}(v))-\{(\overrightarrow{d_1^-}(v)+\overrightarrow{d_1^+}(v))+(\overleftarrow{d_2^-}(v)+\overleftarrow{d_2^+}(v))\} = \beta\cdot 2k$. By the choice of σ' and noting the difference of indegrees and outdegrees of each vertex remain the same in the construction of D_2 from D, $\prod_{e\sim v} \sigma'(e)$ depends on the parity of β. Thus it is sufficient to prove that $\alpha \equiv \beta \pmod 2$. Since (D, f) is a $4k$-flow in (G, σ), we have that

$$\overleftarrow{d_1^-}(v)+(-1)\cdot\overleftarrow{d_2^-}(v)+(2k+1)\cdot\overleftarrow{d_1^+}(v)+(2k-1)\cdot\overleftarrow{d_2^+}(v) = \overrightarrow{d_1^-}(v)+(-1)\cdot\overrightarrow{d_2^-}(v)+(2k+1)\cdot\overrightarrow{d_1^+}(v)+(2k-1)\cdot\overrightarrow{d_2^+}(v).$$

Hence, $\beta = \overrightarrow{d_1^+}(v) - \overleftarrow{d_1^+}(v) + \overrightarrow{d_2^+}(v) - \overleftarrow{d_2^+}(v) \equiv \overrightarrow{d_1^+}(v) + \overleftarrow{d_1^+}(v) + \overrightarrow{d_2^+}(v) + \overleftarrow{d_2^+}(v) \equiv \alpha \pmod 2$.

Assume that for a signed graph (G, σ), there exists an inversing-equivalent signature σ' and an orientation D on G such that, with respect to (G, σ'), such that $(2k-1)(\overleftarrow{d_D^+}(v) - \overrightarrow{d_D^+}(v)) = \overleftarrow{d_D^-}(v) - \overrightarrow{d_D^-}(v)$. Let $f : E(G) \to \mathbb{Z}$ be a mapping satisfying that $f(e) = -1$ if e is a negative edge of (G, σ') and $f(e) = 2k-1$ otherwise. It is easy to see that (D, f) is a circular $\frac{4k}{2k-1}$-flow of (G, σ') and thus of (G, σ). □

We note the following dual version of its conclusion in the case of planar graphs.

Lemma 5.5.6. *A signed Eulerian planar graph (G, σ) admits a modulo $2k$-orientation if and only if its dual signed graph (G^*, σ^*) admits a homomorphism to C_{-2k}.*

The dual statement of Theorem 5.5.2, which is about homomorphism to $-C_{-k}$, can be stated as follows. But we note that $(G, +)$ admits a homomorphism to $C_{+(2k+1)}$ if and only if $(G, -)$ admits a homomorphism to $C_{-(2k+1)}$.

Theorem 5.5.7. *A signed graph (G, σ) admits a homomorphism to $-C_{-k}$ if and only if the followings are satisfied: There is a partition $E_1, E_2, \ldots E_k$ of edges of G such that each E_i is the set of* *positive edges of a signature which is switching equivalent to σ; There is an orientation D on G such that for each pair i, j and for each cycle C of G, and with the clockwise orientation on C, the difference between the number of the forward and the number of the backward edges of C that are colored i is the same as the difference among edges that are colored j.*

In an inductive approach using Theorem 5.5.2 then one may try to prove the existence of two disjoint inversing-equivalent signatures σ' and σ'' such that removing all negative edges in the signature σ' from G, the remaining graph together with the signature σ'' is connective enough that is $f(k-1)$-edge-connected. Applying induction and taking the $k-1$ signatures on this graph, one would be left with a smaller task of completing it to a solution for the original graph. Theorem 5.5.7 provides a similar potential for an inductive approach.

6 | Circular flow indexes of some classes of signed graphs

This chapter is based on the following paper:

[LNWZ22] J. Li, R. Naserasr, Z. Wang, and X. Zhu. "Circular flow in mono-directed signed graphs". In: *In preparation* (2022)

In this chapter, we study the circular flow indexes of some families of signed graphs.

Recall that the famous Tutte's 5-flow conjecture and 3-flow conjecture (in Section 1.4), which remain unsolved, state that every 2-edge-connected graph admits a nowhere-zero 5-flow and every 4-edge-connected graph admits a nowhere-zero 3-flow, respectively. For these two conjectures, P. Seymour [Sey81] showed that every 2-edge-connected graph admits a circular 6-flow and F. Jaeger [Jae79] proved that every 4-edge-connected graph admits a circular 4-flow. Note that the celebrated 4-color theorem, in terms of flows, is equivalent to stating that every 2-edge-connected planar graph admits a circular 4-flow. Moreover, W.T. Tutte conjectured more ambitiously that in place of the condition of planarity with "Petersen-minor-free", the statement is still correct. This conjecture is known as Tutte's 4-flow conjecture. It has been verified that every 2-connected 3-regular Petersen-minor-free graph admits a circular 4-flow, due to an equivalent result that every bridgeless cubic Petersen-minor-free graph is 3-edge-colorable (see Section 1.2 and [ESST16] for details). Some extensions of these classical results and conjectures to signed graphs are studied in Section 6.1. First, we could restate Seymour's 6-flow theorem and the 5-flow conjecture in Theorem 6.1.1 and Conjecture 6.1.2 in the language of the circular flows in mono-directed signed graphs. Based on the group connectivity results of [JLPT92], we show in Theorems 6.1.4 and 6.1.5 that 3-edge-connectivity allows signed graphs to have circular 6-flows and 4-edge-connectivity allows signed graphs to admit circular 4-flows.

In Section 6.2, we show in Theorem 6.2.3 that if a graph G contains 3 edge-disjoint spanning trees, then given any signature on G, there exists an ϵ such that the signed graph (G, σ) admits a circular $(4-\epsilon)$-flow. By the Nash-Williams and Tutte theorem, every 6-edge-connected signed graph (G, σ) has $\Phi_c(G, \sigma) < 4$.

We have already seen that the edge-connectivity plays a central role in studying the existence of flows of graphs. Thus highly edge-connected (signed) graphs are of special interests in the study of flow theory. Jaeger's circular flow conjecture [Jae88] states that every $4k$-edge-connected graphs admits a circular $\frac{2k+1}{k}$-flow, where the case of $k=1$ is Tutte's 3-flow conjecture. However, Jaeger's circular-flow conjecture has been disproved in [HLWZ18] when $k \geqslant 3$. But it has been verified positively for $6k$-edge-connected graphs in [LTWZ13]. Recently, similar results are obtained for other edge-connectivities in [LWZ20], i.e., every $(6k-2)$-edge-connected graphs admits a circular $\frac{4k}{2k-1}$-flow and every $(6k+2)$-edge-connected graphs admits a circular r-flow where r is strictly smaller than $\frac{2k+1}{k}$. In Section 6.3, we obtain analogous circular flow results on signed graphs of given edge-connectivity. Adapting their methods to signed graphs, in Theorem 6.3.5, we provide upper bounds of circular flow indexes of signed graphs based on the conditions of edge-connectivities in form of $6k+i$ for each $i \in [6]$. In particular, we show that every $(6k-1)$-edge-connected signed graph admits a circular $\frac{4k}{2k-1}$-flow and every $(6k+2)$-edge-connected signed graph admits a circular $\frac{2k+1}{k}$-flow. Moreover, restricted to planar graphs, Jaeger's circular flow conjecture has a dual form that every planar graph of girth at least $4k$ admits a homomorphism to the odd cycle C_{2k+1}. This conjecture has attracted special attention (see [Zha02; DP17; BKKW04], etc) but is still open. The result of circular $\frac{2k+1}{k}$-flows of [LTWZ13] implies, in particular, that every planar graph of girth at least $6k$ admits a homomorphism to C_{2k+1}. In Section 6.3.1, we study an analogous problem of homomorphism of signed bipartite planar graphs to negative even cycles by using circular flows. Specifically, we show in Theorem 6.3.7 that every $(6k-2)$-edge-connected signed Eulerian graph admits a circular $\frac{4k}{2k-1}$-flow, and thus, by duality (Theorem 5.1.2) and the bipartite folding lemma, every signed bipartite planar graph of negative-girth at least $6k-2$ admits a homomorphism to C_{-2k}.

6.1 Circular 4-flows and 6-flows

Whether graphs or signed graphs, the main question normally is the existence of a flow where values assigned to edges are limited. The starting point is to avoid 0, which is known as the nowhere-zero flow, based on which, numerous conjectures arise, for example, the famous 3-flow and 5-flow conjectures.

Note that our definition uses the normal direction on the edges, and thus the roles that different signs of edges play are about the different limitations on the values. The forbidden values, moreover, are antipodal values in a circular view. Therefore, sometimes stronger results can be stated by using only the values that are available for both positive and negative edges. In particular, Seymour's 6-flow theorem and Tutte's 5-flow conjecture can be restated in the following forms.

Theorem 6.1.1. [Seymour's 6-flow theorem restated] *Every 2-edge-connected signed graph has a circular 12-flow (D, f) such that for each edge e, $f(e) \in \{1, 2, 3, 4, 5\}$.*

To observe this, we take a circular 6-flow (D, f) of a 2-edge-connected graph G and view it as a circular 12-flow in (G, σ) for any signature σ on G. We can also directly apply Observation 5.4.3 and Lemma 5.4.18. Similarly, we have the following restatement.

Conjecture 6.1.2. [Tutte's 5-flow conjecture restated] *Every 2-edge-connected signed graph has a circular 10-flow.*

Without changing the condition of 2-edge-connectivity to a higher connectivity, one cannot do any better than changing r to $2r$. This is shown through a subdivision operation in Lemma 5.4.18.

Note that in Jaeger's conjecture about circular $\frac{2k+1}{k}$-flow (or similarly, circular $\frac{4k}{2k-1}$-flow), the limitation can go up to a limit of only one or two values on each edge. Different limitations on values for edges are also studied in the literature. For example, we have seen the $\mathbb{Z}_k$-connectivity of graphs in Section 2.1.2 and we refer to [JLPT92] for more details.

Theorem 6.1.3. [JLPT92]

(1) Every 3-edge-connected graph is $\mathbb{Z}_6$-connected.

(2) Every graph with two edge-disjoint spanning trees is $\mathbb{Z}_4$-connected. In particular, every 4-edge-connected graph is $\mathbb{Z}_4$-connected.

Next, we shall make use of these group connectivity results of graphs to obtain circular 4-flow and 6-flow results of signed graphs.

Theorem 6.1.4. *Every 3-edge-connected signed graph admits a circular 6-flow.*

Proof. Let (G,σ) be a 3-edge-connected signed graph. Thus the underlying graph G is $\mathbb{Z}_6$-connected by Theorem 6.1.3 (1). Recall the equivalent definition (in Proposition 2.1.2) of a graph being $\mathbb{Z}_\ell$-connected: a graph G is $\mathbb{Z}_\ell$-connected if and only if for any $g: E(G) \to \mathbb{Z}_\ell$, there exists a modulo ℓ-flow (D,f) such that $f(e) \not\equiv g(e) \pmod{\ell}$ for every edge $e \in E(G)$.

Therefore, we define $g: E(G) \to \mathbb{Z}_6$ to be a mapping satisfying that

$$g(e) = \begin{cases} 0, & \text{if } e \text{ is a positive edge,} \\ 3, & \text{if } e \text{ is a negative edge.} \end{cases}$$

Thus the graph G admits a modulo 6-flow (D,f) such that $f(e) \not\equiv g(e) \pmod{6}$ for each edge e. It is easy to verify that such (D,f) is a circular modulo 6-flow in (G,σ). □

Similarly, we can directly apply Theorem 6.1.3 (2) to obtain the following.

Theorem 6.1.5. *Every 4-edge-connected signed graph admits a circular 4-flow.*

In the next section, we will give a direct proof of Theorem 6.1.5 and rather emphasize on the idea of considering the interactions of edge-disjoint spanning trees in proving the existence of circular flow.

6.2 Circular flow via the Nash-Williams and Tutte theorem

The following theorem, first proved, independently, by Nash-Williams [Nas61] and Tutte [Tut61], but normally referred as the Nash-William and Tutte theorem, provides a necessary and sufficient condition for a graph to have at least t disjoint spanning trees.

Theorem 6.2.1. [Nash-William and Tutte theorem] *Given a graph G, it contains t edge-disjoint spanning trees if and only if for every partition $\mathcal{P}$ of vertices in p parts, there are at least $t(p-1)$ edges connecting the parts of $\mathcal{P}$.*

For $t = k$ and G being $2k$-edge-connected, the conditions of the theorem will hold for every partition $\mathcal{P}$. Thus we have the following corollary.

Corollary 6.2.2. *Every $2k$-edge-connected graph contains a system of k edge-disjoint spanning trees.*

A direct proof of Theorem 6.1.5: Let (G,σ) be a 4-edge-connected signed graph, thus by Corollary 6.2.2, we may assume that G contains two edge-disjoint spanning trees T_1 and T_2. By Lemma 5.2.4, without loss of generality, we can apply some inversing operations such that all negative edges are contained in T_1.

For each edge $e \in E(G \backslash T_1)$, let C_e^1 be the unique cycle in $T_1 + e$. Define $C^1 = \triangle_{e \in E(G\backslash T_1)} C_e^1$. Similarly, let C_e^2 be the unique cycle in $T_2 + e$ for each $e \in E(T_1)$, and let $C^2 = \triangle_{e\in E(T_1)} C_e^2$. Note that both of C^1 and C^2 are Eulerian subgraphs of G. Then we choose an arbitrary orientation D on G and when restricted to subgraphs of G, we still use the notation D to denote their orientations induced by D. The graph C^1 admits a circular 2-flow (D,f_1) such that $|f_1(e)| = 1$ for each $e \in E(C^1)$, and the graph C^2 admits a circular 2-flow (D,f_2) such that $|f_2(e)| = 1$ for each $e \in E(C^2)$. Let $f = 2f_1 + f_2$.

Hence, for each negative edge e (which must be in $E(T_1)$), $|f(e)| = |2f_1(e) + f_2(e)| \in \{1, 3\}$, and for each positive edge e, $|f(e)| \in \{1, 2, 3\}$. Therefore, (D, f) is a circular 4-flow in (G, σ). □

Later in Section 7.2, a sequence of signed bipartite planar simple graphs is constructed whose limit of the circular chromatic is 4. As the girth of the underlying graph is 4, their dual graphs are 4-edge-connected. Thus we have a sequence of 4-edge-connected signed graph for which the limit of the circular flow index is 4. Therefore, one cannot expect to replace the upper bound of 4 for the circular flow index of 4-edge-connected graphs in Theorem 6.1.5 with a smaller number. However, it follows from a later result in Theorem 7.2.1 that for signed Eulerian planar graph 4 is not attainable either. In other words, for each such a signed graph $\hat{G}$, there exist a positive number $\epsilon(\hat{G})$ such that $\Phi_c(\hat{G}) = 4 - \epsilon(\hat{G})$. We do not know if this is also the case for the circular flow index of all 4-edge-connected signed graphs. In the next theorem, we show that under a slightly different edge-connectivity, this would be the case.

Theorem 6.2.3. *For any signed graph (G, σ) that contains 3 edge-disjoint spanning trees, $\Phi_c(G, \sigma) < 4$.*

Proof. Let T_1, T_2 and T_3 be three edge-disjoint spanning trees of the underlying graph G. By Lemma 5.2.4, we can find an inversing-equivalent signed graph (G, σ') of (G, σ) where all the negative edges are in T_1. We have seen in the proof of Theorem 6.1.5 that every signed graph that contains at least two edge-disjoint spanning trees admits a circular 4-flow. So the signed graph (G, σ') admits a circular 4-flow. We aim to build a modulo 4-flow (D, f) of (G, σ') with more restrictions as follows:

- For each negative edge e in T_1, $f(e) \in \{0, 1, 3\}$,
- For each positive edge e in $T_1 \cup T_2$, $f(e) \in \{1, 2, 3\}$,
- For each positive edge e in $G \backslash \{T_1 \cup T_2\}$, $f(e) = 2$.

Let D be an orientation on G. We define a mapping $g : E(G) \to \mathbb{Z}_4$ as follows:

$$g(e) = \begin{cases} 0, & \text{for each edge } e \in E(T_2), \\ 2, & \text{for each edge } e \in E(G \backslash T_2). \end{cases}$$

Then we define $\beta^* : V(G) \to \mathbb{Z}_4$ satisfying that $\beta^*(v) \equiv \partial_D g(v) \pmod 4$. Such a mapping β^* could be verified to be a $\mathbb{Z}_4$-boundary. We consider the subgraph $H = T_1 \cup T_2$ of G. The graph H, which contains 2 edge-disjoint spanning trees, is $\mathbb{Z}_4$-connected by Theorem 6.1.3 (2). Thus by Proposition 2.1.2, for any given $\mathbb{Z}_4$-boundary β, there exists a mapping $f : E(H) \to \mathbb{Z}_4$ such that $\partial_D f(v) \equiv \beta(v) \pmod 4$ for each vertex $v \in V(H)$. Let $f^* : E(H) \to \{1, 2, 3\}$ denote the mapping corresponding to β^*. To extend the mapping f^* to the whole graph G, we assume that $f^*(e) = 0$ for each edge $e \in E(G \backslash H)$. Let $f = f^* - g$ and thus $\partial_D f(v) \equiv 0 \pmod 4$ for every vertex $v \in V(G)$. Thus (D, f) is a circular modulo 4-flow in (G, σ'). Moreover, for each edge $e \in E(T_1)$, $f(e) \in \{1-2, 2-2, 3-2\}$ which is $\{0, -1, +1\}$; for each edge $e \in E(T_2)$, $f(e)$ is in $\{1-0, 2-0, 3-0\}$ which is $\{1, 2, 3\}$; for each edge $e \in E(G \backslash H)$, $f(e)$ is in $\{0-2\}$ which is $\{2\}$. So such (D, f) satisfies all the conditions required above.

Because the edges of T_3 are not tight, (G, σ') has no tight cut with respect to such a circular 4-flow (D, f'). Hence, by Lemma 5.4.6, (G, σ) admits a circular r-flow where $r < 4$. □

In particular, applying Corollary 6.2.2, we have the following result. We note that this is a special case of a more general result (Theorem 6.3.5 (2) when $p = 1$) stated in the next section.

Corollary 6.2.4. *For every* 6*-edge-connected signed graph* (G, σ)*,* $\Phi_c(G, \sigma) < 4$.

Moreover, it has been proved in [HLL18] that every graph that contains 4 edge-disjoint spanning trees admits a circular 3-flow. Simply by Lemma 5.2.4, we can show that every signed graph with 5 edge-disjoint spanning trees admits a circular 3-flow, and consequently, every 10-edge-connected signed graph admits a circular 3-flow. But this result is not optimal. In the next section, we will apply other techniques to show a stronger result (Theorem 6.3.5 (4) when $p = 1$) that 8-edge-connectivity is sufficient for signed graph to admit a circular 3-flow.

Note that C. Thomassen has proved in [Tho12] that every 8-edge-connected graph is $\mathbb{Z}_3$-connected and furthermore it has been proved in [LTWZ13] that 6-edge-connected graph is $\mathbb{Z}_3$-connected. However, we cannot make use of those group connectivity result and apply a similar idea as in Theorem 6.1.4 to show that every 8-edge-connected (or even stronger, 6-edge-connected) signed graph admits a circular 3-flow. It is because our definition of circular (modulo) 3-flow in signed graphs is quite different from the notion of $\mathbb{Z}_3$-connected graphs.

6.3 Circular flows of highly edge-connected signed graphs

In this section, we shall study the circular flow indexes of some classes of signed graphs with given edge-connectivity conditions. The results are obtained by considering stronger statements based on the following definitions.

Definition 6.3.1. [LWZ20] Given a graph G, a function $\beta : V(G) \to \{0, \pm 1, \ldots, \pm k\}$ is a $(2k, \beta)$-*boundary* of G if for every vertex $v \in V(G)$,

$$\beta(v) \equiv d(v) \pmod 2, \quad \text{and} \quad \sum_{v \in V(G)} \beta(v) \equiv 0 \pmod{2k}.$$

Given a subset $A \subset V(G)$, we define $\beta(A) \in \{0, \pm 1, \ldots, \pm k\}$ such that $\beta(A) \equiv \sum_{v \in A} \beta(v) \pmod{2k}$.

Given a $(2k, \beta)$-boundary β, an orientation D of G is called a $(2k, \beta)$-*orientation* if for every vertex $v \in V(G)$,

$$\overleftarrow{d_D}(v) - \overrightarrow{d_D}(v) \equiv \beta(v) \pmod{2k}.$$

The following theorem and its corollary play a key role in proving our results in this section.

Theorem 6.3.2. [LTWZ13; LWZ20] *Let* G *be a graph with a* $(2k, \beta)$*-boundary* β*, where* $k \geqslant 3$*. Let* z_0 *be a vertex of* $V(G)$ *such that* $d(z_0) \leqslant 2k-2+|\beta(z_0)|$*. Assume that* D_{z_0} *is an orientation on* $E(z_0)$ *(edges incident to* z_0*) which achieves boundary* $\beta(z_0)$ *at* z_0*. Let* $V_0 = \{v \in V(G) \backslash \{z_0\} \mid \beta(v) = 0\}$*. If* $V_0 \neq \emptyset$*, we let* v_0 *be a vertex of* V_0 *with the smallest degree. Assume that* $d(A) \geqslant 2k - 2 + |\beta(A)|$ *for any* $A \subset V(G) \backslash \{z_0\}$ *with* $A \neq \{v_0\}$ *and* $|V(G) \backslash A| > 1$*. Then the partial orientation* D_{z_0} *can be extended to a* $(2k, \beta)$*-orientation on the entire graph* G*.*

The next key theorem follows from Theorem 6.3.2.

Theorem 6.3.3. [LWZ20] *Let* G *be a* $(3k-3)$*-edge-connected graph, where* $k \geqslant 3$*. For any* $(2k, \beta)$*-boundary of* G*,* G *admits a* $(2k, \beta)$*-orientation.*

In general, this theorem holds for both even and odd k as remarked in [LWZ20].

Next, we will extend a theorem of [LWZ20], which allows us to apply $(4p, \beta)$-orientations of graphs to study circular $\frac{2p}{q}$-flows of signed graphs. The proof is based on similar ideas to the proof of Theorem 1.2 in [LWZ20]. Recall that $d^+_{\hat{G}}(v)$ denotes the number of positive edges incident to v in the signed graph $\hat{G}$, and when the graph we refer to is clear from the context, we may write $d^+(v)$ as well.

Theorem 6.3.4. *Given positive integers p and q satisfying that $p \geqslant q$, a signed graph $\hat{G}$ admits a $\frac{2p}{q}$-flow if and only if the graph $(2p-2q)G$ admits a $(4p, \beta)$-orientation with $\beta(v) \equiv 2p \cdot d^+(v) \pmod{4p}$ for each vertex $v \in V(G)$,.*

Proof. Given a signed graph $\hat{G}$, we present a one-to-one correspondence between a $(4p, \beta)$-orientation on the graph $(2p-2q)G$ and a $\frac{2p}{q}$-flow in the signed graph $\hat{G}$.

By Tutte's theorem, a $\frac{2p}{q}$-flow in $\hat{G}$ is equivalent to a modulo $4p$-flow in $\hat{G}$ satisfying the following conditions.

- For each positive edge e, $f(e) \in \{2q, 2q+2, \ldots, 4p-2q\}$.
- For each negative edge e, $f(e) \in \{2p-2q, \ldots, 2, 0, -2, \ldots, -(2p-2q)\}$.

For each $e \in E(\hat{G})$, let $[e]$ denote the set of its corresponding $2p-2q$ parallel edges in $(2p-2q)G$.

For a given $(4p, \beta)$-boundary β where $\beta(v) \equiv 2p \cdot d^+(v) \pmod{4p}$ for each $v \in V(G)$, assume that D is a $(4p, \beta)$-orientation on the graph $(2p-2q)G$. This can also be viewed as a mapping of edges of $(2p-2q)G$ to $\mathbb{Z}_{4p}$ where each edge is assigned 1. We choose an arbitrary orientation D' on G. For each $e \in E(G)$, we define a mapping $I_e : [e] \to \{\pm 1\}$ as follows: $I_e(e') = 1$ if $e' \in [e]$ in D is oriented as same as e in D' and $I_e(e') = -1$ otherwise. Then we define a mapping $f_1 : E(G) \to \mathbb{Z}_{4p}$ as follows:

$$f_1(e) = \sum_{e_i \in [e]} I_e(e_i).$$

Note that for each edge $e \in E(G)$, $f_1(e) \in \{2p-2q, 2p-2q-2, \ldots, 2, 0, -2, \ldots, -(2p-2q)\}$, i.e., $|f_1(e)|$ is even and it satisfies that $|f_1(e)| \leqslant 2p-2q$. Thus such (D', f_1) is a modulo $4p$-flow in G satisfying that $\partial_{D'} f_1(v) \equiv \beta(v) \pmod{4p}$ for each vertex $v \in V(G)$. We define another mapping $g : E(G) \to \mathbb{Z}_{4p}$ such that $g(e) = 0$ if e is a negative edge and $g(e) = 2p$ if e is a positive edge. Then, for each $v \in V(G)$,

$$\partial_{D'} g(v) = \begin{cases} 2p, & \text{if } d^+(v) \text{ is odd,} \\ 0, & \text{otherwise.} \end{cases}$$

Thus $\partial g(v) \equiv 2p \cdot d^+(v) \pmod{4p}$ for each $v \in V(G)$.

Let $f = f_1 + g$. Then $f : E(G) \to \mathbb{Z}_{4p}$ satisfies the following claims: for each positive edge e, $f(e) = f_1(e) + 2p \in \{2q, 2q+2, \ldots, 4p-2q\}$ and for each negative edge e, $f(e) = f_1(e) \in \{2p-2q, \ldots, 2, 0, -2, \ldots, -(2p-2q)\}$. Moreover, considering the orientation D' on G,

$$\partial_{D'} f(v) = \partial_{D'} f_1(v) + \partial_{D'} g(v) = \beta(v) + 2p \cdot d^+(v) \equiv 0 \pmod{4p}.$$

Hence, (D', f) is a modulo $4p$-flow in G as desired.

Conversely, let (D', f) be a modulo $4p$-flow in G satisfying the conditions stated as above. We can define a $(4p, \beta)$-orientation on $(2p-2q)G$ by appropriately orienting the edges in $[e]$ for each edge $e \in E(G)$, from the process above. As we observe from the above, the correspondence between the $\frac{2p}{q}$-flow in (G, σ) and the special $(4p, \beta)$-orientation on the graph $(2p-2q)G$ is one-to-one. This completes the proof. □

We are now ready to give our main results about the upper bounds on the circular flow indices of signed graphs based on the edge-connectivity of the underlying graphs. To prove the theorem, rather than study the circular flow in signed graphs directly, we apply Theorem 6.3.4 to study an orientation property of αG for some choice of α. Theorem 6.3.2 and Theorem 6.3.3 are also quite important in our proofs.

Theorem 6.3.5. *Given a signed graph (G, σ), we have the following claims:*

(1) If G is $(6p-1)$-edge-connected, then $\Phi_c(G,\sigma) \leqslant \frac{4p}{2p-1}$.

(2) If G is $6p$-edge-connected, then $\Phi_c(G,\sigma) < \frac{4p}{2p-1}$.

(3) If G is $(6p+1)$-edge-connected, then $\Phi_c(G,\sigma) \leqslant \frac{8p+2}{4p-1}$.

(4) If G is $(6p+2)$-edge-connected, then $\Phi_c(G,\sigma) \leqslant \frac{2p+1}{p}$.

(5) If G is $(6p+3)$-edge-connected, then $\Phi_c(G,\sigma) < \frac{2p+1}{p}$.

(6) If G is $(6p+4)$-edge-connected, then $\Phi_c(G,\sigma) \leqslant \frac{8p+6}{4p+1}$.

Proof. Let (G,σ) be a signed graph and let D be an orientation on G. In each case, we will choose some value of ℓ. We define g to be a mapping of $E(G)$ to $\mathbb{Z}_{4\ell}$ satisfying that:

$$g(e) = \begin{cases} 2\ell, & \text{if } e \text{ is a positive edge,} \\ 0, & \text{if } e \text{ is a negative edge.} \end{cases}$$

We define a mapping $\beta : V(G) \to \mathbb{Z}$ satisfying the following:

$$\beta(v) \equiv \partial_D g(v) \pmod{4\ell}.$$

Note that $\sum_{v\in V(G)} \beta(v) \equiv 0 \pmod{4\ell}$. Moreover, $\beta(v) = 2\ell$ if $d^+(v)$ is odd and $\beta(v) = 0$ otherwise. Thus $\beta(v) \equiv 2\ell \cdot d^+(v) \pmod{4\ell}$.

Based on the notations introduced above, we will prove three claims (1), (2) and (3) of this theorem, and each of the remaining three is quite similar to one of the previous proved cases.

(1). To prove that (G,σ) admits a $\frac{4p}{2p-1}$-flow, by Theorem 6.3.4, it suffices to consider the orientation property of the multigraph $2G$. First we observe that $2G$ is a $(12p-2)$-edge-connected graph. Combing the property of β defined above with $\ell = 2p$ and the fact that $\beta(v) \equiv d_{2G}(v) \pmod 2$ for each vertex v, we conclude that β is a $(8p,\beta)$-boundary of $2G$. Thus applying Theorem 6.3.3 with $k = 4p$, we have that $2G$ admits an $(8p,\beta)$-orientation. It completes the proof of this claim by applying Theorem 6.3.4.

The proof of the claim (4) is similar to the claim (1). In this case, we consider $2G$, observing that it is $(12p+4)$-edge-connected. We take $\ell = 2p+1$ and it is easy to verify that β is an $(8p+4,\beta)$-boundary of $2G$. By Theorem 6.3.3, it admits an $(8p+4,\beta)$-orientation, which could be transformed to a $\frac{2p+1}{p}$-flow in (G,σ) by Theorem 6.3.4.

(2). In this claim, we will prove that there exists a sufficiently large $s = s(G)$ such that (G,σ) admits a $\frac{4ps-2}{2ps-s}$-flow. Our claim then follows by observing that $\frac{4ps-2}{(2p-1)s} < \frac{4p}{2p-1}$. By Theorem 6.3.4, based on the notations, we take $\ell = 2ps-1$ and consider the multigraph $(2s-2)G$. We add a vertex z_0 to the graph $(2s-2)G$ connecting it to each vertex of G with $12p-8$ parallel edges and let H denote the resulting graph. Note that $d_H(z_0) = (12p-8)|V(G)|$. We may extend β to z_0 by defining $\beta(z_0) \equiv 0 \pmod{8ps-4}$ and noting that $\beta(v) \equiv d_H(v) \pmod 2$, we observe that the extended function is an $(8ps-4,\beta)$-boundary of the multigraph H.

Next we shall apply Theorem 6.3.2 to obtain an $(8ps-4,\beta)$-orientation on H. We define a partial orientation D_{z_0} at vertex z_0 of H as follows: For each vertex v, orient half of the edges connected to z_0 toward v and the other half away from v. If we choose s large enough, then we will have $d(z_0) \leqslant (8ps-4)-2+|\beta(z_0)|$. For each subset A of $V(G)$ with $|V(G)\backslash A| > 1$, since $(2s-2)G$ is $(12ps-12p)$-edge-connected, we have at least $12ps-12p$ edges connecting A to $V(G)\backslash A$. And since

$z_0 \notin A$, there are $(12p-8)|A|$ edges connecting z_0 to A. Thus $d_H(A) \geqslant 12ps - 12p + (12p-8)|A| \geqslant 12ps - 8$, the latter inequality being the consequence of the fact that we take $|A| \geqslant 1$ and $p \geqslant 1$. Therefore, noting that $\beta(A) \leqslant 4ps - 2$, we have that $d(A) \geqslant 12ps - 8 = (8ps-4) - 2 + |\beta(A)|$.

As the conditions of Theorem 6.3.2 are satisfied for H with z_0 being the special vertex, we use this theorem to get an $(8ps-4, \beta)$-orientation on H. We claim that restriction of this orientation to $(2s-2)G$ is also an $(8ps-4, \beta)$-orientation on it. This is the case because, for each vertex v, the number of edges oriented to z_0 from v and oriented to v from z_0 are chosen to be the same. We can then apply Theorem 6.3.4 to get a $\frac{4ps-2}{2ps-s}$-flow in (G, σ).

The proof of the claim (5) is similar to the claim (2). We shall prove that there exists a sufficiently large $s = s(G)$ such that $\Phi_c(G, \sigma) \leqslant \frac{4ps+2s-2}{2ps} < \frac{2p+1}{p}$. We take $\ell = (2p+1)s - 1$, consider the multigraph $(2s-2)G$ and again add to $(2s-2)G$ a special vertex z_0 such that it is connected to each vertex of G with $12p-2$ parallel edges. Take the partial orientation D_{z_0} as in the claim (2) and we can show that there is a large enough ϵ such that all the conditions of Theorem 6.3.2 are satisfied. Thus we extend this D_{z_0} to all the H and obtain an $(8ps+4s-4, \beta)$-orientation on H, therefore, an $(8ps+4s-4, \beta)$-orientation on $(2s-2)G$. We are done.

(3). To prove that (G, σ) admits a $\frac{8p+2}{4p-1}$-flow, by Theorem 6.3.4, it suffices to consider the existence of the $(16p+4, \beta)$-orientation on $4G$. Observe that the multigraph $4G$ is $(24p+4)$-edge-connected. Take $\ell = 4p+1$ in the previous settings and note that $\beta(v) \equiv d_{4G}(v) \pmod{16p+4}$ and thus β is a $(16p+4, \beta)$-boundary of $4G$. By Theorem 6.3.3, $4G$ admits a $(16p+4, \beta)$-orientation. Therefore, (G, σ) admits a $\frac{8p+2}{4p-1}$-flow by Theorem 6.3.4.

To prove the claim (6), similar to the proof of the claim (3), we consider $4G$ which is $(24p+16)$-edge-connected. Take $\ell = 4p+3$ and thus β is verified to be a $(16p+12, \beta)$-boundary of $4G$. By Theorem 6.3.3, it admits a $(16p+12, \beta)$-orientation, which can be translated to a $\frac{8p+6}{4p+1}$-flow in (G, σ) by Theorem 6.3.4. □

6.3.1 Signed Eulerian graphs

Note that in Theorem 6.3.5, we always consider the $(2\ell, \beta)$-orientation on the multigraph αG where α is an even number. That is because we can take advantage of the fact that αG is Eulerian (assuming that G is connected). Here we give a circular flow result for signed Eulerian graphs with given high edge-connectivity conditions.

Theorem 6.3.6. *Given positive integers p, a signed Eulerian graph $\hat{G}$ admits a $\frac{4p}{2p-1}$-flow if and only if the underlying graph G admits a $(4p, \beta)$-orientation with $\beta(v) \equiv 2p \cdot d^+(v) \pmod{4p}$ for each vertex $v \in V(G)$.*

Proof. Given a signed Eulerian graph $\hat{G}$, we present a one-to-one correspondence between a $(4p, \beta)$-orientation on the graph G satisfying that $\beta(v) \equiv 2p \cdot d^+(v) \pmod{4p}$ and a $\frac{4p}{2p-1}$-flow in $\hat{G}$. By Lemma 5.5.4, a $\frac{4p}{2p-1}$-flow in $\hat{G}$ is equivalent to a modulo $4p$-flow (D, f) such that for each positive edge e, $f(e) \in \{2p-1, 2p+1\}$ and for each negative edge e, $f(e) \in \{-1, 1\}$.

For a given $(4p, \beta)$-boundary β satisfying that $\beta(v) \equiv 2p \cdot d^+(v) \pmod{4p}$ for each $v \in V(G)$, assume that D is a $(4p, \beta)$-orientation on the graph G. This can also be viewed as a mapping of edges of G to $\mathbb{Z}_{4p}$ where each edge is assigned 1. Let D' be an orientation on G. We define a mapping $f_1 : E(G) \to \mathbb{Z}_{4p}$ such that $f_1(e) = 1$ if e is oriented in D the same as in D' and $f_1(e) = -1$ otherwise. Thus such (D', f_1) is a modulo $4p$-flow in G satisfying that $\partial_{D'} f_1(v) \equiv \beta(v) \pmod{4p}$ for each vertex $v \in V(G)$. We define another mapping $g : E(G) \to \mathbb{Z}_{4p}$ such that $g(e) = 0$ if e is a positive edge and $g(e) = 2p$ if e is a negative edge. Thus $\partial_{D'} g(v) \equiv 2p \cdot d^+(v) \pmod{4p}$ for each $v \in V(G)$. Let $f = f_1 + g$. Then $f : E(\hat{G}) \to \mathbb{Z}_{4p}$ satisfies the following conditions: for each positive edge e, $f(e) = f_1(e) + 2p \in \{2p-1, 2p+1\}$ and for each negative edge e, $f(e) = f_1(e) \in \{-1, 1\}$. Moreover, $\partial_{D'} f(v) = \partial_{D'} f_1(v) + \partial_{D'} g(v) = \beta(v) + 2p \cdot d^+(v) \equiv 0 \pmod{4p}$. Hence, (D', f) is a modulo $4p$-flow in $\hat{G}$ as we required.

Conversely, let (D', f) be a modulo $4p$-flow in G satisfying the conditions stated as above. We can define a $(4p, \beta)$-orientation on G by appropriately orientating each edge $e \in E(G)$, from the process above. As we observe from the above, the correspondence between the $\frac{4p}{2p-1}$-flow in (G, σ) and the $(4p, \beta)$-orientation on the graph G with $\beta(v) \equiv 2p \cdot d^+(v) \pmod{4p}$ is one-to-one. This completes the proof. □

The next theorem follows from Theorems 6.3.3 and 6.3.6.

Theorem 6.3.7. *For any signed Eulerian graph (G, σ), if G is $(6p-2)$-edge-connected, then $\Phi_c(G, \sigma) \leqslant \frac{4p}{2p-1}$.*

By the duality between the circular coloring of signed bipartite planar graphs and the circular flow in signed Eulerian planar graphs, we conclude that every signed bipartite planar graph of girth at least $6p-2$ admits a circular $\frac{4p}{2p-1}$-coloring, i.e., it admits a homomorphism to C_{-2k}. Furthermore, we claim that we can replace the girth condition with the same negative-girth condition and obtain the same conclusion. We first prove the next lemma.

Lemma 6.3.8. *Let G be a $(6k-2)$-edge-connected Eulerian graph and let z_0 be a vertex of degree $6k-2$ of G. For any $(4k, \beta)$-boundary β and its corresponding pre-orientation D_{z_0} on the edges incident to z_0 satisfying that $\overleftarrow{d_{D_{z_0}}}(z_0) - \overrightarrow{d_{D_{z_0}}}(z_0) \equiv \beta(z_0) \pmod{4k}$, D_{z_0} can be extended to a $(4k, \beta)$-orientation on G.*

Proof. Let G be a $(6k-2)$-edge-connected Eulerian graph and z_0 be a vertex of degree $6k-2$. Let $N_G(z_0)$ denote the set of neighbors of z_0. Assume β is a $(4k, \beta)$-boundary and D_{z_0} is the corresponding pre-orientation on the edges incident to z_0. We aim to apply Theorem 6.3.2, thus we need a "proper" β^* such that $|\beta^*(z_0)| = 2k$. Let D'_{z_0} be a pre-orientation obtained from D_{z_0} by changing one in-arc, say (w, z_0), of z_0 to an out-arc and let β' be defined as follows:

$$\beta'(v) = \begin{cases} \beta(v) + 2 & \text{if } v = z_0, \\ \beta(v) - 2, & \text{if } v = w, \\ \beta(v), & \text{otherwise.} \end{cases}$$

It is easy to verify that β' is a $(4k, \beta)$-boundary of G and D'_{z_0} achieves β' at the vertex z_0. We claim that the pre-orientation D_{z_0} is extendable if and only if the pre-orientation D'_{z_0} is extendable. It is simply because the extended orientations, say D and D', will be the same when restricted to the subgraph $G - z_0$. By repeatedly applying this flipping operation, we may find a $(4k, \beta)$-boundary β^* satisfying that $|\beta^*(z_0)| = 2k$ and its corresponding pre-orientation $D^*_{z_0}$. Since G is $(6k-2)$-edge-connected, all the conditions (with respect to β^*) of Theorem 6.3.2 are satisfied, thus $D^*_{z_0}$ can be extended to G. Therefore, for any $(4k, \beta)$-boundary β of G, its corresponding pre-orientation D_{z_0} which achieves β at z_0 could be extended to a $(4k, \beta)$-orientation on G. □

Applying the bipartite folding lemma (Lemma 2.3.15) and Lemma 6.3.8, we have the following.

Theorem 6.3.9. *Every signed bipartite planar graph of negative-girth at least $6p-2$ admits a circular $\frac{4p}{2p-1}$-coloring.*

Proof. Assume to the contrary that (G, σ) is a minimum counterexample. By Lemma 2.3.15, we may assume that (G, σ) is a signed bipartite plane graph of negative-girth $6p-2$ in which each facial cycle is a negative $(6p-2)$-cycle and (G,σ) admits no circular $\frac{4p}{2p-1}$-coloring. Let $\hat{G}^* = (G^*, \sigma^*)$ be the dual signed graph of (G,σ). Hence, the signed graph $\hat{G}^*$ is Eulerian, $(6p-2)$-regular and moreover, each of its negative edge-cuts has size at least $6p-2$. If G^* is $(6p-2)$-edge-connected, then we are done by Theorem 6.3.7. We may assume that $\hat{G}^*$ has a small positive even edge-cut and we choose an edge-cut (X, X^c) with X being inclusion-wise minimal among all the possibilities. That is to say, every subset Y of X has $|E(Y,Y^c)| \geqslant 6p-2$. Let $\hat{H}$ and $\hat{H}^c$ denote the signed subgraphs of $\hat{G}^*$ induced by X and X^c, respectively.

First, clearly, $\hat{G}^*/\hat{H}$ admits a circular $\frac{4p}{2p-1}$-flow. Otherwise, its dual signed graph (which is a subgraph of (G,σ)) admits no circular $\frac{4p}{2p-1}$-coloring, contradicting the minimality of (G,σ). By Theorem 6.3.6, we know that $\hat{G}^*/\hat{H}$ admits a $(4p,\beta)$-orientation with $\beta(v) \equiv 2p \cdot d^+(v) \pmod{4p}$. Let D be such a $(4p,\beta)$-orientation on $\hat{G}^*/\hat{H}$.

Now we consider $G_1 = \hat{G}^*/\hat{H}^c$ and we denote by z_0 the new vertex obtained by contraction. Let D_{z_0} denote the orientation of D restricted on the edges incident to z_0 (i.e., $E(X,X^c)$) and let β be a $(4p,\beta)$-boundary (with $\beta(v) \equiv 2p{\cdot}d^+(v) \pmod{4p}$) of G_1 satisfying that $\beta(z_0) = \overleftarrow{d}_{D_{z_0}}(z_0) - \overrightarrow{d}_{D_{z_0}}(z_0)$. Hence, we know that D_{z_0} achieves β at z_0. In order to apply Lemma 6.3.8, we add $6p-2-d(z_0)$ many edges connecting z_0 with one vertex of $G_1 - z_0$, and orient them half toward z_0 and half away from z_0. We denote the resulting graph by G_1' and the resulting pre-orientation at z_0 by D'_{z_0}. Note that β is still a $(4p,\beta)$-boundary of G_1' and it is still achieved at z_0 by D'_{z_0}. Now we apply Lemma 6.3.8 to G_1' and we conclude that D'_{z_0} can be extended to a $(4p,\beta)$-orientation on G_1', thus also a $(4p,\beta)$-orientation on G_1. So the $(4p,\beta)$-orientation of $\hat{H}^c$ is extended to $\hat{H}$ and thus $\hat{G}^*$ admits a $(4p,\beta)$-orientation with $\beta(v) \equiv 2p \cdot d^+(v) \pmod{4p}$. By Theorem 6.3.6, G^* admits a circular $\frac{4p}{2p-1}$-flow, thus by duality, a contradiction. □

6.4 Further questions

Recall a restatement of Tutte's 5-flow conjecture (Conjecture 6.1.2): Every 2-edge-connected signed graph admits a circular 10-flow. Based on Seymour's 6-flow theorem, we showed in Theorem 6.1.1 that every 2-edge-connected signed graph admits a circular 12-flow.

It has been proved in [PZ03] that for any rational number r between 2 and 5, there exists a graph G with circular flow index being r. Thus by Lemma 5.4.18 and regarding a graph G as a signed graph $(G,+)$, we have the following corollary:

Proposition 6.4.1. *For any rational number $r \in [2,10]$, there exists a 2-edge-connected signed graph $\hat{G}$ whose circular flow index is r.*

Improving the edge-connectivity conditions, we have seen in Theorem 6.1.4 that the 3-edge-connectivity is sufficient for a signed graph to admit a circular 6-flow but we expect this condition can also imply the existence of circular 5-flows of signed graphs. Noting that there is a reduction of Tutte's 5-flow conjecture to 3-edge-connected cubic graphs [Sey81], we have the following stronger conjecture.

Conjecture 6.4.2. *Every 3-edge-connected signed graph admits a circular 5-flow.*

Similar to the restatement of Tutte's 5-flow conjecture, by applying Observation 5.4.3 and Lemma 5.4.18, we can reformulate Tutte's 4-flow conjecture as follows. Note that Petersen graph, denoted by P, admits a nowhere-zero 5-flow but do not admit a nowhere-zero 4-flow, we have examples of 2-connected signed graphs, for example, $T_2(P,+)$, which do not admit a circular 8-flow.

Conjecture 6.4.3. [Tutte's 4-flow conjecture restated] *Every 2-edge-connected signed Petersen-minor-free graph admits a circular 8-flow.*

Furthermore, we showed in Theorem 6.1.5 that every 4-edge-connected signed graph admits a circular 4-flow and this is the best bound of the circular flow index for the class of 4-edge-connected signed graphs that we can expect. The tightness is based on the construction of a sequence of signed bipartite planar graphs of girth 4 on n vertices whose circular chromatic numbers is $4 - \frac{4}{\lceil \frac{n+2}{2} \rceil}$. See Theorem 7.2.8 for more details.

W.T. Tutte has conjectured that every 4-edge-connected graph admits a nowhere-zero 3-flow. Later, M. Kochol proved in [Koc01] that Tutte's 3-flow conjecture is equivalent to the following statement.

Conjecture 6.4.4. [Tutte's 3-flow conjecture restated] *Every 5-edge-connected graph admits a nowhere-zero 3-flow.*

Thus we may propose the next conjecture which implies Tutte's 3-flow conjecture if it is true.

Conjecture 6.4.5. *Every 5-edge-connected signed graph admits a circular 3-flow.*

In the end, we state a few more general questions, analogous to Jaeger's circular flow conjecture.

Question 6.4.6. *Given an integer $k \geqslant 1$, what is the smallest integer $f_1(k)$ such that every $f_1(k)$-edge-connected signed graphs admits a circular $\frac{2k+1}{k}$-flow?*

Question 6.4.7. *Given an integer $k \geqslant 1$, what is the smallest integer $f_2(k)$ such that every $f_2(k)$-edge-connected signed graphs admits a circular $\frac{4k}{2k-1}$-flow?*

We provide upper bounds for each of $f_1(k)$ and $f_2(k)$ in Theorem 6.3.5, that is to say, $f_1(k) \leqslant 6k + 2$ and $f_2(k) \leqslant 6k - 1$. When restricted to signed Eulerian graphs, we have the following question.

Question 6.4.8. *Given an integer $k \geqslant 1$, what is the smallest integer $g(k)$ such that every $g(k)$-edge-connected signed Eulerian graphs admits a circular $\frac{4k}{2k-1}$-flow?*

Regarding this question, we proved in Theorem 6.3.7 that every $(6k-2)$-edge-connected signed Eulerian graph admits a circular $\frac{4k}{2k-1}$-flow, thus $g(k) \leqslant 6k-2$. Restricted to planar graphs, it also implies that every signed bipartite planar graph of girth $6k-2$ admits a homomorphism to C_{-2k}.

Noting that in graph, large odd-edge-connectivity is sufficient for graphs to admit circular $\frac{p}{q}$-flows. Here, for signed Eulerian graphs, we consider to replace the condition of edge-connectivity with the negative-edge-connectivity, i.e., the size of a minimum negative edge-cut. Dual to the bipartite analogue of Jaeger-Zhang conjecture (see Question 1.5.3), we propose a stronger question:

Question 6.4.9. *Given an integer $k \geqslant 1$, what is the smallest integer $g'(k)$ such that every negative-$g'(k)$-edge-connected signed Eulerian graphs admits a circular $\frac{4k}{2k-1}$-flow?*

Part IV

Homomorphisms of Signed (Bipartite) Planar Graphs

7 | Circular $(4-\epsilon)$-colorings of signed bipartite planar graphs

This chapter is based on the following papers:

[KNNW22] F. Kardoš, J. Narboni, R. Naserasr, and Z. Wang. "Circular $(4-\epsilon)$-coloring of some classes of signed graphs". In: *SIAM J. Discrete Math. Accepted subject to revision. arXiv:2107.12126* (2022)

[NWZ21] R. Naserasr, Z. Wang, and X. Zhu. "Circular chromatic number of signed graphs". In: *Electron. J. Combin.* 28.2 (2021), Paper No. 2.44, 40. DOI: 10.37236/9938

Note that in Theorem 3.3.12, to capture the 4-color theorem, it is sufficient to bound (from above) the circular chromatic number of a special subclass of signed bipartite planar graphs by $\frac{16}{5}$. Recall that any signed bipartite graph with a digon has the circular chromatic number of exactly 4. Thus we are highly motivated to study the circular chromatic number of the class of signed bipartite planar simple graphs (see Question 3.3.15). Let $\mathcal{SBP}$ denote the class of signed bipartite planar graphs and for each even k, we denote by $\mathcal{SBP}_k$ the class of signed bipartite planar graphs of negative-girth at least k.

In this chapter, on the one hand, we obtain that $\chi_c(\mathcal{SBP}_4) = 4$ and thus answer Question 3.3.15. On the other hand, we prove that the supremum 4 cannot be reached by any of the signed bipartite planar graphs. More precisely, in Theorem 7.2.1, applying the tight cycle argument, we give an upper bound of $4 - \frac{4}{\lfloor \frac{n+2}{2} \rfloor}$ for the circular chromatic number of a signed bipartite planar simple graph on n vertices. Moreover, we construct a sequence of signed bipartite planar simple graphs that attain the bound of Theorem 7.2.8. Based on the notion of signed indicators, we introduce Lemma 7.2.4 and Corollary 7.2.6 to determine the circular chromatic numbers of our sequence of signed graphs.

7.1 $\mathcal{C}_{<4}$-closed operations

Let $\mathcal{C}_{<4}$ be the class of signed graphs of circular chromatic number strictly smaller than 4. In this section, we present two graph operations that preserve membership in this class $\mathcal{C}_{<4}$.

We first observe that Theorem 4.1.2 could also be viewed as an operation that preserves membership in this class: For each $(G, \sigma) \in \mathcal{C}_{<4}$ and any pair of distinct vertices x and y of (G, σ), if we add a degree-2 vertex u and join it to two distinct vertices x and y with edges of arbitrary signs, then the resulting signed graph is also in $\mathcal{C}_{<4}$.

A slight modification and generalization of this one is based on the following notation. Let (G, σ) be a signed graph and let u be a vertex of (G, σ). We define $F_u(G, \sigma)$ to be the signed graph obtained from (G, σ) by contracting all the edges incident to u and keeping signs of all other edges as it is. One could easily observe that for two switching-equivalent signatures σ and σ' of G, the signed graphs $F_u(G, \sigma)$ and $F_u(G, \sigma')$ might not be switching equivalent. The claim of next theorem is that the inverse operation of F_u is a $\mathcal{C}_{<4}$-closed operation.

Theorem 7.1.1. *Given a signed graph (G, σ) and a vertex u of (G, σ), if $\chi_c(F_u(G, \sigma)) < 4$, then $\chi_c(G, \sigma) < 4$.*

As $F_u(G, \sigma)$ and $F_u(G, \sigma')$ might not be switching equivalent even if (G, σ) and (G, σ') are switching equivalent, while applying this theorem in our main proof it is important to choose a suitable signature (switching equivalent to σ). In particular, if in G two neighbors of u, say x and y, are adjacent with a positive edge, then $F_u(G, \sigma)$ will have a positive loop and so its circular chromatic number is ∞. Similarly, if in G two neighbors of u have another common neighbor v which sees one with a positive edge and the other with a negative edge, then $F_u(G, \sigma)$ has a digon and does not belong to $\mathcal{C}_{<4}$.

The proof of this theorem is quite similar to the proof of Theorem 4.1.2. We consider a circular $(4-\epsilon)$-coloring of $F_u(G, \sigma)$. Then we consider a corresponding coloring on $(G-u, \sigma)$ noting that all neighbors of u are colored with a same color. We then modify the coloring as in the proof of Theorem 4.1.2 to find a color for u. We omit the details.

Next we define an edge-operation which also preserves membership in $\mathcal{C}_{<4}$. Let $\hat{G}$ be a signed graph with a positive edge uv. We define $F_{uv}(\hat{G})$ to be the signed graph obtained from $\hat{G}$ as follows. First we add a copy u' of u, that is to say for every neighbor w (except v) of u we join u' to w with an edge which is of the same sign as uw. Similarly, we add a copy v' of v. Then we add two more vertices x and y with the following connections: xu, yv as positive edges and xu', yv', and xy as negative edges. See Figure 7.1.

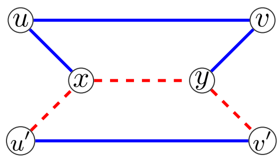


Figure 7.1. The operation F_{uv}

With similar techniques to the proof of Theorem 4.1.2 (and Theorem 7.1.1), we prove that $\mathcal{C}_{<4}$ is closed under the operation F_{uv}.

Theorem 7.1.2. *Given a signed graph $\hat{G}$ and a positive edge uv of $\hat{G}$, if $\chi_c(\hat{G}) < 4$, then $\chi_c(F_{uv}(\hat{G})) < 4$.*

Proof. Let φ be a circular $(4-\epsilon)$-coloring of $\hat{G}$ for a positive real number ϵ. We assume, without loss of generality, that $\varphi(u) = 0$ and $1 \leqslant \varphi(v) \leqslant 2 - \frac{\epsilon}{2}$. We view φ as a partial coloring of $F_{uv}(\hat{G})$. Our goal would be to modify φ to a circular $(4-\frac{\epsilon}{4})$-coloring of $\hat{G}$ in such a way that we can extend it on the four new vertices u', v', x and y. This would be done in two steps. We first scale the circle to increase its circumference by $\frac{\epsilon}{2}$ and then insert an interval of length $\frac{\epsilon}{4}$ into the circle after which we must modify images of some vertices. The details are as follows.

First we define the $(4-\frac{\epsilon}{2})$-coloring φ' as follows:

$$\varphi'(w) = \frac{4-\frac{\epsilon}{2}}{4-\epsilon}\varphi(w).$$

This was the scaling part. Let $\gamma = \frac{4-\frac{\epsilon}{2}}{4-\epsilon}$ and note that

$$\gamma = 1 + \frac{\epsilon}{8-2\epsilon} > 1 + \frac{\epsilon}{8} > 1.$$

Next we define ψ and show that it is a circular $(4-\frac{\epsilon}{4})$-coloring of $F_{uv}(\hat{G})$. On the vertices of $\hat{G}$, the mapping ψ is defined as follows.

$$\psi(w) = \begin{cases} \varphi'(w), & \text{if } \varphi'(w) < 1 - \frac{\epsilon}{8}, \\ \varphi'(w) + \frac{\epsilon}{4}, & \text{otherwise.} \end{cases}$$

It is then extended to the remaining four vertices by:

$$\psi(u') = \frac{\epsilon}{8},\ \psi(v') = \varphi'(v) + \frac{\epsilon}{8},\ \psi(x) = 1,\ \text{and}\ \psi(y) = \varphi'(v) + \frac{\epsilon}{4} - 1.$$

We need to show that ψ is a circular $(4-\frac{\epsilon}{4})$-coloring of $F_{uv}(\hat{G})$. Restriction of ψ on $\hat{G}$ is a circular coloring because if w_1 and w_2 are two adjacent vertices of $\hat{G}$, then in coloring φ either (1) $\varphi(w_1)$ and $\varphi(w_2)$ are at distance at least 1, or (2) $\varphi(w_1)$ and $\overline{\varphi(w_2)}$ are at distance at least 1. This distance then is increased to at least γ in φ'. Then in defining ψ based on φ' either the distance remains the same, or it increases by $\frac{\epsilon}{4}$, or one end moves by a value of $\frac{\epsilon}{8}$. Thus, in all the cases the resulting distance is still larger than 1.

It remains to consider the connection to and among new vertices, u', v', x, and y. By the definition of ψ, the five edges incident to x or y are satisfying the conditions of the circular $(4-\frac{\epsilon}{4})$-coloring. It remains to verify the condition for edges incident with u' and v' but not incident with x or y.

We first consider the edges incident with u'. Recall that u' is a copy of u. Let w be a neighbor u in $\hat{G}$. Based on the sign of wu' we consider two cases.

- wu' is a positive edge.

  We need to show that the distance between $\psi(w)$ and $\psi(u')$ is at least 1. Using the definition of circular coloring based on the circle, we consider both clockwise and anticlockwise distances on the circle. The anticlockwise path of the circle from u' to w passes through u and since u and w are already proved to be at distance at least 1, the anticlockwise distance from $\psi(u')$ to $\psi(w)$ is larger than 1. For the clockwise direction, since uw is a positive edge we have $\varphi(w) \geqslant 1$. Thus, by the definition of ψ, we $\psi(w) = \varphi'(w) + \frac{\epsilon}{4}$ whereas $\psi(u') = \frac{\epsilon}{8}$. Therefore, the clockwise distance of $\psi(u')$ and $\psi(w)$ is larger than the clockwise distance of $\varphi(u) = 0$ and $\varphi'(w)$ which is at least 1.

- wu' is a negative edge.

  In circular $(4-\epsilon)$-coloring φ, the distance of $\varphi(u)$ and $\varphi(w)$ is at most $1-\frac{\epsilon}{2}$. Again we consider two possibilities depending on if the distance is obtained in clockwise direction starting from 0 or anticlockwise. For clockwise direction, we observe that $\varphi'(w) < 1 - \frac{\epsilon}{8}$. Thus in defining ψ the distance of $\psi(u)$ and $\psi(w)$ remains the same as the distance of $\varphi'(u)$ and $\varphi'(w)$, and the distance of $\psi(u')$ to $\psi(w)$ is actually shorter. If the distance of $\varphi(w)$ and $\varphi(u)$ is obtained on anticlockwise direction starting from 0, then this distance is at most $1-\frac{\epsilon}{2}$. Therefore, the distance of $\varphi'(w)$ and $\varphi'(u)$ is at most $(1-\frac{\epsilon}{2})\gamma$ which is strictly smaller than $1-\frac{\epsilon}{4}$. As $\psi(u') = \frac{\epsilon}{8}$, the distance between $\psi(w)$ and $\psi(u')$ remains strictly smaller than $1-\frac{\epsilon}{8}$, thus the negative edge wu' satisfies the condition.

We now consider the edges incident with v'. Observe that since $1 \leqslant \varphi(v) \leqslant 2 - \frac{\epsilon}{2}$, we have $\gamma \leqslant \varphi'(v) \leqslant \gamma(2 - \frac{\epsilon}{2})$. By the definition of ψ, and because $\varphi'(v) \geqslant 1 - \frac{\epsilon}{8}$, we have:

$$\gamma + \frac{\epsilon}{8} \leqslant \psi(v') = \varphi'(v) + \frac{\epsilon}{8} \leqslant \gamma(2 - \frac{\epsilon}{2}) + \frac{\epsilon}{8} = 2 - \frac{\epsilon}{8}.$$

As v' is a copy of v, based on the sign of wv we consider two cases.

- wv' is a positive edge.

  We need to show that the distance between $\psi(w)$ and $\psi(v')$ is at least 1. Since wv is a positive edge, there are two possibilities: (1) $\varphi(w) \in [0, 1 - \frac{\epsilon}{2}]$, (2) $\varphi(w) \in [2, 4 - \epsilon]$. For case (1), $\varphi'(w) = \gamma\varphi(w) < 1 - \frac{\epsilon}{8}$, then $\psi(w) = \varphi'(w)$ and thus the distance between $\psi(w)$ and $\psi(v')$ is larger than $\gamma + \frac{\epsilon}{8}$. For case (2), $\psi(w) = \varphi'(w) + \frac{\epsilon}{4}$. Thus the distance between $\psi(w)$ and $\psi(v')$ is at least $1 + \frac{\epsilon}{8-2\epsilon} + \frac{\epsilon}{8}$, hence strictly larger than 1.

- wv' is a negative edge.

  As wv is a negative edge in $\hat{G}$, in any circular $(4-\epsilon)$-coloring φ, the distance of $\varphi(v)$ and $\varphi(w)$ is at most $1 - \frac{\epsilon}{2}$ and then the distance of $\varphi'(v)$ and $\varphi'(w)$ is at most $\gamma(1 - \frac{\epsilon}{2}) < 1 - \frac{\epsilon}{4}$. Also, we have that $\frac{\epsilon}{2} \leqslant \varphi(w) \leqslant 3 - \epsilon$. By the definition of ψ, if $\varphi'(w) \geqslant 1 - \frac{\epsilon}{8}$, then $\psi(w) = \varphi'(w) + \frac{\epsilon}{4}$ and thus the distance between $\psi(w)$ and $\psi(v')$ is at most $\gamma(1 - \frac{\epsilon}{2}) + \frac{\epsilon}{8} < 1 - \frac{\epsilon}{8}$. It remains to show that if $\varphi'(w) < 1 - \frac{\epsilon}{8}$, then the distance between $\psi(w)$ and $\psi(v')$ is smaller than $1 - \frac{\epsilon}{8}$. In this case, $\psi(w) = \varphi'(w)$. Therefore, compared with the distance between $\varphi'(w)$ and $\varphi'(v)$, the distance between $\psi(w)$ and $\psi(v')$ is increased by $\frac{\epsilon}{8}$, therefore, it is at most $\gamma(1 - \frac{\epsilon}{2}) + \frac{\epsilon}{8} < 1 - \frac{\epsilon}{8}$. □

These operations will be used in the proof of the main result.

7.2 Signed bipartite planar simple graphs

In this section, we shall prove the following theorem, which implies that $\mathcal{SBP}_4 \subset \mathcal{C}_{<4}$.

Theorem 7.2.1. *If (G, σ) is a signed bipartite planar simple graph on n vertices, then*

$$\chi_c(G, \sigma) \leqslant 4 - \frac{4}{\lfloor \frac{n+2}{2} \rfloor}.$$

Moreover, this upper bound is tight for each value of $n \geqslant 2$.

As in Section 4.1.1, we first show that the circular chromatic number of any signed bipartite planar simple graph is strictly smaller than 4 in Theorem 7.2.2. Then we use the notion of tight cycle to get an improved upper bound. Finally, we show that this upper bound is tight.

To this end, we will work with a minimum counterexample. One of the properties of a minimum counterexample follows from the bipartite folding lemma (Lemma 2.3.15). Recall that a (signed) plane graph is a (signed) planar graph together an embedding on the plane. For a plane graph, a *separating* ℓ-cycle is an ℓ-cycle which is not a face. Lemma 2.3.15 allows us to "fold" positive facial cycles in signed bipartite plane graphs such that the negative girth condition is not changed.

Theorem 7.2.2. *For any signed bipartite planar simple graph (G, σ), we have $\chi_c(G, \sigma) < 4$.*

Proof. Assume that (G, σ) is a minimum counterexample with respect to $|V(G)|$, i.e., for no $\epsilon > 0$, (G, σ) admits a circular $(4 - \epsilon)$-coloring and $|V(G)|$ is minimized. By the minimality of (G, σ) and the bipartite folding lemma, we may assume that (G, σ) admits a planar embedding such that every facial cycle of (G, σ) is a negative 4-cycle. From here on, we will consider (G, σ) together with such a planar embedding. Moreover, since any subgraph of (G, σ) is also a signed bipartite planar simple graph, it follows from Theorem 4.1.2 that $\delta(G) \geqslant 3$.

We proceed by proving some structural properties of (G, σ) in the form of claims.

Claim 1 *Every vertex of even degree in (G, σ) must be in a separating 4-cycle.*

Proof of the claim: Let u be vertex of (G, σ) and of even degree. Assume to the contrary that u is not in any separating 4-cycle. Let C be the boundary of the face in $(G - u, \sigma)$ which contains u. This cycle C in the embedding of (G, σ) bounds $d(u)$ negative faces, each of which is a negative 4-cycle. As $d(u)$ is even, C is a positive cycle. Since switching does not affect the circular chromatic number, we may assume that σ is a signature in which all the edges of C are positive.

Let (G', σ') be the signed graph obtained from (G, σ) by contracting all edges incident with u and by replacing each set of parallel edges of a same sign with a single edge of the same sign. Observe that, as u is in no separating 4-cycle, (G', σ') has no digon. Thus (G', u') is a signed simple graph. Furthermore, it is a signed bipartite planar simple graph which has less vertices than (G, σ). Thus it admits a circular $(4 - \epsilon)$-coloring for some positive ϵ. But then Theorem 7.1.1 implies that $\chi_c(G, \sigma) < 4$. ♢

Claim 2 *For every pair of adjacent vertices each of an odd degree in (G, σ), at least one is in a separating 4-cycle.*

Proof of the claim: The proof of this claim is similar to the previous one. Towards the contradiction, let x and y be two adjacent vertices of odd degrees, neither of which is in a separating 4-cycle. We consider the facial cycle C which is obtained after deleting x and y, and once again conclude that C must be a positive cycle as it bounds $d(x) + d(y)$ many negative faces in (G, σ). Without loss of generality, we assume that σ assigns positive signs to all the edges of C and that $\sigma(xy) = -$. Let $x_1, x_2, \ldots, x_\ell$ be the neighbors of x distinct from y in the cyclic order of the embedding and, similarly, let $y_k, y_{k-1}, \ldots, y_1$ be the neighbors of y, distinct from x, in the cyclic order (see Figure 7.2). Thus x_1y_1 and $x_\ell y_k$ are both edges of C and hence both are positive. We have two assertions on the neighborhood of x and y.

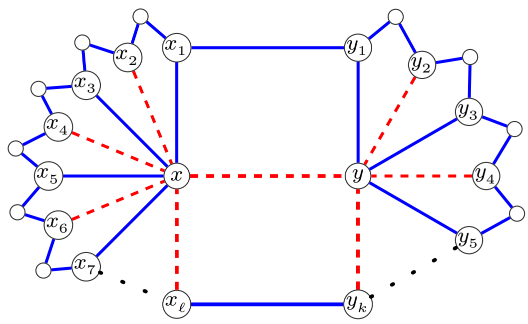


Figure 7.2. $\{x, y\}$-neighborhood

The first observation is that x_1y_1 and $x_\ell y_k$ are the only edges connecting some x_i to some y_j. That is because any other connection x_iy_j would create a separating 4-cycle xx_iy_jy but we have assumed (towards a contradiction) that x and y are in no such a 4-cycle. The second is that $\sigma(xx_1) = \sigma(yy_1)$ and that $\sigma(xx_\ell) = \sigma(yy_k)$. To see that $\sigma(xx_1) = \sigma(yy_1)$, we consider the face xx_1y_1y. We already know that xy is a negative edge and that x_1y_1 is a positive edge. For this face to be a negative 4-cycle then we must have $\sigma(xx_1) = \sigma(yy_1)$. That $\sigma(xx_\ell) = \sigma(yy_k)$ follows from the same argument by considering the face $xx_\ell y_k y$.

To complete the proof of the claim, we consider two signed graphs, we consider two signed graphs as follows. The first one is obtained from $(G - xy, \sigma)$ by contracting all the edges incident to

x where the new vertex is denoted u, and by contracting all the edges incident to y where the new vertex is denoted v, by replacing each set of parallel edges of a same sign with a single edge of the same sign and then adding a positive edge to connect u and v. We denote the result by $\hat{G}'$ and note that it is a signed bipartite planar graph. Moreover, as neither of x, y is in a separating 4-cycle, $\hat{G}'$ has no digon and thus it is simple. We note furthermore that in $\hat{G}'$ the vertex u is connected to the vertex v with a positive edge (resulted from x_1y_1 and $x_\ell y_k$). By the minimality of (G,σ), we conclude that $\chi_c(\hat{G}') < 4$.

The second signed graph we consider is obtained from (G,σ) by identifying positive neighbors of x into a new vertex u, the negative ones (except for y) into a new vertex u', and by identifying positive neighbors of y into a new vertex v, the negative ones (except for x) into a new vertex v'. Among a set of parallel edges of the same sign we delete all but one. Let $\hat{G}''$ be the resulting signed graph. We note that $\hat{G}''$ is not necessarily planar anymore. It follows from the discussion on the neighborhood of x and y that in $\hat{G}''$ there is no edge connecting u' to v and, similarly, no edge connecting u to v'. Moreover, u is connected to v only with a positive edge and u' is connected to v' only with a positive edge.

Overall we observe that $\hat{G}''$ is a (proper) subgraph of $F_{uv}(\hat{G}')$. It follows from Theorem 7.1.2 that $F_{uv}(\hat{G}')$ and, therefore, $\hat{G}''$ is in $\mathcal{C}_{<4}$, but $\hat{G}''$ is a homomorphic image of (G,σ) which implies $\chi_c(G,\sigma) \leqslant \chi_c(\hat{G}'')$. ◊

Claim 3 *There is no separating* 4*-cycle in* (G,σ)*.*

Proof of the claim: Towards a contradiction, assume that there is a separating 4-cycle and let C be a separating 4-cycle with the minimum number of vertices inside. Let v_1, v_2, v_3, and v_4 be the four vertices of C in this cyclic order. Let u be a vertex inside C. As (G,σ) is bipartite, u can be adjacent to at most two vertices of C. Since (G,σ) has minimum degree at least 3, u must have a neighbor, say v, which is not on C and thus inside C. By Claim 1 and Claim 2, at least one of u or v, say u, is in a separating 4-cycle, denoted C_u. Since C contains the minimum number of vertices inside, C_u cannot be all inside C. Thus u is adjacent to two vertices of C. Noting that G is bipartite, and by symmetry, we may assume v_1 and v_3 are adjacent to u. Then of the two 4-cycles $uv_1v_2v_3$ and $uv_1v_4v_3$ one contains v and thus is a separating 4-cycle with less vertices inside than C. This contradicts the choice of C and, hence, proves the claim. ◊

To complete the proof of the theorem, we observe that, by Claims 1 and 3, all vertices must be of odd degree, and, by Claim 2, no two of them can be adjacent, a contradiction. □

Tight examples

We now construct tight examples and give the tight accurate bound in Theorem 7.2.8.

Definition 7.2.3. Let Γ_1 be a positive 2-path connecting u_1 and v_1. For $i \geqslant 2$,

- if i is even, then let Γ_i be obtained from Γ_{i-1} by
  - adding two vertices u_i and v_i,
  - connecting u_i to u_{i-1} by a positive edge, u_i to v_{i-1} by a negative edge,
  - connecting v_i to u_{i-1} by a negative edge, v_i to v_{i-1} by a positive edge;
- if i is odd, then let Γ_i be obtained from Γ_{i-1} by
  - adding two vertices u_i and v_i,
  - connecting each of u_i and v_i to each of u_{i-1} and v_{i-1} by a positive edge.

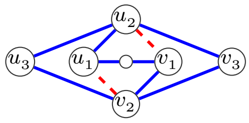


Figure 7.3. Γ_3

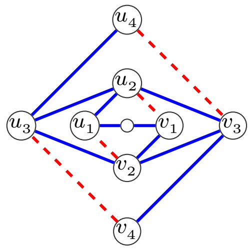


Figure 7.4. Γ_4

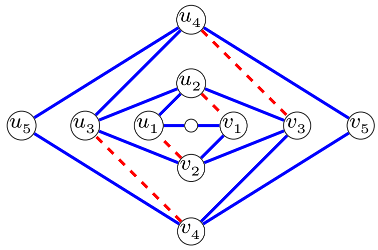


Figure 7.5. Γ_5

For example, Γ_3, Γ_4 and Γ_5 are illustrated in Figure 7.3, 7.4, and 7.5 respectively.

Since this construction is similar to a construction in Section 4.1.1(see Figures 4.3, 4.4, and 4.5), with similar idea used in the proof of Proposition 4.1.4, we can also provide the circular chromatic number of each of the signed graphs in this sequence. Here, we give an alternative proof using the notion of signed indicator. First we prove some properties of this sequence of signed graphs when regarded as signed indicators.

Lemma 7.2.4. *Assume $i \geqslant 1$, $0 < \epsilon < \frac{1}{i}$ and $r = 4 - 2\epsilon$. Let $\mathcal{I}_i = (\Gamma_i, u_i, v_i)$ be a signed indicator. The following claims hold:*

- *If i is odd, then $Z(\mathcal{I}_i, r) = [0, \frac{r}{2} - i\epsilon]$.*
- *If i is even, then $Z(\mathcal{I}_i, r) = [i\epsilon, \frac{r}{2}]$.*

Proof. We prove this lemma by induction on i. For $i = 1$, this is trivially true. Assume $i \geqslant 2$ and the lemma holds for $i' < i$.

Case 1. i is even.

Assume ϕ is a circular r-coloring of Γ_i with $\phi(u_{i-1}) = 0$. As $Z(\mathcal{I}_{i-1}, r) = [0, \frac{r}{2} - (i-1)\epsilon]$, we may assume that $\phi(v_{i-1}) \in [0, \frac{r}{2} - (i-1)\epsilon]$. The possible colors for u_i are $[1, \frac{r}{2} + 1 - i\epsilon]$, and the possible colors for v_i are $[3 - \epsilon, r) \cup [0, 1 - i\epsilon]$. So the possible distances between $\phi(u_i)$ and $\phi(v_i)$ are $[i\epsilon, \frac{r}{2}]$, i.e., $Z(\mathcal{I}_i, r) = [i\epsilon, \frac{r}{2}]$.

Case 2. i is odd.

Assume ϕ is a circular r-coloring of Γ_i with $\phi(u_{i-1}) = 0$. As $Z(\mathcal{I}_{i-1}, r) = [(i-1)\epsilon, \frac{r}{2}]$, we may assume that $\phi(v_{i-1}) \in [(i-1)\epsilon, \frac{r}{2}]$. The possible colors for u_i and v_i are $[1 + (i-1)\epsilon, 3 - 2\epsilon]$. So the possible distances between $\phi(u_i)$ and $\phi(v_i)$ are $[0, 2 - (i+1)\epsilon] = [0, \frac{r}{2} - i\epsilon]$, i.e., $Z(\mathcal{I}_i, r) = [0, \frac{r}{2} - i\epsilon]$.

It completes the proof. □

Corollary 7.2.5. *For any $\epsilon > 0$, there is a signed bipartite planar simple graph Γ with $\chi_c(\Gamma) > 4 - 2\epsilon$.*

Proof. Let $\frac{1}{2\epsilon} < i < \frac{1}{\epsilon}$. Let Γ'_i be obtained from the disjoint union of Γ_{2i-1} and Γ_{2i} by identifying u_{2i-1} in Γ_{2i-1} and u_{2i} in Γ_{2i} into a single vertex u'_i, and identifying v_{2i-1} in Γ_{2i-1} and v_{2i} in Γ_{2i} into a single vertex v'_i. It follows from the construction that Γ'_i is a signed bipartite planar simple graph.

Let $\mathcal{I}'_i = (\Gamma'_i, u'_i, v'_i)$. Then for $r = 4 - 2\epsilon$, $\frac{r}{2} - (2i-1)\epsilon = 2 - 2i\epsilon < 2i\epsilon$. Hence

$$Z(\mathcal{I}'_i, r) = Z(\mathcal{I}_{2i-1}, r) \cap Z(\mathcal{I}_{2i}, r) = [0, \frac{r}{2} - (2i-1)\epsilon] \cap [2i\epsilon, \frac{r}{2}] = \varnothing.$$

So Γ'_i is not circular r-colorable. □

Corollary 7.2.6. *If $i = 2k$, then for any graph G,*

$$\chi_c(G(\mathcal{I}_i)) = \frac{4k\chi_c(G)}{k\chi_c(G)+1} = 4 - \frac{4}{k\chi_c(G)+1}.$$

Proof. Assume that $\chi_c(G(\mathcal{I}_i)) = r = 4 - 2\epsilon$. For $i = 2k$, by Lemma 7.2.4, $Z(\mathcal{I}_i) = [i\epsilon, \frac{r}{2}]$. Thus $\mathcal{I}_i$ is a plus indicator. Note that $i\epsilon = k(4-r)$. We apply Lemma 3.3.9 with $\chi_c(G) = r'$, we have that $\frac{r}{r'} = k(4-r)$ by solving which we obtain the formula in the statement. □

Corollary 7.2.7. *We have that*

- $\chi_c(\Gamma_{2k}) = 4 - \frac{4}{2k+1}$.
- $\chi_c(\Gamma_{2k+1}) = 4 - \frac{2}{k+1}$.

Proof. For even values of $i = 2k$, the formula for the circular chromatic number of Γ_{2k} follows from Corollary 7.2.6 by taking G to be K_2.

Let $r = 4 - 2\epsilon$. A similar computation can be done by taking Γ_{2k+1} as an indicator $\mathcal{I}_-$ in Lemma 3.3.11 with $\mathcal{I}_+$ being free to choose (with respect to $t = (2k+1)\frac{4-r}{2}$ where $r = \chi_c(\Omega(\mathcal{I}_+, \mathcal{I}_{2k+1})))$. Applying this indicator to $(K_2, -)$, then we get the formula that

$$r = (2k+1)\frac{4-r}{2}\chi_c(K_2, -),$$

and thus $r = \frac{4k+2}{k+1}$. □

Next, using the notion of tight cycle, we improve the bound of Theorem 7.2.2 and then show that this improved bound is tight.

Theorem 7.2.8. *For any signed bipartite planar simple graph (G, σ) on n vertices, we have:*

- *For each odd value of n, $\chi_c(G, \sigma) \leqslant 4 - \dfrac{8}{n+1}$.*
- *For each even value of n, $\chi_c(G, \sigma) \leqslant 4 - \dfrac{8}{n+2}$.*

Moreover, these bounds are tight for each value of $n \geqslant 2$.

Proof. As stated in Proposition 3.1.11, we know that $\chi_c(G, \sigma) = \frac{p}{q}$ where p is twice the length of a cycle in G. As G is a bipartite graph, the length of each cycle is even. Thus $p = 4k$ for some positive integer k such that $2k \leqslant n$.

Since $\chi_c(G, \sigma) < 4$ we have $\frac{p}{q} < 4$, in other words, $4k < 4q$. As k and q are integers we have $k+1 \leqslant q$. Thus $\chi_c(G, \sigma) \leqslant \frac{4k}{k+1} = 4 - \frac{4}{k+1}$. The upper bounds claimed in the theorem then follows by noting that $n \geqslant 2k$ and that $n \geqslant 2k+1$ when n is odd.

To prove that the bounds are tight, we consider a homomorphic image Γ_i^* of the above examples Γ_i for when $n = 2i$ is even. To build Γ_i^* from Γ_i, one must identify the last two vertices u_i and v_i of Γ_i after a suitable switching. Note that the number of vertices of Γ_i^* is $2i$. Then by adding an isolated vertex to Γ_i, we get an example that works for $n = 2i+1$. By Corollary 7.2.7, we are done. □

Here we provided a bound of the circular chromatic number for the class of signed bipartite planar simple graphs. In the next chapters, we will investigate several subclasses of signed bipartite planar graphs of given larger negative-girths. The study also fits well in the framework of the study of the analogue of Jaeger-Zhang conjecture.

8 | Edge-density of C_{-4}-critical signed graphs

This chapter is based on the following paper:

[NPW22] R. Naserasr, L. A. Pham, and Z. Wang. "Density of C_{-4}-critical signed graphs". In: *J. Combin. Theory Ser. B* 153 (2022), pp. 81–104. DOI: 10.1016/j.jctb.2021.11.002

Using the T_ℓ-construction introduced in Section 2.3.3, we have seen another reformulation of 4-color theorem (Theorem 2.3.7): For any planar graph G, the signed graph $T_2(G,+)$ admits a homomorphism to C_{-4}. Observing that for a simple graph G the negative-girth of $T_2(G,+)$ is at least 6 (corresponding to a triangle of G), R. Naserasr, E. Rollova, and E. Sopena conjectured in [NRS15] that every signed bipartite planar graph of negative-girth at least 6 admits a homomorphism to C_{-4}. However, in Section 8.5, we disprove this conjecture. Furthermore, as an application of the results of this section, we prove that if the condition on negative-girth is increased to 8, then the result holds.

In Section 2.4, we have introduced the notion of (H,π)-critical signed graphs. In this chapter, we will focus on C_{-4}-critical signed graphs. Since the girth condition of Definition 2.4.2 eliminates the cases which do not satisfy conditions of the no-homomorphism lemma, the first observation is that C_{-4}-critical signed graphs are all bipartite. We show in Theorem 8.3.1 that every C_{-4}-critical signed graph on n vertices must have at least $\lceil \frac{4n}{3} \rceil$ edges with a sole exception of a signed bipartite graph on 7 vertices which has only 9 edges (depicted in Figure 8.4). The main method that we use in the proof is the potential technique, developed in [KY14a] and then further used in [DP17] and [PS22], to estimate the potential and thus the sparsity of such critical graphs.

In the next section, we present the necessary terminology and also include a useful result (Theorem 8.1.1) about determining the existence of an edge-sign preserving homomorphism to C_{-4}. In Section 8.2, we start with some extremal examples of C_{-4}-critical signed graphs, for example, the smallest C_{-4}-critical signed graph with respect to the number of vertices, namely Γ in Figure 8.3 and the smallest C_{-4}-critical signed graph with respect to the edge-density, namely $\hat{W}$ in Figure 8.4. Furthermore, we consider some other signed bipartite graphs and some constructions on them, and prove that either it contains $\hat{W}$ as a subgraph or it maps to C_{-4}. Those structures will appear in the proof of the main theorem when we apply some operations to the minimum counterexample.

After the preparations, in Section 8.3, we prove our main result on the minimum number of edge-density of C_{-4}-critical signed graphs. We use the potential technique to find the forbidden configurations in a minimum counterexample and apply the discharging technique to obtain a contradiction in the end. Furthermore, in Section 8.4 we introduce some techniques to build C_{-4}-critical signed graphs of low edge-density. Among other constructions, the construction $S(C_{2k+1})$ helps to conclude the tightness of our bound. Furthermore, there are two other constructions to build C_{-4}-critical signed graphs with almost tight edge-density.

We conclude this chapter with the application of our result to signed bipartite planar graphs in Section 8.5. Combining the bipartite analogue of Folding lemma and the Euler formula, in Theorem 8.5.1 we show that every signed bipartite planar graph of negative girth at least 8 maps to C_{-4}. Moreover, using an example of a signed planar graph which is not 4-colorable, constructed in [KN21], and applying the T_2-construction to it, we could conclude that girth 8 is the best possible. In order to bound the circular chromatic number of the class $\mathcal{SBP}_8$, we build a sequence of signed bipartite planar graphs of negative-girth 8 whose circular chromatic numbers are approaching $\frac{8}{3}$ and thus we have that $\chi_c(\mathcal{SBP}_8) = \frac{8}{3}$.

8.1 Homomorphism to C_{-4}

As C_{-4} is the primary subject of this work, we will use the labeling of Figure 8.1 when referring to this signed graph on its own, but as a subgraph of another signed graph it will have an induced labeling.

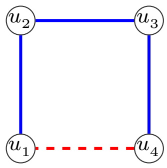


Figure 8.1. C_{-4}

While the switching homomorphism and the edge-sign preserving homomorphism are closely related, these two notions are also fundamentally different. In particular, for our main target C_{-4}, it follows from Corollary 2.3.6 that deciding if a signed graph (G, σ) admits a homomorphism to it is an NP-complete problem. We refer to [BFHN17; BS18; DFM+20] for more on this subject. However, for edge-sign preserving homomorphism to the signed graph in Figure 8.1, there is the following duality theorem.

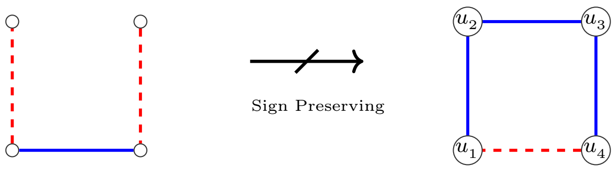


Figure 8.2. C_{-4} and its edge-sign preserving dual

Theorem 8.1.1. [CNS20] *Given a signed bipartite graph (G, σ), we have $(G, \sigma) \xrightarrow{s.p.} C_{-4}$ if and only if $(P_4, \pi) \overset{s.p.}{\not\to} (G, \sigma)$ where (P_4, π) is the signed path of length 3 given in Figure 8.2.*

We note that this leads to a polynomial time algorithm for the problem of determining if an input signed graph admits an edge-sign preserving homomorphism to C_{-4}.

Combined with Theorem 2.2.5, this theorem says that in order to map a signed bipartite graph (G, σ) to C_{-4} it is necessary and sufficient to find a switching σ' of σ where no positive edge is incident with a negative edge at each end of it.

8.2 C_{-4}-critical signed graphs

Setting (H, π) to be C_{-4} in the definition of (H, π)-critical signed graphs, we have the following definition.

Definition 8.2.1. A signed graph (G, σ) is said to be C_{-4}-*critical* if the followings hold:

- (G, σ) is a signed bipartite simple graph;
- $(G, \sigma) \nrightarrow C_{-4}$;
- $(G', \sigma) \to C_{-4}$ for every proper subgraph G' of G.

We first give some general structural properties of a C_{-4}-critical signed graph.

Lemma 8.2.2. *Every C_{-4}-critical signed graph is 2-connected.*

This is an easy consequence of the fact that C_{-4} is vertex transitive.

Recall that a k-thread of G is a path of length k whose internal vertices are all of degree 2 in G. It is easily observed that the maximum length of a thread in an (H, π)-critical graph is bounded by a function of (H, π). For C_{-4}-critical signed graphs, we have:

Lemma 8.2.3. *A C_{-4}-critical signed graph $\hat{G}$ does not contain a 3-thread.*

Proof. Assume to the contrary that G has a 3-thread $P = x_0x_1x_2x_3$. Recall that a C_{-4}-critical signed graph is bipartite. As x_0 and x_3 are connected by a path of length 3, they are in different parts of G. Since $\hat{G}$ is C_{-4}-critical, the signed graph $\hat{G}' = \hat{G} - \{x_1, x_2\}$ maps to C_{-4}. Let φ be such a mapping. Observe that, by Lemma 8.2.2, $\hat{G}'$ is connected, thus φ preserves the bipartition of G'. In particular, $\varphi(x_0)$ and $\varphi(x_3)$ are in two different parts of C_{-4} and thus adjacent. We note that φ has possibly applied switchings on some vertices of G'. Working with the resulting signature obtained from the same switching on $\hat{G}$, we let $\hat{P}$ be the signed graph induced on P. If $\hat{P}$ has the same sign as the edge $\varphi(x_0)\varphi(x_3)$, then φ can be extended by mapping $\hat{P}$ to this edge as well. Otherwise, φ can be extended by mapping $\hat{P}$ to the rest of the C_{-4} (that is $C_{-4} - \varphi(x_0)\varphi(x_3)$). □

In this lemma, length 3 for a forbidden thread is the best one can do. We may see examples of C_{-4}-critical signed graphs with vertices of degree 2, that correspond to 2-threads. However, we still have some restriction on such threads:

Observation 8.2.4. *Given a C_{-4}-critical signed graph, a vertex of degree 2 cannot be on a C_{+4}.*

In fact, this is generally true for any core, note that any (H, π)-critical signed graph must be a core.

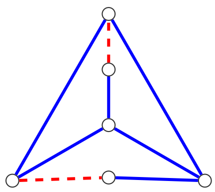


Figure 8.3. The smallest C_{-4}-critical signed graph Γ

Now we can consider some small examples which are C_{-4}-critical. It can be easily verified that any signed bipartite graph with at most two vertices in one of the two parts maps to C_{-4}. Thus the first example of C_{-4}-critical signed graph must have at least six vertices. Let Γ be the signed graph obtained from K_4 by subdividing two nonadjacent edges, each once, with a signature assigned in such a way that each triangle of the K_4 become a negative 4-cycle (see Figure 8.3). It is not hard to see that Γ is an example of a C_{-4}-critical signed graph on six vertices. In fact, up to switching, it is the unique C_{-4}-critical signed graph on six vertices. We further note that Γ has edge-density exactly $\frac{4}{3}$.

An example of higher interest, which is also a signed graph on a subdivision of K_4, is the signed graph $\hat{W}$ of Figure 8.4 which is depicted in two different ways. This signed graph is proved in [CNS20] to have smallest maximum average degree among all signed bipartite graphs that does not map to C_{-4}, that is a maximum average degree of $\frac{18}{7}$. Using the extended notion of C_{-4}-critical signed graphs we introduced here, we will prove $\hat{W}$ to be the sole exception among the signed bipartite graphs of average degree less than $\frac{8}{3}$.

Figure 8.4. C_{-4}-critical signed graph $\hat{W}$ depicted in two ways

We give two different proofs for the fact that $\hat{W}$ does not map to C_{-4}. Each proof takes advantage of one of the presentations in Figure 8.4, and leads to different development of ideas.

Proposition 8.2.5. *The signed graph $\hat{W}$ of Figure 8.4 does not map to C_{-4}. Moreover, up to a switching equivalence, this is the only signature on this graph with this property.*

Proof. Based on the presentation on the left side, if $\hat{W}$ maps to C_{-4}, then the outer 6-cycle, as it is a negative cycle, must map surjectively to C_{-4}, but then v_0 must be identified with one of v_2, v_4, v_6, thus creating a negative cycle of length 2 and, therefore, contradicting the no-homomorphism lemma.

The equivalence class of signatures on this graph is determined by the signs of its three facial 4-cycles as depicted in the left side of the figure. If one of these facial 4-cycles is positive, then degree 2 vertex on this face can be mapped to v_0, after a switching if needed. The resulting image then easily maps to C_{-4}. □

An alternative proof. Based on the presentation on the right side, observe that each pair among y_1, y_2, y_3 is connected by a positive 2-path (through x_1) and by a negative 2-path. Thus identifying any two of them would create C_{-2}. In other words, in any homomorphic image of $\hat{W}$ which is a signed simple graph, the vertices y_1, y_2 and y_3 must have distinct images.

For the moreover part, given a signature on W we may switch it so that x_1y_1, x_1y_2, and x_1y_3 are positive edges. After such a switching, if each of x_2, x_3, and x_4 is incident to one positive and one negative edge, then we have a (switching) isomorphic copy of $\hat{W}$. Otherwise, one of the vertices x_2, x_3, and x_4 can map to x_1. After such a mapping, we have a signed bipartite graph on six vertices. If this signed graph does not map to C_{-4}, then it must contain Γ as a subgraph. However, Γ has eight edges while our six-vertex graph has only seven edges. □

Our goal here is to study the edge-density of C_{-4}-critical signed graphs. We show that, with the exception of $\hat{W}$, every such signed graph has edge density at least $\frac{4}{3}$. In our proof then not only $\hat{W}$ but some constructions based on $\hat{W}$ will be of importance.

Automorphisms of $\hat{W}$ split its vertices to three orbits: $\{x_1\}$, $\{y_1, y_2, y_3\}$ and $\{x_2, x_3, x_4\}$ and split its edges to two orbits: those incident to x_1 and those on the outer 6-cycle. We will need to consider two signed graphs obtained from $\hat{W}$ by subdividing one of its edges twice and then assigning a signature on the edges of this path so that the sign of the path is the same as the sign of the edge it has replaced. Since there are two orbits of the edges on $\hat{W}$, essentially we have only two signed graphs obtained in this way. Presentations of these two signed graphs, each after a switching, are given in Figures 8.5 and 8.6. The signed graph of Figure 8.5, Ω_1, is obtained from $\hat{W}$ by subdividing the edge x_1y_3 twice (where all three edges are assigned positive signs) and then switching at the vertex set $\{x_2, x_3, y_3\}$. The signed graph of Figure 8.6, Ω_2, is obtained from $\hat{W}$ by subdividing the edge x_4y_1 twice (where all three edges are assigned positive signs) and then switching at the vertex set $\{x_2, x_4, y_2\}$.

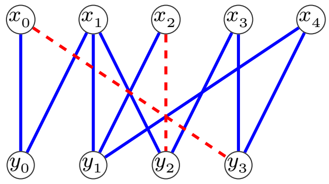


Figure 8.5. Ω_1

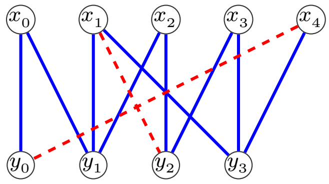


Figure 8.6. Ω_2

It is easily observed that each of the two signed graphs with the signature presented in the Figures 8.5 and 8.6 satisfies the conditions of Theorem 8.1.1, and, therefore, each of them maps to C_{-4}. In the next two lemmas, we show that one cannot make either of these two signed graphs C_{-4}-critical by only adding a vertex of degree 2.

Lemma 8.2.6. *Let Ω_1 be the signed graph of Figure 8.5. If we add a vertex v to one part of Ω_1 and connect it with two vertices in the other part (with any signature), the resulting signed graph admits a homomorphism to C_{-4}.*

Proof. Let Ω_1 be the signed bipartite graph of Figure 8.5 consisting of a bipartition (X, Y) where $X = \{x_0, x_1, x_2, x_3, x_4\}$ and $Y = \{y_0, y_1, y_2, y_3\}$.

If the two edges incident to the new vertex v are of the same sign, by switching at that new vertex, if needed, we consider them both positive. The resulting signed graph has a signature satisfying Theorem 8.1.1, therefore, maps to C_{-4}. Hence we assume that the two edges incident to v are of different signs and consider two cases depending on to which part the vertex v belongs.
Case 1. v is added to the X part. We consider three possibilities.

- v is adjacent to y_3. By switching at v, if necessary, we assume that vy_3 is negative. The only possible problem against Theorem 8.1.1 is by the positive edge vy_2. In that case, to resolve the issue, we apply a switching at x_2.
- v is not adjacent to y_3 but v is adjacent to y_2. We consider vy_2 to be negative and we are done.
- v is adjacent to both of y_0 and y_1. We take vy_1 as a negative edge and we are done after a switching at x_2.

Case 2. v is added to the Y part. We consider three possibilities.

- v is adjacent to one or both of x_0 and x_2. We switch at one of x_0 and x_2, one which is adjacent to v. Then by a switching at v (if needed) we have both edges incident to v of positive signs. The resulting signed graph satisfies the conditions of Theorem 8.1.1.

- v is adjacent to x_3 but not adjacent to x_0 and x_2. We assume vx_3 is a negative edge. We switch at x_3 and the conditions of Theorem 8.1.1 are satisfied.

- v is adjacent to both of x_1 and x_4. Similarly, we assume vx_4 is negative. We switch at x_0 and we are done. □

Lemma 8.2.7. *Let Ω_2 be the signed graph of Figure 8.6. If we add a vertex v to one part of Ω_2 and connect it with two vertices in the other part (with any signature), the resulting signed graph either contains $\hat{W}$ and maps to it or admits a homomorphism to C_{-4}.*

Proof. Let Ω_2 be the signed bipartite graph of Figure 8.6 consisting of a bipartition (X, Y) where $X = \{x_0, x_1, x_2, x_3, x_4\}$ and $Y = \{y_0, y_1, y_2, y_3\}$.

As in the previous lemma, we can assume that of the two edges incident to v exactly one is negative. Again, we consider two cases depending on to which part v belongs.

Case 1. v is added to the X part. We consider three possibilities.

- v is adjacent to y_2. By a switching at v, if needed, we assume vy_2 is negative. The only reason Theorem 8.1.1 may not work is by the positive edge vy_0. This can be taken care of by switching at y_0.

- v is adjacent to y_0 but not adjacent to y_2. By considering vy_0 as the negative edge incident to v, the resulting signed graph satisfies the condition of Theorem 8.1.1.

- v is adjacent to both y_1 and y_3. We may assume vy_3 is a negative edge. The subgraph induced by $x_1, x_2, x_3, y_1, y_2, y_3$ and v is (switching) isomorphic to $\hat{W}$. To see this isomorphism, using labeling of W as in the Figure 8.4, it is enough to switch at x_2 and y_2, and relabel v as x_4 while keeping all other labels the same. Finally, to see that the full graph of this case maps to $\hat{W}$, it is enough to extend the previous isomorphism (which we name ψ) to a mapping. This is done, for example, by setting $\psi(x_0) = (+, x_1)$ and $\psi(y_0) = (+, y_2)$ and $\psi(x_4) = (+, x_3)$.

Case 2. v is added to the Y part. We consider four possibilities.

- v is adjacent to x_1. We choose vx_1 to be negative. The only obstacle against Theorem 8.1.1 then can come from the positive edge vx_4, but we can switch at y_0 to resolve this issue.

- v is adjacent to x_4 but not adjacent to x_1. We choose vx_4 to be negative. Then we already have a signature satisfying conditions of Theorem 8.1.1.

- v is adjacent to x_0 but to neither of x_1 and x_4. Assuming that vx_0 is negative, we can switch at y_0 to apply Theorem 8.1.1.

- v is adjacent to both x_2 and x_3. We assume vx_3 is negative. The subgraph induced by $x_1, x_2, x_3, y_1, y_2, y_3$ and v is (switching) isomorphic to $\hat{W}$. One such isomorphism ϕ is defined as follows: $\phi(x_1) = (-, y_2)$, $\phi(x_2) = (+, y_1)$, $\phi(x_3) = (+, y_3)$, $\phi(y_1) = (-, x_2)$, $\phi(y_2) = (+, x_1)$, $\phi(y_3) = (+, x_3)$, $\phi(v) = (+, x_4)$. To complete this isomorphism to a homomorphism of the full graph to $\hat{W}$ we map x_0, x_4 and y_0 as follows: $\phi(x_0) = (-, y_2)$, $\phi(x_4) = (+, y_3)$, $\phi(y_0) = (-, x_1)$.

We note that for a better correspondence we have used same or similar labels for vertices of the graphs. In the mappings ψ and ϕ thus the vertices of the domains are those of the graphs we work with but the images are those of $\hat{W}$ as labeled in Figure 8.4. □

We are now ready to state and prove our main result on the structure of C_{-4}-critical signed graphs.

8.3 Edge-density of C_{-4}-critical signed graphs

The main theorem we shall prove is as follows.

Theorem 8.3.1. *If $\hat{G}$ is a C_{-4}-critical signed graph that is not isomorphic to $\hat{W}$, then*

$$|E(G)| \geqslant \frac{4|V(G)|}{3}.$$

Thus the natural *potential* function of graphs we may work with is:

$$p(G) = 4|V(G)| - 3|E(G)|.$$

We note that the potential of a signed graph is the potential of its underlying graph.

Observation 8.3.2. *We have $p(K_1) = 4$, $p(K_2) = 5$, $p(P_3) = 6$ and $p(C_4) = 4$. Thus any signed bipartite graph on at most 4 vertices has potential at least 4.*

In the rest of this section, we let $\hat{G} = (G, \sigma)$ be a minimum counterexample to Theorem 8.3.1. That is to say:

- $\hat{G}$ is a C_{-4}-critical signed graph with $p(\hat{G}) \geqslant 1$ and $\hat{G}$ is not isomorphic to $\hat{W}$;
- Any C_{-4}-critical signed graph $\hat{H}$, $\hat{H} \neq \hat{W}$, with $|V(\hat{H})| < |V(\hat{G})|$ satisfies that $p(\hat{H}) \leqslant 0$.

Given a signed graph $\hat{H}$, we denote a signed graph obtained from $\hat{H}$ by adding a new vertex and joining it to two vertices of $\hat{H}$ (where the signs of the two new edges are arbitrary) by $P_2(\hat{H})$. Normally, to denote a path as a graph we use P_n where n is the number of the vertices.

In the following lemma, we list the plausible potential of the subgraphs of the minimum counterexample $\hat{G}$.

Lemma 8.3.3. *Let $\hat{G} = (G, \sigma)$ be a minimum counterexample to Theorem 8.3.1 and let $\hat{H}$ be a subgraph of $\hat{G}$. Then*

1. *$p(\hat{H}) \geqslant 1$ if $\hat{G} = \hat{H}$;*
2. *$p(\hat{H}) \geqslant 3$ if $\hat{G} = P_2(\hat{H})$;*
3. *$p(\hat{H}) \geqslant 4$ otherwise.*

Proof. The first claim is our assumption on $\hat{G}$. If $\hat{G} = P_2(\hat{H})$, then $p(\hat{G}) = p(\hat{H}) + 4 \times 1 - 3 \times 2$, and then, since $p(\hat{G}) \geqslant 1$, we have $p(\hat{H}) \geqslant 3$. We now prove that for any other subgraph of $\hat{G}$, $p(\hat{H}) \geqslant 4$.

Suppose to the contrary that $\hat{G}$ contains a proper subgraph $\hat{H}$ which does not satisfy $\hat{G} = P_2(\hat{H})$, and satisfies $p(\hat{H}) \leqslant 3$. Among all such subgraphs, let $\hat{H}$ be chosen so that $|V(\hat{H})| + |E(\hat{H})|$ is maximized. We first claim that $\hat{H}$ is an induced subgraph of $\hat{G}$. Otherwise, assume that e is an edge connecting two vertices of $\hat{H}$. As adding an edge to a graph only decreases the potential, the assumption of the maximality implies that either $\hat{G} = \hat{H} + e$ or $\hat{G} = P_2(\hat{H} + e)$. In the first case, $p(\hat{H}) = p(\hat{G}) + 3 \geqslant 4$; in the second case, $p(\hat{H}) = p(\hat{H} + e) + 3 \geqslant 6$ (since $p(\hat{H} + e) = p(\hat{G}) - 4 + 6 \geqslant 3$), both contradicting the assumption that $p(\hat{H}) \leqslant 3$.

By Observation 8.3.2, $|V(\hat{H})| \geqslant 5$. As $\hat{G}$ is C_{-4}-critical and $\hat{H}$ is a proper subgraph, there is a homomorphism φ of $\hat{H}$ to C_{-4}. Since C_{-4} is vertex transitive, we may assume that φ preserves the bipartition of $\hat{H}$ induced by the bipartition of $\hat{G}$. This is automatic if H is connected, but important if H is not connected.

Observe that the mapping φ may have applied switching on some vertices of $\hat{H}$. Applying switching on the same set of vertices of $\hat{G}$, we get a switching equivalent signed graph. For simplicity, and without loss of generality, we may assume that $\hat{G}$ was given with this signature already.

Define $\hat{G}_1$ to be a signed (multi)graph obtained from $\hat{G}$ by first identifying vertices of $\hat{H}$ which are mapped to the same vertex of C_{-4} under φ, and then identifying all parallel edges of the same sign. Observe that $\hat{G}_1$ is a homomorphic image of $\hat{G}$ and that $\varphi(\hat{H})$ is (isomorphic to) the image of $\hat{H}$ in this mapping. Recall that in the mapping of $\hat{G}$ to $\hat{G}_1$ the bipartition is preserved. Therefore, $\hat{G}_1$ is bipartite. Since homomorphism is an associative relation, and since $\hat{G} \nrightarrow C_{-4}$, we have $\hat{G}_1 \nrightarrow C_{-4}$. This can only be for one of two reasons: Either $\hat{G}_1$ contains a digon (i.e., C_{-2}), or $\hat{G}_1$ contains a C_{-4}-critical subgraph. We consider the two cases separately:

Case 1. $\hat{G}_1$ contains a C_{-2}.

This implies that $\hat{G}$ contains a negative path $\hat{P}$ of length 2 with both endpoints in $\hat{H}$ and with its internal vertex in $V(\hat{G})\backslash V(\hat{H})$. We have that

$$p(\hat{H} + \hat{P}) = p(\hat{H}) + 4 \times 1 - 3 \times 2 = p(\hat{H}) - 2 < p(\hat{H}). \tag{8.1}$$

Recall that $\hat{H}$ is a maximum proper subgraph satisfying that $\hat{G} \neq P_2(\hat{H})$ and $p(\hat{H}) \leqslant 3$. Noting that $\hat{H} \subsetneq \hat{H} + \hat{P}$ and $\hat{H} + \hat{P}$ is a subgraph of $\hat{G}$, and by the maximality of $\hat{H}$, there are two possibilities: either $\hat{H} + \hat{P}$ is not a proper subgraph of $\hat{G}$, i.e., $\hat{G} = \hat{H} + \hat{P}$, or $\hat{G} = P_2(\hat{H} + \hat{P})$. The former case is impossible as $\hat{G} \neq P_2(\hat{H})$. So $\hat{G} = P_2(\hat{H} + \hat{P})$ and then

$$p(\hat{H} + \hat{P}) = p(\hat{G}) - 4 \times 1 + 3 \times 2 \geqslant 1 - 4 + 6 = 3 \geqslant p(\hat{H}), \tag{8.2}$$

which is in contradiction with (8.1).

Case 2. $\hat{G}_1$ contains a C_{-4}-critical subgraph $\hat{G}_2$.

We classify the vertices of $\hat{G}_2$ into two parts: Those of the images of $V(\hat{H})$, and the remaining vertices. We denote the former set by X_1, more precisely $X_1 = \varphi(V(\hat{H})) \cap V(\hat{G}_2)$, and the latter set by A, more precisely $A = V(\hat{G}_2)\backslash X_1$. The subgraph induced by X_1 is denoted by $\hat{X}$, in other words $\hat{X} = \varphi(\hat{H}) \cap \hat{G}_2$. Observe that since $\hat{G}_2 \nrightarrow C_{-4}$ and $\varphi(\hat{H}) \subset C_{-4}$, $A \neq \varnothing$.

Since $|V(\hat{H})| \geqslant 5$ and φ is a mapping of $\hat{H}$ to C_{-4}, at least two vertices are identified and, therefore, $|V(\hat{G}_2)| \leqslant |V(\hat{G}_1)| < |V(\hat{G})|$. As $\hat{G}$ is a minimum counterexample to Theorem 8.3.1, we have either $p(\hat{G}_2) \leqslant 0$ or $\hat{G}_2 = \hat{W}$. Since $p(\hat{W}) = 1$, in all cases we have $p(\hat{G}_2) \leqslant 1$.

We now define a subgraph $\hat{G}_3$ of $\hat{G}$ as follows: Vertices of $\hat{G}_3$ are those vertices of $\hat{G}$ each of which is either a vertex of $\hat{G}_2$ or a vertex of $\hat{H}$. That is to say $V(\hat{G}_3) = A \cup V(\hat{H}) = \{V(\hat{G}_2) \cup V(\hat{H})\}\backslash X_1$. To give the edge set of $\hat{G}_3$, we first choose a set E' of the edges of $\hat{G}$ as follows: If a vertex $u \in A$ is adjacent to a vertex $v \in X_1$, then we choose a vertex $v' \in V(\hat{H})$ such that first of all $\varphi(v') = v$, second of all $uv' \in E(G)$. From the construction of $\hat{G}_1$, it is clear that there is such a vertex v'. We note that, there might be more than one choice for v', in which case we select exactly one at random, and then let uv' be an edge in E'. The edge set of $\hat{G}_3$ is then defined to be the set of edges of $\hat{G}$ that are either induced by A, or by $V(H)$ or edges in E', with signature induced from the fixed signature of $\hat{G}$. In other words, $E(\hat{G}_3) = E(\hat{G}_2 - \hat{X}) + E(\hat{H}) + E'$. Since each connection between the vertices of $\hat{X}$ and $\hat{G}_2 - \hat{X}$ has a unique corresponding edge in E', it follows that $|E(\hat{G}_3)| = |E(\hat{G}_2)| - |E(\hat{X})| + |E(\hat{H})|$ and, therefore, $p(\hat{G}_3) = p(\hat{G}_2) - p(\hat{X}) + p(\hat{H})$.

Since $\hat{G}$ and $\hat{G}_2$ are both C_{-4}-critical signed graphs, $\hat{G}_2$ is not a subgraph of $\hat{G}$, that is to say $\hat{X} \neq \varnothing$. As $\hat{X}$ is a subgraph of C_{-4}, by Observation 8.3.2, $p(\hat{X}) \geqslant 4$. Then we obtain that

$$p(\hat{G}_3) = p(\hat{G}_2) - p(\hat{X}) + p(\hat{H}) \leqslant 1 - 4 + p(\hat{H}) = p(\hat{H}) - 3 \leqslant 0. \tag{8.3}$$

Since $\hat{G}_3$ is a subgraph of $\hat{G}$ and $\hat{H} \subsetneq \hat{G}_3$ (because $A \neq \varnothing$), by the maximality of $\hat{H}$ and noting that $p(\hat{G}_3) < p(\hat{H})$, either $\hat{G} = \hat{G}_3$ or $\hat{G} = P_2(\hat{G}_3)$. If $\hat{G} = \hat{G}_3$, then $p(\hat{G}_3) \geqslant 1$; if $\hat{G} = P_2(\hat{G}_3)$, then $p(\hat{G}_3) \geqslant 3$, each of which is contradicting (8.3). □

Towards proving Theorem 8.3.1, next we show that the underlying graph G of the minimum counterexample $\hat{G}$ does not contain two 4-cycles sharing edges. Let Θ_1 denote the graph of Figure 8.7 and Θ_2 denote the graph of Figure 8.8.

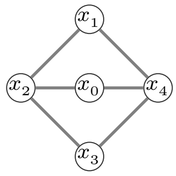


Figure 8.7. Θ_1

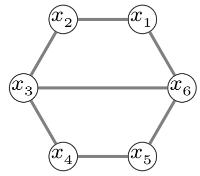


Figure 8.8. Θ_2

Lemma 8.3.4. *Given a minimum counterexample $\hat{G}$ to Theorem 8.3.1, the underlying graph G does not contain either Θ_1 or Θ_2 as a subgraph.*

Proof. By contradiction, assume Θ_1 is a subgraph of G and let $x_0, x_1, \ldots, x_4$ be the labeling of its vertices in G as well. Observe that $p(\Theta_1) = 2$. Thus, by Lemma 8.3.3, $G = (\Theta_1, \sigma)$ for some signature σ. We note that there are three 4-cycles in Θ_1, of which at least one must be a positive 4-cycle. By Observation 8.2.4 and as $d(x_0) = d(x_1) = d(x_3) = 2$, no signature on Θ_1 would result in a C_{-4}-critical signed graph. Thus $\hat{G}$ can not contain a copy of Θ_1 as a subgraph.

Assume to the contrary that Θ_2 is a subgraph of G and let $x_1, x_2, \ldots, x_6$ be its vertices in G as well. Observe that $p(\Theta_2) = 3$, thus, by Lemma 8.3.3, either G has only six vertices, or it has seven vertices and $G = P_2(\Theta_2)$. By Lemma 8.2.3, no signature on Θ_2 would result in a C_{-4}-critical signed graph. If we add one or more edges to Θ_2, then we will have a graph on 6 vertices and at least $\frac{4}{3} \times 6 = 8$ edges. This cannot form a counterexample. We note that after adding an edge one may assign a signature to get the only C_{-4}-critical signed graph on six vertices Γ.

The remaining possibility is that $G = P_2(\Theta_2)$. Let w be the added vertex. If w is not adjacent to one of x_1 or x_2, then we have a contradiction by Lemma 8.2.3. Similarly, w must also be adjacent to one of x_4 and x_5. As G is bipartite and by the symmetries of Θ_2, we may assume that w is adjacent to x_1 and x_5. Thus the underlying graph of G is the same as that of $\hat{W}$, and by Proposition 8.2.5 it must be (switching) isomorphic to $\hat{W}$. Therefore, $\hat{G}$ can not contain a copy of Θ_2 as a subgraph. □

In the next lemma, we imply further structure on the neighborhood of a 2-thread.

Lemma 8.3.5. *Let vv_1u be a 2-thread in $\hat{G}$. Suppose that v is a vertex of degree 3 and let v_2, v_3 be the other two neighbors of v. Then the path v_2vv_3 must be contained in a negative 4-cycle in $\hat{G}$.*

Proof. Suppose to the contrary that the path v_2vv_3 is not contained in a negative 4-cycle. If needed, by switching at v_2 or v_3, we may assume that both vv_2 and vv_3 are of positive sign. Then by identifying v_2 and v_3 to a new vertex v_0, we get a homomorphic image $\hat{G}_1$ of $\hat{G}$. Observe that, since v_2 and v_3 are in the same part of G, G_1 is also bipartite. Furthermore, by our assumption, $\hat{G}_1$ does not contain a C_{-2} and, therefore, $g_{ij}(\hat{G}_1) \geqslant g_{ij}(C_{-4})$ for every $ij \in \mathbb{Z}_2^2$.

As $\hat{G}$ does not map to C_{-4}, its homomorphic image, $\hat{G}_1$, does not map to it either. Thus there must be a C_{-4}-critical subgraph $\hat{G}_2$ of $\hat{G}_1$. By Lemmas 8.2.3 and 8.2.2, neither of the vertices v and v_1 is a vertex of $\hat{G}_2$. On the other hand, $v_0 \in V(\hat{G}_2)$, as otherwise $\hat{G}_2$ is a proper subgraph of $\hat{G}$ which does not map to C_{-4}, contradicting the fact that $\hat{G}$ is C_{-4}-critical. Since $\hat{G}$ is a minimum counterexample to the Theorem and $|V(\hat{G}_2)| < |V(\hat{G})|$, we have either $p(\hat{G}_2) \leqslant 0$ or $\hat{G}_2 = \hat{W}$ in which case $p(\hat{G}_2) \leqslant; 1$.

Let $\hat{G}_3$ be the signed graph obtained from $\hat{G}_2$ by splitting v_0 back to v_2 and v_3, adding the vertex v and adding the positive edges vv_2 and vv_3 back. Note that $\hat{G}_3$ is a subgraph of $\hat{G}$. We observe that

$$p(\hat{G}_3) = p(\hat{G}_2) + 4 \times 2 - 3 \times 2 = p(\hat{G}_2) + 2 \leqslant 3. \tag{8.4}$$

Furthermore, the equality is only possible if $\hat{G}_2 = \hat{W}$. As $v_1 \notin V(\hat{G}_3)$, we know that $\hat{G}_3 \neq \hat{G}$. By Lemma 8.3.3, we must have $p(\hat{G}_3) = 3$ and $\hat{G} = P_2(\hat{G}_3)$. And since equality in (8.4) must hold, we also have $\hat{G}_2 = \hat{W}$.

As $\hat{G}_2 = \hat{W}$, vertices of $\hat{G}_2$ are of degree 2 or 3, and, thus, the splitting operation on v_0 (that we considered in order to build $\hat{G}_3$) is the same as subdividing one of its edges twice. Recall that there are two types of edges in $\hat{W}$ up to (switching) isomorphism. Thus the subdivided signed graph $\hat{G}_3$ is one of the two signed graphs: either Ω_1 of Figure 8.5 or Ω_2 of Figure 8.6. Thus either $\hat{G} = P_2(\Omega_1)$ or $\hat{G} = P_2(\Omega_2)$. In the former case, by Lemma 8.2.6, $\hat{G}$ maps to C_{-4}. In the latter case, by Lemma 8.2.7, either $\hat{G}$ maps to C_{-4} or it contains $\hat{W}$ as a proper subgraph but this contradicts the fact that $\hat{G}$ is C_{-4}-critical. □

By combining Lemma 8.3.4 and Lemma 8.3.5, we have our main forbidden configuration as follows:

Corollary 8.3.6. *A vertex of degree* 3 *in the minimum counterexample* $\hat{G}$ *does not have two neighbors of degree* 2.

We are now ready to prove Theorem 8.3.1.

Proof. (of Theorem 8.3.1) We will employ the discharging technique. We assign an initial charge of $c(v) = d(v)$ to each vertex of the minimum counterexample $\hat{G}$. Observe that the total charge is $2|E(G)|$. We apply the following discharging rule:

"Every vertex of degree 2 receives a charge of $\frac{1}{3}$ from each of its neighbors."

For each vertex v, let $c'(v)$ be the charge of v after the discharging procedure. Since there is no 3-thread in G, each degree-2 vertex v receives a total of $\frac{2}{3}$ from its two neighbors and thus $c'(v) = 2+\frac{2}{3} = \frac{8}{3}$. Each degree-3 vertex u has at most one neighbor of degree 2, so $c'(u) \geqslant 3-\frac{1}{3} = \frac{8}{3}$. Each vertex w of degree at least 4 has charge $c'(w) \geqslant d(w) - \frac{d(w)}{3} = \frac{2d(w)}{3} \geqslant \frac{8}{3}$. Thus the total charge is at least $\frac{8|V(G)|}{3}$. That contradicts the assumption that $p(\hat{G}) = 4|V(G)| - 3|E(G)| \geqslant 1$. □

Applying this result in terms of maximum average degree of the (underlying) graph, denoted $mad(G)$, we have the following.

Corollary 8.3.7. *Given a signed bipartite (simple) graph $\hat{G}$, if $mad(G) < \frac{8}{3}$ and $\hat{G}$ does not contain $\hat{W}$ as a subgraph, then $\hat{G} \to C_{-4}$.*

8.4 Constructions of sparse C_{-4}-critical signed graphs

We have already seen in Section 2.3.3 that $\chi(G) \leqslant k$ if and only if $T_{k-2}(G, +) \to C_{-k}$. Next we extend this connection based on the notion of 0-free $2k$-coloring of signed graphs. A signed graph (G, σ) is K^s_{2k}*-critical* if it does not admit a 0-free $2k$-coloring but each of its proper subgraphs does. Using this notion, Theorem 2.3.5 can be extended to the following theorem. A proof of this theorem is also obtained by revising the proofs of Theorem 2.3.5 and Theorem 2.4.3 given in Section 2.3.3.

Theorem 8.4.1. *A signed multigraph $\hat{G}$ admits a 0-free $2k$-coloring if and only if $T_{2k-2}(\hat{G}) \to C_{-2k}$. Moreover, $\hat{G}$ is K^s_{2k}-critical if and only if $T_{2k-2}(\hat{G})$ is C_{-2k}-critical.*

Given a graph G, we denote by $\tilde{G}$ the signed multigraph obtained from G by replacing each edge with a digon. It is easily observed that:

Observation 8.4.2. *A graph G is k-colorable if and only if the signed multigraph $\tilde{G}$ admits a 0-free $2k$-coloring.*

Next we develop another technique to build C_{-2k}-critical signed graphs.

Theorem 8.4.3. *Given a graph G, we have $\chi(G) \leqslant k$ if and only if $T_{2k-2}(\tilde{G}) \to C_{-2k}$. Moreover, G is $(k+1)$-critical if and only if $T_{2k-2}(\tilde{G})$ is C_{-2k}-critical.*

Proof. The first part of the theorem follows from Theorem 8.4.1 and Observation 8.4.2.

For the moreover part, we first assume that $T_{2k-2}(\tilde{G})$ is C_{-2k}-critical and need to show that G is $(k+1)$-critical. For this it is enough to show that every proper subgraph H of G is k-colorable. Since $T_{2k-2}(\tilde{H})$ is a proper subgraph of $T_{2k-2}(\tilde{G})$ and any proper subgraph of $T_{2k-2}(\tilde{G})$ maps to C_{-2k}, by the first part of the theorem, H is k-colorable.

Next we assume G is $(k+1)$-critical. Let e be an edge of $T_{2k-2}(\tilde{G})$. Our goal is to show that $T_{2k-2}(\tilde{G}) - e$ maps C_{-2k}. Let uv be the edge in G which corresponds to the thread (of $T_{2k-2}(\tilde{G})$) to which e belongs.

Let $G' = G/uv$ be the graph obtained from G by contracting the edge uv and let $H = G - uv$. We observe that G' is k-colorable because first of all H is k-colorable, and, secondly, in any such coloring u and v must receive the same color.

Next we consider two signed graphs obtained from $T_{2k-2}(\tilde{H})$: (1) The signed graph T^{+}_{uv} is obtained by identifying u and v. (2) The signed graph T^{-}_{uv} is obtained by switching at u and then identifying u and v. One may easily observe that these two signed graphs are (switching) isomorphic, and in fact each of them can be regarded as $T_{2k-2}(\tilde{G'})$. Since G' is k-colorable, and by the first part of the theorem, $T_{2k-2}(\tilde{G'})$ maps to C_{-2k}.

If e is deleted from the negative uv-thread, then we consider a mapping of T^{+}_{uv} to C_{-2k}. This mapping can also be viewed as a mapping of $T_{2k-2}(\tilde{H})$ to C_{-2k} where u and v are identified without any switching on them. As the positive uv-thread is of even length $2k-2$, this mapping can easily be extended to a mapping of $T_{2k-2}(\tilde{G}) - e$ to C_{-2k}. When e is on the positive uv-thread, we will consider a mapping of T^{-}_{uv} to C_{-2k}. We note that after switching at u, the negative path becomes positive. The mapping can then be extended as in the previous case. □

The T_{2k-2}-construction, when applied on signed multigraphs $\tilde{G}$, connects the k-coloring problem of G to C_{-2k}-coloring problem of signed graphs. Thus C_{-2k}-coloring problem captures k-coloring problem for all the values of k. We note that $T_2(\tilde{G})$ is the same as $S(G)$ defined in Section 2.3.4.

The first construction of C_{-4}-critical signed graphs is based on the T_{2k-2}-construction of signed multigraphs $\tilde{G}$. By Theorem 8.4.3 and noting that odd cycles are the only 3-critical graphs, we have $T_2(\tilde{C}_{2k+1})$ (i.e., $S(C_{2k+1})$) as an example of C_{-4}-critical signed graph for each value of k. See Figures 8.9 and 8.10 for $T_2(\tilde{C}_3)$ and $T_2(\tilde{C}_5)$. The signed bipartite graph $\hat{G}_{2k+1} = T_2(\tilde{C}_{2k+1})$ has $6k+3$ vertices and $8k+4$ edges. Thus $T_2(\tilde{C}_{2k+1})$ is an example of a C_{-4}-critical signed graph for which the bound of Theorem 8.3.1 is tight.

Some proper identification of certain vertices of $T_2(\tilde{C}_{2k+1})$ will also result in constructions of C_{-4}-critical signed graphs. Let $\hat{G}'_{2k+1}$ be the signed (bipartite) graph obtained from $\hat{G}_{2k+1}$ by identifying two vertices of degree 2 which are at distance 2 and their common neighbor is adjacent to both with positive edges. See Figure 8.11 for an illustration of $\hat{G}'_5$. However, there is an exception that $\hat{G}'_3$ contains $\hat{W}$ as a proper subgraph and, thus, is not C_{-4}-critical. For $k \geqslant 2$, $\hat{G}'_{2k+1}$ does not map to C_{-4} because it is a homomorphic image of $\hat{G}_{2k+1}$. Moreover, it has average degree of $\frac{8|V(G'_{2k+1})|+2}{3|V(G'_{2k+1})|}$, it does not contain $\hat{W}$ as a subgraph and any proper subgraph of it has average degree strictly less than $\frac{8}{3}$. Thus, by Corollary 8.3.7, it is a C_{-4}-critical signed graph for which the bound of Theorem 8.3.1 is tight. Further identification of vertices of degree 2 would lead to other examples for which the bound of Theorem 8.3.1 is either tight or nearly tight.

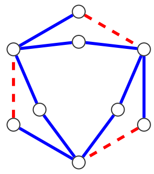


Figure 8.9. $T_2(\tilde{C}_3)$

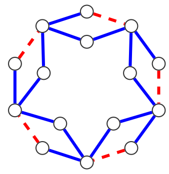


Figure 8.10. $T_2(\tilde{C}_5)$

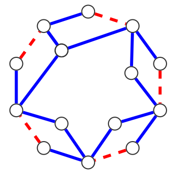


Figure 8.11. $\hat{G}'_5$

We state the next method to build C_{-4}-critical signed graphs from two given C_{-4}-critical signed graphs. Let $\hat{G}_1$ and $\hat{G}_2$ be two C_{-4}-critical signed graphs each with a vertex of degree 2. Suppose u is a vertex of degree 2 in $\hat{G}_1$ with u_1 and u_2 as its neighbors, and v is a vertex of degree 2 in $\hat{G}_2$ with v_1 and v_2 as its neighbors. As $\hat{G}_1$ is a C_{-4}-critical signed graph, $\hat{G}_1 - u$ maps to C_{-4}. But any such mapping must map u_1 and u_2 to the same vertex of C_{-4} and must have applied a switching on $\hat{G}_1 - u$ so that with the same switching on $\hat{G}_1$, the path u_1uu_2 is negative. We consider $\hat{G}_1$ with this signature and do the same on $\hat{G}_2$. Hence, no switching applies to the vertices u_1, u_1, v_1, v_2 in each of homomorphisms, say φ_1 and φ_2, of $\hat{G}_1 - u$ and $\hat{G}_2 - v$ to C_{-4} and moreover $\varphi_1(u_1) = \varphi_1(u_2)$ and $\varphi_2(v_1) = \varphi_2(v_2)$.

We then build a signed graph $\mathcal{F}(\hat{G}_1, \hat{G}_2) = \hat{G}$ from disjoint union of $\hat{G}_1$ and $\hat{G}_2$ by deleting u and v, and adding a positive edge u_1v_1 and a negative edge u_2v_2. Following from the discussion above, $\mathcal{F}(\hat{G}_1, \hat{G}_2) = \hat{G}$ is C_{-4}-critical. In Figure 8.12, we have depicted the signed graph obtained from this operation on two disjoint copies of $\hat{W}$ (see Figure 8.4). We note that this is an example of a C_{-4}-critical signed graph on 12 vertices for which the bound of Theorem 8.3.1 is tight. One may note that, furthermore, the same technique can be applied to build a new C_{-2k}-critical signed graph from two C_{-2k}-critical signed graphs each having a vertex of degree 2. Moreover, towards building a C_{-2k}-critical signed graph of lower edge-density, instead of connecting u_iv_i directly, one may use paths of length $k-1$, one of positive sign, one of negative sign.

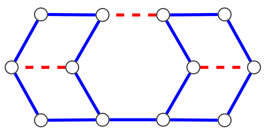


Figure 8.12. $\mathcal{F}(\hat{W}, \hat{W})$

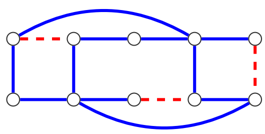


Figure 8.13. $\mathcal{H}(\Gamma, \Gamma)$

The famous Hajós construction of k-critical graphs can also be adapted to build C_{-4}-critical signed graphs. Let $\hat{G}_1$ and $\hat{G}_2$ be two C_{-4}-critical signed graphs with x_1y_1 being a positive edge of $\hat{G}_1$ and x_2y_2 being a negative edge of $\hat{G}_2$. Then $\hat{G}_1 - x_1y_1$ admits a homomorphism $\phi = (\phi_1, \phi_2)$ to C_{-4}. Since C_{-4} is vertex transitive, and since $(-\phi_1, \phi_2)$ is the same as (ϕ_1, ϕ_2), we may consider only the mappings for which $\phi(x_1) = (+, u_2)$ (where u_2 refers to the labeling of C_{-4} in Figure 8.1). Then we must have $\phi_1(y_1) = -$ as otherwise, ϕ is also a mapping of $\hat{G}_1$ to C_{-4}. Furthermore, for any other edge e, if we take a mapping ϕ' of $\hat{G} - e$ satisfying $\phi'(x_1) = (+, u_2)$, then we must have $\phi(y_1) = +$. Similarly, consider a C_{-4}-critical signed graph $\hat{G}_2$ with a negative edge x_2y_2. Then by a similar argument, for any mapping ψ of $\hat{G}_2 - x_2y_2$ for which $\psi(x_2) = (+, u_2)$, we must have $\psi(y_2) = +$.

We now build a new C_{-4}-critical signed graph $\hat{H} = \mathcal{H}(\hat{G}_1, \hat{G}_2)$ as follows: $\hat{H}$ is obtained from vertex disjoint copies of $\hat{G}_1$ and $\hat{G}_2$ by deleting x_1y_1, x_2y_2 and identifying x_1 with x_2 to get a vertex

x and y_1 with y_2 to get a vertex y. We observe that if there exists a homomorphism φ of $\hat{H}$ to C_{-4}, then, by symmetries, we may assume $\varphi(x) = (+, u_2)$. Then the restriction on $\hat{G}_1$ implies $\varphi_1(y) = -$ and the restriction on $\hat{G}_2$ implies $\varphi_1(y) = +$, a contradiction, implying that $\hat{H}$ does not map to C_{-4}. Removing an edge from one part of $\hat{H}$ then leads in mappings of the two different parts that can be merged together, which shows that $\hat{H}$ is C_{-4}-critical. An example of this construction, using two disjoint copies of the unique C_{-4}-critical signed graph Γ on 6 vertices (see Figure 8.3) is given in Figure 8.13.

The signed graph $\mathcal{H}(\hat{G}_1, \hat{G}_2)$ has $|V(\hat{G}_1)| + |V(\hat{G}_2)| - 2$ vertices. Using the techniques mentioned above one can easily build C_{-4}-critical signed graphs of orders $9, 10, 11, 12$. Then applying Hajós construction to a previously built C_{-4}-critical signed graph and Γ (on 6 vertices), one can build a C_{-4}-critical signed graph on any number n of vertices for $n \geqslant 9$.

In the end, we will use the construction given above to discuss the relation between the minimum number of edges of k-critical graphs and C_{-k}-critical signed graphs with given number of vertices. Given positive integers k and n ($n \geqslant k+2$), let $f(n,k)$ be the minimum number of edges of a k-critical graph on n vertices. We refer to [KY14b] for an almost precise value of $f(n,k)$ and for historical background on the study of this function. Similarly, we define $g(n,k)$ to be the minimum number of edges of a C_{-k}-critical signed graph on n vertices. As noted above, $g(n,4)$ is well-defined for $n \geqslant 9$. It can be similarly shown that $g(n,k)$ is well-defined for $n \geqslant N_k$ where N_k is an integer depending on k only.

Theorem 2.4.3 (in Section 2.4) and Theorem 8.4.3 imply the following relations between $f(n,k)$ and $g(n,k)$.

- By Theorem 2.4.3,
$$g(n + (k-3)f(n,k), k) \leqslant (k-2)f(n,k). \tag{8.5}$$

- By Theorem 8.4.3,
$$g(n + 2(\ell-1)f(n,\ell), 2\ell) \leqslant 2(\ell-1)f(n,\ell). \tag{8.6}$$

Authors of [DP17] and [PS22] suggest that for $k=5$ and $k=7$ the inequality (8.5) is almost tight. Our work here shows that for C_{-4}-critical signed graphs the inequality of (8.6) provides a tight bound. For $k=6$, the two inequalities provide similar bounds where the only difference is in the constant (in the favor of inequality of (8.5)). For other values of $k = 2\ell$ the inequality of (8.5) provides a better bound than (8.6) and it is tempting to suggest that (8.5) gives a nearly tight bound for $g(n,k)$ for $k \geqslant 5$. A point of hesitation here is that, while the notion of k-critical graphs is widely studied and the value and behavior of $f(n,k)$ are almost determined, the notion of critical signed graphs, aside from its relation to $(2k+1)$-critical graphs (with no sign), is a new notion and hardly anything is known about it. In particular, what can then be said about the minimum number of edges of a K^s_{2k}-critical signed graph? Constructions other than $\tilde{G}$, combined with Theorem 8.4.1, may provide better bounds on $g(n,2k)$.

8.5 Application to signed bipartite planar graphs

As a bipartite analogue of Jaeger-Zhang conjecture, it was conjectured in [NRS15] that every signed bipartite planar graph of negative-girth at least $4k-2$ maps to C_{-2k}. Here we use the bipartite folding lemma (Lemma 2.3.15) to prove that every signed bipartite planar graph of negative-girth at least 8 maps to C_{-4} and we show that this bound is tight, thus disproving the exact claim of the conjecture for the case $k=2$.

Recall that Lemma 2.3.15 shows that given a signed bipartite planar graph (G, σ) with an embedding on the plane, if the length of the shortest negative cycles of (G, σ) is at least $2k$ and a face F is not a negative cycle of length $2k$, then there is a homomorphic image of (G, σ) which identifies two vertices at distance 2 of F and such that its shortest negative cycles are also of length at least $2k$.

Observe that this identification preserves both of the planarity and the bipartiteness. Thus, repeatedly applying the lemma, we get a homomorphic image where all faces are negative cycles of length $2k$. Taking $k = 4$, starting from a signed bipartite planar graph whose shortest negative cycles are of length at least 8, we get a homomorphic image $\hat{G}$ with a planar embedding where all faces are (negative) 8-cycles. Applying the Euler formula on this graph, we have $|E(G)| \leqslant \frac{4}{3}(|V(G)| - 2)$. By taking $\hat{G}$ to be a smallest signed bipartite planar graph which does not map to C_{-4} and whose shortest negative cycle is of length 8, we conclude that on the one hand, by the argument above, it has at most $\frac{4}{3}(|V(G)| - 2)$ edges, but on the other hand, $\hat{G}$ must be C_{-4}-critical, and thus, by Theorem 8.3.1, has at least $\frac{4}{3}|V(G)|$ edges. This contradiction is a proof of Theorem 8.5.1.

Theorem 8.5.1. *Every signed bipartite planar graph of negative-girth at least* 8 *maps to* C_{-4}. *Moreover, this girth condition is the best possible.*

We now show that the negative-girth condition being at least 8 in this theorem is tight. For this, it would be enough to build a signed planar simple graph (G, σ) which is not $\{\pm 1, \pm 2\}$-colorable. Then, by Theorem 8.4.1, $T_2(G, \sigma)$ is a signed bipartite planar graph of negative girth at least 6 which does not map to C_{-4}.

Theorem 8.5.2. *There exists a bipartite planar graph G of girth* 6 *with a signature σ such that* $(G, \sigma) \nrightarrow C_{-4}$.

The smallest example we have built so far in this way have 150 vertices. However, this example has the extra property that vertices on one part of the (bipartite) graph are all of degree 2. Perhaps simpler examples can be built which do not satisfy this property. Using the language of the circular coloring of signed graphs, we could restated the above Theorem.

Theorem 8.5.3. *We have that* $\chi_c(\mathcal{SBP}_8) = \frac{8}{3}$.

That $\chi_c(\mathcal{SBP}_8) \leqslant \frac{8}{3}$ is a consequence of Theorem 8.5.1 and Corollary 3.4.4. We will conclude Theorem 8.5.3 by a sequence of signed bipartite planar graphs of negative-girth 8 whose circular chromatic number is approaching $\frac{8}{3}$. Using a special case of Theorem 3.3.18 for $\ell = 2$, we have that $\chi_c(T_2^*(G, \sigma)) = \frac{4\chi_c(G,\sigma)}{2+\chi_c(G,\sigma)}$. We have seen in Section 7.2 that there is a series of signed bipartite graphs Γ_i and for each integer i, it has the circular chromatic number $4 - \frac{4}{i+1}$. Thus the signed bipartite graph $T_2^*(\Gamma_i)$ satisfies that

$$\chi_c(T_2^*(\Gamma_i)) = \frac{4 \times (4 - \frac{4}{i+1})}{2 + (4 - \frac{4}{i+1})} = \frac{8i}{3i+1}.$$

Note that $T_2^*(\Gamma_i)$ is of negative-girth 8 and when i goes into infinity, the limit of $\chi_c(T_2^*(\Gamma_i)$ is $\frac{8}{3}$.

The C_{-4}-coloring problem even when restricted to the class of signed bipartite planar graphs remains an NP-complete problem proved in [DFM+20]. Thus, one does not expect to find an efficient classification of signed bipartite planar graphs which map to C_{-4}. However, some strong sufficient conditions could be provided. One such condition is based on the restatement of the 4-color theorem given in Theorem 2.3.7. Another is Theorem 8.5.1 of this work that shows no negative cycle of length $2, 4, 6$ is a sufficient condition. As a generalization of Theorem 2.3.7 which also captures essential cases of Theorem 8.5.1, we propose the following:

Conjecture 8.5.4. *Let G be a bipartite planar graph of girth at least* 6. *Let σ be a signature on G such that in (G, σ) all* 6*-cycles are of the same sign. Then* $(G, \sigma) \rightarrow C_{-4}$.

We note that, while one may use Lemma 2.3.15 to reduce facial 4-cycles of a signed graph which is the subject of Theorem 8.5.1, there could be separating 4-cycles in a signed bipartite planar graph to which this theorem may apply. Therefore, the conjecture does not capture all cases to which Theorem 8.5.1 applies.

9 | Homomorphisms of signed bipartite planar graphs to $(K_{3,3}, M)$

This chapter is based on the following paper:

[NW21] R. Naserasr and Z. Wang. *Signed bipartite circular cliques and a bipartite analogue of Grötzsch's theorem.* 2021. arXiv: 2109.12618 [math.CO]

9.1 Introduction

Recall there is a strengthening of the 4-color theorem that every signed bipartite planar simple graph admits a homomorphism to $(K_{4,4}, M)$. Following a special case for $k = 3$ of Theorem 3.3.12, we have the next reformulation of the Grötzsch theorem.

Theorem 9.1.1. [Grötzsch's theorem restated] *For any triangle-free planar simple graph G, $\chi_c(S(G)) \leqslant 3$.*

As an analogue of the Grötzsch theorem, we are interested in the homomorphism to $(K_{3,3}, M)$, a core subgraph of $(K_{4,4}, M)$. Note that in Corollary 3.4.3 the homomorphism problem to $(K_{3,3}, M)$ is equivalent to the circular 3-coloring problem of signed bipartite planar graphs. We further note that considering Theorem 2.3.10, the problem of mapping signed bipartite planar graphs to $(K_{3,3}, M)$ captures the 3-coloring problem of ordinary planar graphs. We will prove the following theorem, noting that our proof is based on the 4-color theorem.

Theorem 9.1.2. *Every signed bipartite planar graph of negative-girth at least 6 admits a homomorphism to $(K_{3,3}, M)$. Moreover, the girth condition is best possible.*

We view Theorem 9.1.2 as a parallel theorem to Grötzsch's theorem. Later, in Section 9.3, we propose a question as potentially common strengthening of these two theorems.

Another line of this study is to bound the circular chromatic number of given classes of signed graphs. Following the results in Chapters 7 and 8, we have that $\chi_c(\mathcal{SBP}_4) = 4$ and $\chi_c(\mathcal{SBP}_8) = \frac{8}{3}$ and both of equalities are attained by some tight (infinite or finite) examples. Here, addressing the case $\ell = 6$, we bound $\chi_c(\mathcal{SBP}_6)$ in Theorem 9.2.4. The upper bound of 3 is implied from Theorem 9.1.2 but its tightness has not been verified yet.

9.2 Circular 3-coloring of signed bipartite planar graphs

The proof of the upper bound of Theorem 9.1.2 is based on some results regarding the connection between the edge-coloring results and homomorphism to signed projective cubes introduced in Section 1.3. We first state some results we may use.

The following result is implied by an edge-coloring result of [DKK16], whose proof is based on the results of [Gue03], eventually based on the 4-color theorem.

Theorem 9.2.1. [NRS13] *If (G,σ) is a signed planar graph satisfying that $g_{ij}(G,\sigma) \geqslant g_{ij}(SPC(5))$ for $ij \in \mathbb{Z}_2^2$, then $(G,\sigma) \to SPC(5)$.*

As the signed projective cube $SPC(5)$ is bipartite and of negative-girth 6, i.e., $g_{10}(SPC(5)) = 6$ and $g_{01}(SPC(5)) = g_{11}(SPC(5)) = \infty$, it means that every signed bipartite planar graph of negative-girth at least 6 admits a homomorphism to $SPC(5)$. A characterization of mapping signed graphs to the signed projective cubes is also given in [NRS13].

Theorem 9.2.2. [NRS13] *Given a positive integer k and a signed graph (G,σ), $(G,\sigma) \to SPC(k)$ if and only if there exists a partition of the edges of G, say $E_1, E_2, \ldots, E_{k+1}$, such that for each $i \in \{1,2,\ldots,k+1\}$, the signature σ_i, which assigns $-$ to the edges in E_i, is switching equivalent to σ.*

The next theorem, following from Theorems 9.2.1 and 9.2.2, is the basic tool that we will use to prove Theorem 9.1.2 and then obtain the upper bound in Theorem 9.2.4.

Theorem 9.2.3. *Given a signed bipartite planar graph (G,σ) of negative-girth at least 6, there are 6 disjoint subsets of edges, $E_1, E_2, \ldots, E_6$, such that each of the signed graphs (G,σ_i), $i \in \{1,2,\ldots,6\}$, where E_i is the set of negative edges of (G,σ_i), is switching equivalent to (G,σ).*

In other words, the signature packing number of any signed bipartite planar graph of negative-girth at least 6 is at least 6 (see [NY21] for more details on signature packing numbers). Now we are ready to prove the main result as follows.

Theorem 9.2.4. *We have that $\frac{14}{5} \leqslant \chi_c(\mathcal{SBP}_6) \leqslant 3$.*

Proof. We first prove that every signed bipartite planar graph of negative-girth 6 admits a homomorphism to $(K_{3,3}, M)$ and then conclude that $\chi_c(\mathcal{SBP}_6) \leqslant 3$.

Let (G,σ) be a signed bipartite planar graph of negative girth at least 6 with a bipartition (A,B). By Theorem 9.2.3, there are disjoint subsets $E_1, E_2, \ldots, E_6$ of edges of (G,σ) such that for each $i \in [6]$, the signature σ_i, whose negative edges are E_i, is equivalent to σ.

We consider the signed graph (G,σ_1) where the set of negative edges is E_1. Let G' be the graph obtained from G by contracting all the edges in E_1. In this notion of contracting, we delete the contracted edge (those in E_1) but all other edges remain. Thus in theory we may have loops and parallel edge in the resulting graph. However, we show next that not only G' has no loop, it has no triangle either. In other words, we claim that every odd cycle of G' is of length at least 5.

To see this, let C' be an odd cycle of G'. This cycle is obtained from a cycle, denoted C, of G by contracting some edges (of E_1). As G is bipartite, the cycle C must be of even length. Thus the number of the contracted edges, i.e., $|E(C) \cap E_1|$, is odd. Therefore, C is a negative cycle in the signed graph (G,σ_1). As all the (G,σ_i), $i = 1,2,\ldots,6$, are equivalent, C is negative in all of them which means it has an odd number of edges from each of E_i's. As these sets are disjoint, and as for $i = 2,3,\ldots 6$, they still present in G', the cycle C' has an odd number of edges from each E_i for $i = 2,3,\ldots 6$. In particular, that is at least one edge from each, and noting again that they are disjoint sets, we conclude that C' is of length at least 5.

Having shown that G' is a triangle-free planar graph with no loop (might have parallel edges), we may apply the Grötzsch theorem to obtain a 3-coloring $\varphi : V(G') \to \{1,2,3\}$ of G'. Let (X,Y) be the bipartition of $(K_{3,3}, M)$. Label the vertices $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$ such that $\{x_1y_1, x_2y_2, x_3y_3\}$ is the set of negative edges.

We define the mapping ψ of (G,σ_1) to $(K_{3,3}, M)$ as follows:

$$\psi(u) = \begin{cases} x_i, & \text{if } u \in A \text{ and } \varphi(u) = i \\ y_i, & \text{if } u \in B \text{ and } \varphi(u) = i. \end{cases}$$

It remains to show that ψ is an edge-sign preserving mapping of (G, σ_1) to $(K_{3,3}, M)$. For any positive edge uv of (G, σ_1), without loss of generality, we may assume that $u \in A, v \in B$ and that $\varphi(u) = i$, $\varphi(v) = j$. Noting that uv is also an edge of G', as φ is a proper 3-coloring, we have that $i \neq j$. Thus $\psi(u)\psi(v) = x_i x_j$ is a positive edge in $(K_{3,3}, M)$. For any negative edge uv of (G, σ_1), without loss of generality, assume $u \in A, v \in B$. As $uv \in E_1$ is contracted to a vertex to obtain G', $\varphi(u) = \varphi(v) = i$. So $\psi(u)\psi(v) = x_i y_i$ is a negative edge. Hence, ψ is an edge-sign preserving homomorphism of (G, σ_1) to $(K_{3,3,M})$. This completes the proof of the upper bound.

For the lower bound, we consider an example of signed planar simple graph (G, σ) satisfying $\chi_c(G, \sigma) = \frac{14}{3}$, given in Section 4.4. Then it follows from Lemma 3.3.18 that $\chi_c(T_2^*(G, \sigma)) = \frac{14}{5}$. It is easily observed that, since (G, σ) is a signed simple planar graph, $T_2^*(G, \sigma)$ has (negative) girth at least 6 and obviously it is a signed bipartite graph. □

9.3 Questions and remarks

We have proved in Theorem 9.2.4 that every signed bipartite planar graph of negative-girth at least 6 admits a circular 3-coloring. However, the best example of signed bipartite planar graph of negative-girth 6 we know has circular chromatic number $\frac{14}{5}$ and we cannot determine the exact value for $\mathcal{SBP}_6$. It remains an open problem to build such signed graphs of circular chromatic number between $\frac{14}{5}$ and 3. We would like to ask the following question similar to what we have studied for signed bipartite planar graphs in Chapter 7.

Question 9.3.1. *For any given signed bipartite planar graph $\hat{G}$ of negative-girth 6, does there exist an $\epsilon = \epsilon(\hat{G})$ such that $\hat{G}$ admits a circular $(3 - \epsilon)$-coloring?*

We establish a connection between this problem and a circular coloring problem of the class of signed planar simple graphs, using the notion of indicators. We have seen that $\frac{14}{3} \leqslant \mathcal{SP} \leqslant 6$, where $\mathcal{SP}$ denote the class of signed planar simple graphs, in Section 4.4 and the next question is still unsolved.

Question 9.3.2. *For any given signed planar simple graph $\hat{G}$, does there exist an $\epsilon = \epsilon(\hat{G})$ such that $\hat{G}$ admits a circular $(6 - \epsilon)$-coloring?*

Let us mention that a negative answer to Question 9.3.2 would imply a negative answer to Question 9.3.1. For any signed planar simple graph $\hat{G}$, $T_2(\hat{G})$ is in the class $\mathcal{SBP}_6$. If there exists a signed planar simple graph $\hat{G}$ with $\chi_c(\hat{G}) = 6$, then $\chi_c(T_2(\hat{G})) = 3$ which answers Question 9.3.1 negatively.

There are some other perspectives to improve what we have proved in this chapter. The first is about the use of the 4-color theorem in our proof of the upper bound of 3 for the circular chromatic number of the subclass $\mathcal{SBP}_6$. Could one find a relatively short proof of this without using the 4-color theorem? Or can one show that, on the contrary, this result implies the 4-color theorem? We recall that, in Section 2.3.4, reformulations of the 4-color theorem using special classes of planar graphs of large girth are given. So this would not be a surprise.

The second question would potentially strengthen our result to include the Grötzsch theorem as a special case. One possibility is observed by reformulating the Grötzsch theorem itself as follows.

Theorem 9.3.3. [Grötzsch's theorem restated] *If G is a planar graph satisfying that $K_3 \nrightarrow G$, then $G \to K_3$.*

We recall that $K_3 \to G$ if and only if $S(K_3) \to S(G)$. Thus a potential strengthening of our result, which would also include the Grötzsch theorem, is as follows.

Conjecture 9.3.4. *If (G, σ) is a signed bipartite planar graph with the property that $S(K_3) \nrightarrow (G, \sigma)$, then $\chi_c(G, \sigma) \leqslant 3$, i.e., $(G, \sigma) \to (K_{3,3}, M)$.*

Recall the reformulation in Theorem 2.3.10, when G is triangle-free, $S(G)$ contains negative 4-cycles but no 6-cycle. If, furthermore, G is assumed to be of girth 5, then $S(G)$ will contain no 8-cycle either. This calls another line of study that follows Steinberg's conjecture [Ste93]. R. Steinberg proposed that planar graphs with no cycle of length 4 and 5 are 3-colorable. This conjecture was recently disproved in [CHK+17]. However, some supporting results have been proved earlier, with the most notable one being the result of [BGRS05] which shows if cycles of length $4, 5, 6, 7$ are not subgraphs of a planar graph G, then G is 3-colorable. It is natural for us to ask the following question.

Problem 9.3.5. *What is the smallest value of k, $k \geqslant 3$, such that every signed bipartite planar graph with no 4-cycles sharing an edge and no cycles of length $6, 8, \ldots, 2k$, admits a homomorphism to $(K_{3,3}, M)$?*

For further study on this direction we refer to a recent work of [HL22]. In this work, replacing 3-coloring problem with homomorphism (of graphs) to C_{2k+1}, authors consider the question of when forbidden cycles of length $1, 2, \ldots, 2k$, $2k+2, \ldots, f(k)$ and planarity imply a mapping to C_{2k+1}. They conclude that this is only possible when $2k+1$ is a prime number. A natural analogue question is to ask the same for negative even cycles when signed bipartite planar graphs are considered.

10 | Mapping sparse signed graphs to (K_{2k}, M)

This chapter is based on the following paper:

[NŠWX21] R. Naserasr, R. Škrekovski, Z. Wang, and R. Xu. *Mapping sparse signed graphs to (K_{2k}, M)*. 2021. arXiv: 2101.08619 [math.CO]

10.1 Homomorphisms of sparse signed graphs

Given signed graphs (G, σ) and (H, π), we have seen in Lemma 2.2.6 that the girth inequality $g_{ij}(G, \sigma) \geqslant g_{ij}(H, \pi)$ for $ij \in \mathbb{Z}_2^2$ is a necessary condition for the existence of a homomorphism of (G, σ) to (H, π). It's quite natural to ask the following question, which is one of the most central questions in the study of graph homomorphism.

Question 10.1.1. *Given signed graphs (G, σ) and (H, π), under which structural conditions on G the necessary conditions of Lemma 2.2.6 become sufficient?*

For example, Conjectures 1.3.1 and 1.3.2 claim that for $SPC(k)$, the condition of planarity is sufficient.

Conjecture 10.1.2. *Every signed planar graph (G, σ) satisfying that $g_{ij}(G, \sigma) \geqslant g_{ij}(C_{-k})$ for $ij \in \mathbb{Z}_2^2$ admits a homomorphism to $SPC(k-1)$.*

The case $k = 3$ is equivalent to the 4-color theorem, which tells us that every signed planar simple graph $(G, -)$ admits a homomorphism to $(K_4, -)$. For the case $k = 4$, it is a strengthening of the 4-color theorem that has been showed in [NRS13], based on an edge-coloring result of B. Guenin [Gue03] which in turn is based on the 4-color theorem. It claims that every signed planar graph (G, σ) satisfying that $g_{ij}(G, \sigma) \geqslant g_{ij}(K_{4,4}, M)$ for $ij \in \mathbb{Z}_2^2$ admits a homomorphism to $(K_{4,4}, M)$. These two results provide a possible answer to Question 10.1.1 for special targets, that is the condition of "planarity". That is to say, the conditions of no-homomorphism lemma for $(K_4, -)$ and $(K_{4,4}, M)$ are sufficient for signed planar graphs to admit homomorphism to it.

However, planarity is not always a feasible solution for every target. For example, for the signed graph $(K_3, +)$, even with the extra condition of planarity, not only the necessary conditions of the no-homomorphism lemma are not sufficient but it is expected to be far from it, as the 3-coloring problem of planar graphs is known to be an NP-hard problem in [GJS76]. For such cases, two closely related conditions are considered: the first one is having higher girth conditions for planar graphs (for instance, the Grötzsch theorem and Jaeger-Zhang conjecture) and the second one is to have low *maximum average degree*. This approach has been used to prove the following result:

Theorem 10.1.3. [CNS20] *Given a signed graph (H, π), there exists an $\epsilon > 0$ such that every signed graph (G, σ), satisfying $g_{ij}(G, \sigma) \geqslant g_{ij}(H, \pi)$ and $\text{mad}(G) < 2 + \epsilon$, admits a homomorphism to (H, π).*

Since the existence of ϵ is known, the main interest is then to find the best value of ϵ for a given signed graph (H, π). Recall that Question 1.4.4 asks, given a positive integer $\ell \geqslant 2$, what is the smallest value $g(\ell)$ such that every signed planar graph of girth at least $g(\ell)$ admits a circular $\frac{2\ell}{\ell-1}$-coloring. Since a circular $\frac{2\ell}{\ell-1}$-coloring of a signed graph (G, σ) could be regarded as a homomorphism of (G, σ) to the signed circular clique $K^s_{2\ell;\ell-1}$, combined with Euler's formula, the existence of ϵ ($\epsilon > 0$) ensures the existence of $g(\ell)$ of Question 1.4.4. In [CNS20], the best value of $\epsilon = \frac{4}{7}$ is proved for (K_4, e) where only one edge is negative. In this chapter, we find the best value of ϵ for the classes of signed graphs (K_{2n}, M) where M is a perfect matching of the graph under consideration. We highlight the two main results:

Theorem 10.1.4. *Every signed graph with maximum average degree less than $\frac{14}{5}$ admits a homomorphism to (K_6, M). Moreover, the bound $\frac{14}{5}$ is the best possible.*

Note that since we only consider signed simple graphs, the conditions of no-homomorphism lemma with respect to (K_6, M) is always satisfied.

Theorem 10.1.5. *Every signed graph with maximum average degree less than 3 admits a homomorphism to (K_8, M). Moreover, the bound 3 is the best possible.*

Note that the notion of edge-sign preserving homomorphisms of signed graphs is a renaming of the notion of homomorphisms of 2-edge-colored graphs, which have been extensively studied since the 1980s. As (switching) homomorphism is relatively new, using the connection we have seen in Theorem 2.2.8, we may work with the Double Switching Graph of our target signed graphs. In particular, in relation to our work, we may apply Theorem 2.5 of [BKKW04] to the Double Switching Graph $\text{DSG}(K_6, M)$ of (K_6, M) with $t = 3$ to obtain the following and our Theorem 10.1.4 improves this result.

Theorem 10.1.6 (Special case of Theorem 2.5 of [BKKW04])**.** *If G is a graph of girth at least 7 and maximum average degree at most $\frac{28}{11}$, then $(G, \sigma) \to (K_6, M)$ for any signature σ.*

Next we shall explain our motivation of choosing this family of signed graphs, noting that $n = 3$ is of special importance and is the main case of the difficulty in this chapter. First we recall a strengthening of 4-Color Theorem that every signed bipartite planar graph maps to $(K_{4,4}, M)$. The next theorem establishes the connection between the homomorphisms of signed bipartite graphs to signed graphs (K_{2k}, M) and $(K_{k,k}, M)$.

Theorem 10.1.7. *A signed bipartite graph (G, σ) admits a homomorphism to $(K_{k,k}, M)$ if and only if it admits a homomorphism to (K_{2k}, M).*

Proof. Since $(K_{k,k}, M)$ is a subgraph of (K_{2k}, M), if (G, σ) maps to $(K_{k,k}, M)$, then it obviously maps to (K_{2k}, M) as well.

For the other direction, assume that ϕ is a mapping of (G, σ) to (K_{2k}, M). Let (X, Y) be a bipartition of G and let (A, B) be a bipartition of $K_{k,k}$. If ϕ maps any vertex of X to any element in A and any vertex of Y to any element in B, then we are done. Otherwise, we define a mapping ϕ' as follows:

$$\phi'(v) = \begin{cases} \phi(v), & \text{if } v \in X, \phi(v) \in A \text{ or } v \in Y, \phi(v) \in B \\ m(\phi(v)), & \text{otherwise.} \end{cases}$$

where $m(u)$ is the matching vertex of u (in (K_{2k}, M) or $(K_{k,k}, M)$) defined by M. Since G is bipartite, $\phi'(v)$ preserves the adjacency. The images of closed walks then are the same as in ϕ. Thus ϕ' is a homomorphism of (G, σ) to $(K_{k,k}, M)$. □

In particular, for a signed bipartite graph (G, σ), to have "$(G, \sigma) \to (K_{4,4}, M)$" is equivalent to have "$(G, \sigma) \to (K_8, M)$" and to have "$(G, \sigma) \to (K_{3,3}, M)$" is equivalent to have "$(G, \sigma) \to (K_6, M)$". As to study the homomorphism of signed bipartite planar graphs to $(K_{4,4}, M)$ directly is as hard as to study the 4-Color Theorem, motivated by this relationship, we change our target signed graph to be (K_6, M) and more generally (K_{2k}, M). Furthermore, combining Theorems 2.3.10 and 10.1.7, we have the next corollary.

Corollary 10.1.8. *Given an integer $k \geqslant 3$ and a graph G, we have $\chi(G) \leqslant k$ if and only if $S(G) \to (K_{2k}, M)$.*

In light of Theorem 2.2.8, we study the edge-sign preserving homomorphism to $\mathrm{DSG}(K_6, M)$ in place of the switching homomorphism to (K_6, M). Our proof is based on the discharging technique. Assume that (G, σ) is a minimum counterexample to Theorem 10.1.4. To provide a set of forbidden configurations, we need to study the structural properties of the counterexample. We view an edge-sign preserving homomorphism to $\mathrm{DSG}(K_6, M)$ as a special list homomorphism with the list assignments being subsets of the vertices of $\mathrm{DSG}(K_6, M)$ and we call it a $\mathrm{DSG}(K_6, M)$-list coloring. In Section 10.2.2, we develop some basic tools to study this special list coloring. Our set of forbidden configurations are provided in Section 10.3.1. Then in Section 10.3.2, the discharging technique is employed to prove that (G, σ) cannot have maximum average degree less than $\frac{14}{5}$. Generalizations of the theorem to (K_8, M) and even (K_{2k}, M) are considered in Section 10.4 and examples toward the tightness of results are provided in Section 10.5. Note that in contrast to Theorem 2.3.13, we show that the conditions of no-homomorphism lemma are not sufficient for mapping signed planar graphs to (K_8, M) and give a series of signed planar graphs of girth 3 that do not map to (K_8, M) in Theorem 10.5.2.

10.2 Preliminaries

In the sequel, we shall use the labelings of vertices of (K_6, M) and $\mathrm{DSG}(K_6, M)$ as in Figure 10.2. We denote the signature of $\mathrm{DSG}(K_6, M)$ by m^*. In (K_6, M), vertices $2i-1$ and $2i$, for $i \in \{1, 2, 3\}$, are connected by a negative edge where all other edges are positive. The vertex set of $\mathrm{DSG}(K_6, M)$, denoted by C, is partitioned into two sets: C^+ and C^-. Each vertex x in C^α, $\alpha \in \{+, -\}$, is connected to a unique vertex in C^α by a negative edge, this vertex is called the *pair* of x, denoted by *Pair*(x). A *pair of colors* would refer to a vertex of $\mathrm{DSG}(K_6, M)$ and its pair. Given a vertex x^α and its pair y^α, the vertices $x^{-\alpha}$ and $y^{-\alpha}$ form another pair, and moreover, the four vertices together form a *layer* in $\mathrm{DSG}(K_6, M)$, shown as a horizontal line in Figure 10.2. Given a vertex $x \in C$, the only vertex in $\mathrm{DSG}(K_6, M)$ not adjacent to it is called the *inverse* of x. Moreover, given a subset X of C, the inverse of X, denoted X^-, is the set of of inverses of the elements of X.

One of the main ideas of this work is to extend a partial mapping of a signed graph (G, σ) to $\mathrm{DSG}(K_6, M)$ to a mapping of the full graph. Such extension problems lead to the idea of $\mathrm{DSG}(K_6, M)$-list coloring. We use the following drawing to capture this idea: vertices presented using squares are colored and vertices presented using circles are yet to be colored (see Figure 10.3 for example). We refer to [BBF+20] and references there for a general study of list-homomorphism problem of graphs and signed graphs.

However, list assignments considered in this work are rather special and restricted. Normally, in list coloring, elements from the same list are considered to have no difference in the sense of

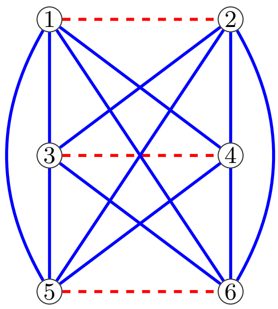


Figure 10.1. (K_6, M)

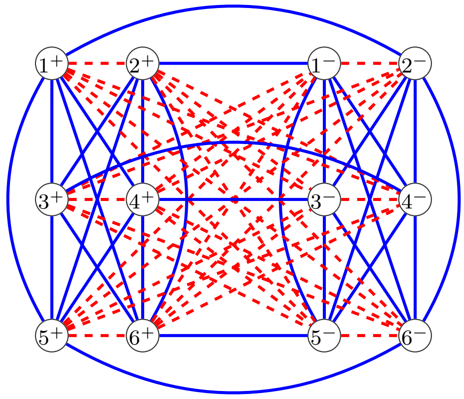


Figure 10.2. DSG(K_6, M)

the permutation. But since our list is a subset of the vertex set of a signed graph, there are structural properties of lists at each vertex. To better describe lists of available colors at vertices and understand the structure of lists, we introduce the following terminology.

In the sequel, given a set L of colors, we always assume that $L \subseteq C$. A set colors of size ℓ is called to be a *ℓ-set*. We denote by L^c the set of elements in $C \backslash L$. A set L of colors is said to be *paired* if for all $x \in L$ but at most one, we have $Pair(x) \in L$. For example, $L_1 = \{1^+, 2^+, 3^+, 4^+, 5^+\}$ and $L_2 = \{1^+, 2^+, 5^-, 6^-\}$ are paired sets of colors while $L_3 = \{1^+, 3^+\}$ and $L_3 = \{1^+, 2^+, 3^-, 5^-\}$ are not. We say a paired set L is *layered* if no three colors of L belong to the same layer and furthermore, we say a layered set L is *one-sided* if all the colors in L are on the same side, i.e., either $L \subseteq C^+$ or $L \subseteq C^-$. A layered set L of size $2k+1$, for $k = 0, 1, 2$, is said to be a *neighbored* $(2k+1)$*-set* if L consists of k pairs on one side and a single element on the other side. Observe that a neighbored 5-set is the set of all vertices adjacent to a vertex v by positive edges. A neighbored 3-set consists of neighbors of a vertex x which are connected to x by positive edges and each of which has a positive path of length 2 to another fixed vertex y. A special case when $|L(v)| = 1$, we know that the vertex v is *precolored*.

Given a signed graph (G, σ), a DSG(K_6, M)*-list assignment* L of (G, σ) is a function that assigns to each vertex of G a set $L(v) \subseteq C$. For a DSG(K_6, M)-list assignment L of (G, σ), a mapping $\phi : v \to L(v)$ such that for each edge $uv \in E(G)$, $m^*(\phi(u)\phi(v)) = \sigma(uv)$ is a special list coloring of (G, σ), as we only consider lists that are subsets of C, when there is no confusion, we call such a list coloring an *L-coloring*. If there exists an L-coloring, we say (G, σ) is *L-colorable*.

Observation 10.2.1. *If (G, σ) is L-colorable, then $(G, \sigma) \xrightarrow{s.p.}$ DSG(K_6, M).*

For a given signed graph (G, σ), a signature on G obtained from σ by switching at a vertex subset X, is denoted by σ^X. Given a DSG(K_6, M)-list assignment L and a vertex subset X of (G, σ), let L^X be a list assignment defined by:

$$L^X(v) = \begin{cases} (L(v))^-, & \text{for } v \in X \\ L(v), & \text{for } v \in V(G)\backslash X. \end{cases}$$

Observation 10.2.2. *For all $X \subset C$, a signed graph (G, σ) is L-colorable if and only if (G, σ^X) is L^X-colorable.*

10.2.1 Tool: signed rooted trees

Let T be a rooted tree with root v. Given a vertex x of T, we define a subtree rooted at x, denoted by T_x, to be the subgraph induced by x and those vertices of T whose unique path to v contains x. Let L be a DSG(K_6, M)-list assignment of a signed rooted tree (T, σ). Toward deciding if (T, σ) is L-colorable and taking advantage of the rooted tree, we have the following notation.

For a vertex x of (T, σ), we define the set of *admissible colors*, denoted $L^a(x)$, to be the set of the colors $c \in L(x)$ such that with the restriction of L onto (T_x, σ) there exists an L-coloring ϕ of (T_x, σ) where $\phi(x) = c$. Thus (T, σ) is L-colorable if and only if $L^a(v) \neq \varnothing$, noting that v is the root of T, and moreover, reducing $L(x)$ to $L^a(x)$ at any time would not affect L-colorability of (T, σ). Sometimes, instead of considering the set of admissible colors at a vertex, it is preferable to consider the set of colors that are forbidden through the restriction from its descendants or neighbors in general. Let xy be an edge of (G, σ) and assume $L(y)$ is the set of colors available at y. Then we define $F_{L(y)}(x)$ to be the set of *forbidden colors* at x because of the edge xy and the list $L(y)$. More precisely, a color c is forbidden on x because of $L(y)$ if for each choice $c' \in L(y)$ either c is not adjacent to c' or $\sigma(xy) \neq m^*(cc')$. For example, if $L(y) = \{1^+, 2^+, 3^-\}$ and xy is a positive edge, then $F_{L(y)}(x) = \{1^+, 2^+, 3^-, 4^-\}$. When the list assignment is clear from the context, we may simply write $F_y(x)$ in place of $F_{L(y)}(x)$. For an L-coloring of a signed subtree (T_x, σ), we have the following relation between the two notions:

$$L^a(x) = L(x) \backslash \bigcup_{\substack{y \\ y \text{ is child of } x}} F_{L^a(y)}(x).$$

Thus in the rest of this work, we may modify list $L(x)$ by removing colors from $F_y(x)$ or simply replacing it by $L^a(x)$ at any necessary time. We first gather some basic rules on these modifications in the following lemmas.

Lemma 10.2.3. *Let (T, σ) be a path vv_1v_2, regarded as a rooted tree with v being the root. Let L be a list assignment satisfying $|L(v_2)| = 1$ and $L(v) = L(v_1) = C$. Then $L^a(v)$ is a paired 10-set.*

Lemma 10.2.4. *Let T be a 3-path $v_1v_2vv_3$, regarded as a rooted tree with v being the root. Let L be a list assignment satisfying $|L(v_1)| = |L(v_3)| = 1$ and $L(v_2) = L(v) = C$. Then $L^a(v)$ is either a neighbored 5-set, or a neighbored 3-set, or it is a one-sided paired 4-set.*

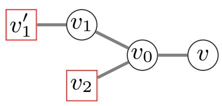


Figure 10.3. A rooted tree (T, σ) for Lemma 10.2.5

Lemma 10.2.5. *Let (T, σ) be a signed rooted tree of Figure 10.3 with the root v. Let L be a list assignment satisfying $|L(v_1')| = |L(v_2)| = 1$ and $L(v) = L(v_0) = L(v_1) = C$. Then we have $|L^a(v)| \geqslant 8$.*

These observations can be verified using the vertex-transitivity of the signed graph DSG(K_6, M).

10.2.2 Properties of L-coloring of signed paths and cycles

In this section, we study the properties of L-coloring of a given signed graph and develop tools of independent interest that will be used in Section 10.3.

In the following lemmas, for a given edge xy, viewed as a rooted tree with y being the root and x being the leaf, we would like to bound the size of $F_x(y)$ in terms of the size of $L(x)$. Note that the larger the set $L(x)$ is, the smaller the set $F_x(y)$ becomes. Often we use the following lemma on an edge xy to evaluate $F_x(y)$ without explicitly stating that the edge xy under consideration is regarded as a rooted tree with the leaf x and the root y.

Lemma 10.2.6. *Let (K_2, σ) be a signed edge xy and let L be its $\mathrm{DSG}(K_6, M)$-list assignment. Then the following statements hold:*

(1) If $L(x)$ is empty, then $F_x(y) = C$.

(2) If $L(x)$ consists of one element, then $F_x(y)$ is a paired 7-set.

(3) If $L(x)$ is a paired 2-set, then $F_x(y)$ is a paired 6-set.

(4) If $L(x)$ is a paired 3-set, then $F_x(y)$ is a paired set of size at most 4. Moreover, if $L(x)$ is not one-sided, then $|(L(y)\backslash F_x(y)) \cap C^+| \geqslant 4$ and $|(L(y)\backslash F_x(y)) \cap C^-| \geqslant 4$.

(5) If $L(x)$ is a paired 4-set, then $F_x(y)$ is a paired set of size at most 4. In particular, if $L(x)$ is not layered, then $F_x(y) = \varnothing$, and if $L(x)$ is one-sided, then $F_x(y)$ is a paired 2-set.

(6) If $L(x)$ is a neighbored 5-set, then $F_x(y)$ is a paired 2-set.

(7) If $L(x)$ is a paired 6-set, then $F_x(y)$ is a paired set of size at most 2. In particular, if $L(x)$ is not layered or one-sided, then $F_x(y) = \varnothing$.

(8) If $L(x)$ is a paired set of size at least 8, then $F_x(y) = \varnothing$.

Proof. We assume that xy is a positive edge. The case when xy is negative is similar.

In the first claim, $L(x) = \varnothing$ means that there is no valid color for x, in other words, all colors are forbidden at y. In the second claim, we may assume $L(x) = \{1^+\}$, then $F_x(y) = \{1^+, 2^+, 1^-, 3^-, 4^-, 5^-, 6^-\}$ which is of size seven. In the third claim, there is only one type of paired 2-set. Without loss of generality, we assume $L(x) = \{1^+, 2^+\}$ and then $F_x(y) = \{1^+, 2^+, 3^-, 4^-, 5^-, 6^-\}$.

Next, suppose that $L(x)$ is a paired 3-set. Without loss of generality, we only consider three possibilities $L(x) = \{1^+, 2^+, 3^+\}$, $L(x) = \{1^+, 2^+, 1^-\}$, and $L(x) = \{1^+, 2^+, 3^-\}$. We easily obtain that for the first possibility $F_x(y) = \{3^-, 5^-, 6^-\}$, for the second possibility $F_x(y) = \{1^+\}$, and for the last possibility $F_x(y) = \{1^+, 2^+, 3^-, 4^-\}$. Observe that $|(L(y)\backslash F_x(y)) \cap C^+| \geqslant 4$ and $|(L(y)\backslash F_x(y)) \cap C^-| \geqslant 4$ when $L(x)$ is not one-sided.

Suppose now $L(x)$ is a paired 4-set. Again without loss of generality, we only need to consider three cases $L(x) = \{1^+, 2^+, 3^+, 4^+\}$ for $L(x)$ being one-sided, $L(x) = \{1^+, 2^+, 1^-, 2^-\}$ for $L(x)$ being not layered, and $L(x) = \{1^+, 2^+, 3^-, 4^-\}$ for $L(x)$ being layered but not one-sided. For the first case $F_x(y) = \{5^-, 6^-\}$, for the second case $F_x(y) = \varnothing$, and for the last case $F_x(y) = \{1^+, 2^+, 3^-, 4^-\}$.

Suppose $L(x)$ is a neighbored 5-set, say $L(x) = \{1^+, 2^+, 3^+, 4^+, 5^-\}$, we easily get $F_x(y) = \{5^-, 6^-\}$.

Suppose $L(x)$ is a paired set of size 6. Without loss of generality, we consider three possibilities $L(x) = C^+$ for $L(x)$ being one-sided, $L(x) = \{1^+, 2^+, 3^+, 4^+, 1^-, 2^-\}$ for $L(x)$ being not layered, and $L(x) = \{1^+, 2^+, 3^+, 4^+, 5^-, 6^-\}$ for $L(x)$ being layered but not one-sided. For the first two possibilities $F_x(y) = \varnothing$, and for the last possibility $F_x(y) = \{5^-, 6^-\}$.

It is an easy observation that if $L(x)$ contains four elements of a layer, then $F_x(y) = \varnothing$. This implies that $F_x(y) = \varnothing$ when $L(x)$ is a paired set of size at least 8. □

We note that if $L(x)$ consists of one color, then $C\backslash F_x(y)$ is a neighbored 5-set and if $L(x)$ is a neighbored 3-set, then $C\backslash F_x(y)$ is a paired 8-set containing neither C^+ nor C^-.

The next observation is straightforward and we will use it frequently in Section 10.3. Note that a paired 8-set, where all the elements compose two layers, do not contain a neighbored 5-set.

Observation 10.2.7. *Let (K_2, σ) be a signed edge xy and let L be a $\mathrm{DSG}(K_6, M)$-list assignment of it satisfying that each of $L(x)$ and $L(y)$ is either a neighbored 5-set or a paired 8-set. Then one can choose $c_x \in L(x)$ and $c_y \in L(y)$ such that c_x and c_y are in different layers and $m^*(c_xc_y) = \sigma(xy)$.*

Lemma 10.2.8. *Let (K_2, σ) be a signed edge xy and let L be its $\mathrm{DSG}(K_6, M)$-list assignment where $L(x)$ is a neighbored 5-set and $L(y)$ is either C^+ or C^-. Then there exists a choice $c_x \in L(x)$ and a 4-subset $L'(y)$ of $L(y)$ such that for every $c_y \in L'(y)$, $m^*(c_xc_y) = \sigma(xy)$.*

Proof. As a neighbored 5-set intersecting both C^+ and C^-, if $\sigma(xy)$ is positive, then we choose $c_x \in L(x) \cap L(y)$, otherwise we choose $c_x \in L(x) \cap (C\backslash L(y))$. Then c_x has four neighbors in $L(y)$ which are adjacent to it by edges of sign $\sigma(xy)$. □

Next we consider the L-colorings of signed nontrivial paths and cycles. The next observation is a useful tool in showing the forbidden configurations in Section 10.3.

Observation 10.2.9. *Let (P_3, σ) be a signed path xyz and let L be a $\mathrm{DSG}(K_6, M)$-list assignment where $L(y) = C$, $L(x) = \{c_x\}$ and $L(z) = \{c_z\}$. Then (P_3, σ) is L-colorable unless one of the following conditions holds:*

(1) c_x and c_z are in the same layer but on different sides, and P_3 is a positive path;

(2) c_x and c_z are in the same layer and on the same side, and P_3 is a negative path.

Lemma 10.2.10. *Let (P_3, σ) be a signed path xyz and let L be its $\mathrm{DSG}(K_6, M)$-list assignment satisfying one of the following conditions:*

(1) $L(y) = C$, $L(x) = \{c_x\}$, and $L(z) = \{c_z\}$ where c_x and c_z are in different layers;

(2) $L(x) = \{c_x\}$, $L(y)$ is a paired 10-set, and $L(z)$ is a neighbored 5-set;

(3) $|L(y)| \geqslant 5$, and each of $L(x)$ and $L(z)$ is a neighbored 5-set.

Then (P_3, σ) is L-colorable.

Proof. The first case is a restatement of Observation 10.2.9. To prove the other two cases, by Observation 10.2.2, we may assume both edges are positive. For the second case, without loss of generality, we may assume that $c_x = 1^+$. If $L(z)$ contains one of $3^-, 4^-, 5^-, 6^-$, say 5^-, without loss of generality, then we take $c_z = 5^-$. Since $L(y)$ is a paired 10-set, it contains one of 2^- or 6^+ either of which completes the coloring. If $L(z)$ contains none of $3^-, 4^-, 5^-, 6^-$, then, as it is a neighbored 5-set, it must contain $3^+, 4^+, 5^+, 6^+$. If the choice of $c_z = 3^+$ does not work, then $L(y)$ is the complement of $\{5^+, 6^+\}$, in which case taking $c_z = 5^+$ would work.

For the third claim, without loss of generality, we assume $L(x) = \{1^+, 2^+, 3^+, 4^+, 5^-\}$. Then except for 5^- and 6^-, every vertex of $\mathrm{DSG}(K_6, M)$ is connected by a positive edge to at least one vertex in $L(x)$. Similarly, there is only one pair, say a, b, of vertices of $\mathrm{DSG}(K_6, M)$ which is not connected by a positive edge to at least one vertex in $L(z)$. Since $|L(y)| \geqslant 5$, there is a color in $L(y)$ different from $5^-, 6^-, a, b$. Assigning this color to y, we can find choices for x and z. □

Lemma 10.2.11. *Let (P_3, σ) be a signed path xyz with $\sigma(xy) = \alpha$ and $\sigma(yz) = \beta$. Given a layered 6-set X, for every $c_x \in X$ and $c_z \in X^{\alpha\beta}$, there exists $c_y \in C$ such that $m^*(c_yc_x) = \sigma(xy)$ and $m^*(c_yc_z) = \sigma(yz)$.*

Proof. By Observation 10.2.2, we assume that $\alpha = \beta = +$. Then $X^{\alpha\beta} = X$. For every $c_x, c_z \in X$, since X is a layered 6-set, either (1) $c_x = \text{Pair}(c_z)$ or (2) c_x and c_z are in different layers. In both cases, Observation 10.2.9 assures that c_y exists. □

In the next lemma, we consider the L-coloring of a signed path on $2k$ vertices.

Lemma 10.2.12. *Let (P_{2k}, σ) be a signed path where $P_{2k} = v_1v_2 \cdots v_{2k}$ for $k \geqslant 1$ and let L be a* DSG(K_6, M)*-list assignment of (P_{2k}, σ) satisfying one of the following conditions:*

(1) $L(v_1)$ is a paired 3-set, $L(v_{2k})$ is a one-sided 4-set, and for each $i \in \{2, \ldots, 2k-1\}$, $L(v_i)$ contains a neighbored 5-set for odd i and $|L(v_i)| \geqslant 10$ for even i.

(2) $L(v_1)$ is a one-sided 4-set, $L(v_{2k})$ is a paired 3-set and for each $i \in \{2, \ldots, 2k-1\}$, $L(v_i)$ contains a neighbored 5-set for odd i and $|L(v_i)| \geqslant 10$ for even i.

Then (P_{2k}, σ) is L-colorable.

Proof. For $k = 1$, we consider a signed edge v_1v_2 with v_1 being its root. By symmetry, we may assume that L is a list assignment satisfying that $L(v_1)$ is a paired 3-set and $L(v_2)$ is a one-sided 4-set. By Lemma 10.2.6 (5), $|F_{v_2}(v_1)| = 2$. For any color $c \in L(v_1)$, we can color v_1 by the remaining color from $L(v_1) \backslash F_{v_2}(v_1)$.

For the first case with $k \geqslant 2$, consider (P_{2k}, σ) as a rooted signed tree with root v_{2k-1}. Since $|L(v_1)| \geqslant 3$, by Lemma 10.2.6 (4), $L^a(v_2)$, which is $L(v_2) \backslash F_{v_1}(v_2)$, contains a paired set of size at least 6. Applying Lemma 10.2.6 (7) and Lemma 10.2.6 (4) alternately, we can propagate this and obtain that $|L^a(v_{2i-1})| = |L(v_{2i-1}) \backslash F_{v_{2i-2}}(v_{2i-1})| \geqslant 5 - 2 = 3$ and $|L^a(v_{2i})| = |L(v_{2i}) \backslash F_{v_{2i-1}}(v_{2i})| \geqslant 10 - 4 = 6$ for $i \in \{2, \ldots, k-2\}$. Finally, by Lemma 10.2.6 (5) and (7), for the root v_{2k-1}, we have $|L^a(v_{2k-1})| = |L(v_{2k-1}) \backslash (F_{v_{2k}}(v_{2k-1}) \cup F_{v_{2k-2}}(v_{2k-1}))| \geqslant 5 - 2 - 2 = 1$.

For the second case with $k \geqslant 2$, similarly, consider (P_{2k}, σ) as a rooted signed tree with root v_{2k-1}. So $|L^a(v_2)| = |L(v_2) \backslash F_{v_1}(v_2)| \geqslant 10 - 2 = 8$, by Lemma 10.2.6 (8) and (6), we can propagate this and obtain that $|L^a(v_{2i})| = |L(v_{2i}) \backslash F_{v_{2i-1}}(v_{2i})| \geqslant 10 - 2 = 8$ and $|L^a(v_{2i+1})| = |L(v_{2i+1}) \backslash F_{v_{2i}}(v_{2i+1})| \geqslant 5 - 0 = 5$ for $i \in \{2, \ldots, k-2\}$. For the root v_{2k-1}, by Lemma 10.2.6 (4) and (8), we have $|L^a(v_{2k-1})| = |L(v_{2k-1}) \backslash (F_{v_{2k}}(v_{2k-1}) \cup F_{v_{2k-2}}(v_{2k-1}))| \geqslant 5 - 4 - 0 = 1$. This completes the proof. □

From now on, we consider the L-coloring of signed cycles. We first state a result of the signed cycle on four vertices.

Lemma 10.2.13. *Let (C_4, σ) be a signed 4-cycle $xyzt$ and let L be a* DSG(K_6, M)*-list assignment of (C_4, σ) where each of $L(x)$ and $L(t)$ is a neighbored 5-set, and each of $L(y)$ and $L(z)$ is a paired 10-set. Then (C_4, σ) is L-colorable.*

Proof. By switching, if necessary, we may assume that xy, yz and zt are all positive edges. If, after necessary switchings, xt is also a positive edge, then we choose colors $c_x \in L(x)$ for x and $c_t \in L(t)$ for t such that they are in different layers but on the same side. This is possible because $L(x)$ and $L(t)$ are both neighbored 5-sets. We may then assume, without loss of generality, $c_x = 1^+$ and $c_t = 3^+$. Then the possibility of colors for the pair of two vertices (y, z) is one of the following cases: $(2^-, 4^-)$, $(4^+, 5^+)$ and $(6^+, 2^+)$. But since in each of $L(y)$ and $L(z)$ only one pair is missing, at least one of these three possibilities works.

Now we assume that xt is negative. With the given $L(x)$ and $L(t)$, we always have a choice for $c_x \in L(x)$ and $c_t \in L(t)$ such that they are on different sides and in different layers. Without loss of generality, assume $c_x = 1^+$ and $c_t = 3^-$. Then the option to extend xt-path through y, z is either coloring vertex y with 2^- and vertex z from $\{5^-, 6^-\}$, or coloring vertex z with 4^+ and vertex yc from

$\{5^+, 6^+\}$. Therefore, if (C_4, σ) is not L-colorable, then either $L(y) = \{1^-, 2^-\}^c, L(z) = \{3^+, 4^+\}^c$ or $L(y) = \{5^+, 6^+\}^c, L(z) = \{5^-, 6^-\}^c$. However, as $L(x)$ and $L(t)$ are both neighbored 5-sets, we also have one of the two following possibilities for c_x and c_t: either they form a pair, or $c_x \in C^-$ and $c_t \in C^+$. Based on whichever the possibility, we have a choice for y and z this time. □

More generally, we have the following result for signed even cycles.

Lemma 10.2.14. *Let (C_{2k}, σ) be a signed even cycle $v_1v_2 \cdots v_{2k}$ with $k \geqslant 2$ and let L be a* DSG(K_6, M)*-list assignment of (C_{2k}, σ) satisfying one of the following conditions:*

(1) For even values of i, $L(v_i)$ is a neighbored 5-set, and for odd values, it is a paired 10-set.

(2) $L(v_1) = C$, $L(v_2)$ is a paired 8-set containing neither C^+ nor C^-, and $L(v_{2k})$ is a neighbored 5-set. For other vertices v_i, if i is even then $L(v_i)$ is a paired 10-set and if i is odd then $L(v_i)$ is a neighbored 5-set.

(3) $L(v_1) = C$, $L(v_2)$ and $L(v_{2k})$ are neighbored 5-sets, $L(v_3)$ is a paired 8-set containing neither C^+ nor C^-. For other vertices, if any, $L(v_i)$ is a neighbored 5-set if i is odd and $L(v_i)$ is a paired 10-set otherwise.

Then (C_{2k}, σ) is L-colorable.

Proof. By Observation 10.2.2, we may assume that $\sigma(v_{2k}v_1) = \sigma(v_1v_2) = \sigma(v_2v_3) = +$ in all cases. We consider each case separately.

(1). Since $L(v_{2k})$ is a neighbored 5-set, either $L(v_{2k}) \cap C^+$ or $L(v_{2k}) \cap C^-$ is a one-sided 4-set. Without loss of generality, we assume that $|L(v_{2k}) \cap C^+| = 4$. Since $L(v_2)$ is also a neighbored 5-set, $L(v_2)$ has at least one element in C^+, without loss of generality, let 1^+ be one such an element and assign it to v_2. Then, using Lemma 10.2.6 (2), update the list of colors available at v_3 to a neighbored 3-set $L'(v_3)$. Furthermore, update the list of available colors at v_{2k} to the one-sided 4-set $L'(v_{2k}) = C^+ \cap L(v_{2k})$ and leave the lists of other vertices as they were given. Applying Lemma 10.2.12 (2) to the signed path (P_{2k-2}, σ) with $P_{2k-2} = v_3 \cdots v_{2k}$ and with the modified list assignment given above, we color all the vertices of P_{2k-2}. If v_{2k} is colored 1^+ or 2^+, then, as $L(v_1)$ is a paired 10-set, we could find a choice to extend the coloring to v_1 and we are done. Else, by symmetry among colors $3^+, 4^+, 5^+, 6^+$, we may assume v_{2k} colored with 3^+. If this coloring is not extendable to v_1, then $L(v_1) = \{5^+, 6^+\}^c$.

If $\{5^+, 6^+\} \cap L(v_2) \neq \varnothing$, then by choosing a color for v_2 from $\{5^+, 6^+\}$ and repeating the same process, this time we are sure to have a choice to color v_1. Thus we have one of the two possibilities for $L(v_2)$: (i). It contains $\{1^+, 2^+, 3^+, 4^+\}$. (ii). It is $\{1^+, 3^-, 4^-, 5^-, 6^-\}$. If the latter happens, we choose a color for $L(v_{2k})$ from C^-, and repeat the previous process with $L'(v_{2k-1})$ being a neighbored 3-set and $L'(v_2) = \{3^-, 4^-, 5^-, 6^-\}$. Hence, we must have $\{1^+, 2^+, 3^+, 4^+\} \subset L(v_2)$. By the symmetry of v_{2k} and v_2, we also have $\{1^+, 2^+, 3^+, 4^+\} \subset L(v_{2k})$. But then once again we repeat the original process by assigning the color in $L(v_2) \cap \{5^-, 6^-\}$ to v_2 and taking $L'(v_{2k}) = \{1^+, 2^+, 3^+, 4^+\}$. So for each choice of color for v_{2k}, we could find a choice for v_1 which is not in $\{5^+, 6^+\}$. We are done.

(2). Since $L(v_{2k})$ is a neighbored 5-set, without loss of generality, we assume that $|L(v_{2k}) \cap C^+| = 4$. Let $X = C^+$ and let $L'(v_2) = L(v_2) \cap X$. We update the list of available colors at v_2 to $L'(v_2)$, observing that it is a one-sided 4-set. Considering the tree v_2v_3 rooted at v_3, by Lemma 10.2.6 (5), since $L(v_3)$ is a neighbored 5-set, we update the list of available colors at v_3 to a neighbored 3-set $L'(v_3)$. Now set $L'(v_{2k}) = X \cap L(v_{2k})$ and $L'(v_i) = L(v_i)$ for $i \in \{4, \ldots, 2k-1\}$. Applying Lemma 10.2.12 (1) to the signed path $P_{2k-2} = v_3 \cdots v_{2k}$ and with the modified list assignment given here, we color all the vertices of P_{2k-2}. Noting that the colors of v_{2k} and v_2 are both chosen from X, and by Lemma 10.2.11, we can complete the coloring to v_1.

(3). We first assume that there is a color $c \in L(v_2)$ such that four positive neighbors (connected by positive edges) of c are in $L(v_3)$. In that case, we color v_2 by c. We update $L(v_{2k})$ by removing colors in the same layer as c and $L(v_3)$ by taking the four positive neighbors of c. To complete the coloring, we color the path $v_3v_4\cdots v_{2k}$ by Lemma 10.2.12. Then by Lemma 10.2.10, we can find a color for v_1 and we are done.

If no such a choice of c exists, then $L(v_2) \subset L(v_3)$, as otherwise any color in $L(v_2)\backslash L(v_3)$ would work. Thus without loss of generality, we assume $L(v_2) = \{1^+, 2^+, 3^+, 4^+, 5^-\}$ and $L(v_3) = \{1^+, 2^+, 3^+, 4^+, 3^-, 4^-, 5^-, 6^-\}$. Next, we examine the choice of 3^+ for v_2. This updates $L(v_3)$ to a neighbored 3-set. If $L(v_{2k})$ contains at most one of 3^- and 4^-, then the updated list $L'(v_{2k}) = L(v_{2k})\backslash\{3^-, 4^-\}$ contains a one-sided 4-set and we are done by applying Lemma 10.2.12. Thus we assume $\{3^-, 4^-\} \subset L(v_{2k})$, this implies that $|L(v_{2k}) \cap \{1^+, 2^+, 5^+, 6^+\}| = 1$, let c_1 be the common element. We consider two cases:

- $c_1 \in \{1^+, 2^+\}$. Then we set $L'(v_{2k}) = \{3^-, 4^-, 5^-, 6^-\}$ and, by using Lemma 10.2.12, we color the path $v_4\cdots v_{2k}$. Then depending on the color of v_4, we choose a color for v_3 from $\{3^+, 4^+, 3^-, 4^-\}$. Based on this choice, we color v_2 from $\{1^+, 2^+, 5^-\}$. By Observation 10.2.9, since $v_{2k}v_1v_2$ is a positive path, any of these choices for v_{2k} and v_2 can be extended to v_1.

- $c_1 \in \{5^+, 6^+\}$. We set $L'(v_{2k}) = \{3^-, 4^-, c_1\}$, and consider the path $v_4\cdots v_{2k}$ (when $k = 2$, this is a single vertex). By Lemma 10.2.12 and with the list assignment $L(v_4), \ldots, L(v_{2k-1}), L'(v_{2k})$, this path admits a coloring. Let c_2 be the color of v_4. Of the elements in $\{3^+, 4^+, 3^-, 4^-\}$ at least one, say c_3, is adjacent to c_2 by a positive edge. Assign c_3 to v_3. If $c_3 \in \{3^+, 4^+\}$, then we choose a color from $\{1^+, 2^+\}$ for v_2; if $c_3 \in \{3^-, 4^-\}$, then there exists a color from $\{3^+, 4^+, 5^-\}$ for v_2. In each case, we can choose a color for v_2 which is not in the same layer as the color of v_{2k}. By Lemma 10.2.10, we can extend this coloring to v_1 and complete the coloring of this even cycle.

Thus in all cases, we find an L-coloring. □

For signed odd cycles, we have similar results based on the given DSG(K_6, M)-list assignments.

Lemma 10.2.15. *Let (C_{2k+1}, σ) be a signed odd cycle where $C_{2k+1} = v_1v_2\cdots v_{2k+1}$ and let L be a* DSG(K_6, M)*-list assignment of (C_{2k+1}, σ) satisfying one of the following conditions:*

(1) $L(v_i)$ is a neighbored 5-set for each even i and $L(v_i)$ is a paired 10-set for each odd i.

(2) $L(v_1) = C$ and $L(v_{2k+1})$ is a neighbored 5-set. For other vertices, $L(v_i)$ is a neighbored 5-set if i is even and $L(v_i)$ is a paired 10-set otherwise.

Then (C_{2k+1}, σ) is L-colorable.

Proof. (1). Applying Lemma 10.2.8 to the signed edge $v_{2k}v_{2k+1}$, since $L(v_{2k})$ is a neighbored 5-set and $L(v_{2k+1})$ is a paired 10-set (thus containing C^+ or C^-), we can assign a color $c_{v_{2k}} \in L(v_{2k})$ to v_{2k} and choose a subset $L'(v_{2k+1})$ of $L(v_{2k+1})$ which is a one-sided 4-set and has the property that for each $c \in L'(v_{2k+1})$ we have $m^*(c_{v_{2k}}c) = \sigma(v_{2k}v_{2k+1})$. Considering the signed edge $v_{2k}v_{2k-1}$, by Lemma 10.2.6 (2), $|L^a(v_{2k-1})| \geqslant 3$. We set $L'(v_{2k-1}) := L^a(v_{2k-1})$ and $L'(v_i) = L(v_i)$ for $i \in \{1, 2, \ldots, 2k-2\}$. We may now apply Lemma 10.2.12 (2) to the signed path (P_{2k}, σ) where $P_{2k} = v_{2k+1}v_1v_2\cdots v_{2k-1}$ and we are done.

(2). The proof of this case is similar to the proof of Lemma 10.2.14 (3). For $k = 1$, by Observation 10.2.7, we can always choose $c_{v_1} \in L(v_1)$ and $c_{v_3} \in L(v_3)$ which are in different layers such that the sign of v_1v_3 is preserved. By Lemma 10.2.10, this coloring can be extended to v_2. Thus we may assume $k \geqslant 2$ and, by Observation 10.2.2, we may assume that $\sigma(v_1v_2) = \sigma(v_1v_{2k+1}) = \sigma(v_2v_3) = +$. Furthermore, without loss of generality, we assume that $L(v_2) = \{1^+, 2^+, 3^+, 4^+, 5^-\}$.

If $L(v_{2k+1})$ contains four elements of C^+, then after taking three of them as $L'(v_{2k+1})$ and setting $L'(v_2) = \{1^+, 2^+, 3^+, 4^+\}$ while keeping the rest of the lists same, we may apply Lemma 10.2.12 to color the path $v_2v_3 \cdots v_{2k+1}$. This coloring then is extendable to v_1 by Lemma 10.2.11.

If $L(v_{2k+1})$ contains $\{5^-, 6^-\}$, then we may take $L'(v_{2k+1})$ to consist of $5^-, 6^-$ and the only element of $L(v_{2k+1})$ in C^+, and complete the coloring as in the previous case. We may, therefore, assume $L(v_{2k+1}) = \{1^-, 2^-, 3^-, 4^-, c\}$ where $c \in \{5^+, 6^+\}$.

Next we consider the case that $\{1^-, 2^-, 3^-, 4^-\} \subset L(v_3)$. By Lemma 10.2.6 (5), there is a 3-subset $L'(v_{2k})$ of $L(v_{2k})$ such that the signed edge $v_{2k}v_{2k+1}$ is L'-colorable for every choice $c_{v_{2k}} \in L'(v_{2k})$ and $L'(v_{2k+1}) = \{1^-, 2^-, 3^-, 4^-\}$. We may now first color the path $v_3v_4 \cdots v_{2k}$ with list assignment $L'(v_3) = \{1^-, 2^-, 3^-, 4^-\}$, $L'(v_{2k})$ defined above, and $L(v_i)$ for all other values of i by Lemma 10.2.12. To complete the coloring, then we color v_2 with 5^-, and v_{2k+1} with a color from $\{1^-, 2^-, 3^-, 4^-\}$. This coloring then is easily extendable to v_1. Thus, without loss of generality, we may assume that $L(v_3) = C \backslash \{1^-, 2^-\}$.

To complete the proof, we consider the list assignment on the path $v_4v_5 \cdots v_{2k+1}$ where v_{2k+1} is assigned $L'(v_{2k+1}) = \{3^+, 4^+, 5^-\}$ and each other vertex is assigned $L(v_i)$. By Lemma 10.2.12, we have an L'-coloring ϕ of this signed path. Given $\phi(v_4)$, one can always pick a color in $\{3^+, 4^+, 3^-, 4^-\}$ for v_3 such that the sign of the edge v_3v_4 is preserved. However, for any such choice, the coloring of v_3 and v_{2k+1} can be extended to v_1 and v_2 by argument similar to the end of the proof of previous lemma. This concludes the proof. □

10.3 Mapping sparse signed graphs to (K_6, M)

In this section, we shall prove the first part of Theorem 10.1.4. With all the connection and preparation, it suffices to prove the following theorem.

Theorem 10.3.1. *Every signed graph (G, σ) satisfying that $\mathrm{mad}(G) < \frac{14}{5}$ admits an edge-sign preserving homomorphism to $\mathrm{DSG}(K_6, M)$.*

To prove Theorem 10.3.1, we assume that (G, σ) is a minimum counterexample, that is to say, $\mathrm{mad}(G) < \frac{14}{5}$, (G, σ) does not map to $\mathrm{DSG}(K_6, M)$ and the number of vertices of G is as small as possible. The proof is typical and organized as follows. In the coming subsection, we present a set of reducible configurations in (G, σ) and then we use discharging arguments to show that at least one reducible configuration listed before exists in (G, σ), which is a contradiction that completes the proof.

10.3.1 Reducible configurations

In order to describe a forbidden configuration better, we use the following standard terminology. A vertex of degree k is called a *k-vertex*. Moreover, a *k^+-vertex* is a vertex with degree at least k and a *k^--vertex* is a vertex of degree at most k. A *k_i-vertex* is a k-vertex with precisely i neighbors of degree 2. When proving that a configuration F is forbidden, we consider F together with all its neighbors that are precolored. A precolored neighbor, say v, of F might see more than one vertex in F, however, for simplicity we view such a configuration with multiple copies of v, one for each neighbor in F, and with all copies receive the same color as v. In a special case that F is a tree, this allows us to view the subgraph induced by F and its neighbors as a tree.

As (K_6, M) is a vertex-transitive signed graph up to switching operation, equivalently, $\mathrm{DSG}(K_6, M)$ is vertex-transitive as a 2-edge-colored graph, we have the following.

Lemma 10.3.2. *The signed graph (G, σ) is 2-connected, and in particular, we have $\delta(G) \geqslant 2$.*

Let L be a DSG(K_6, M)-list assignment of (G, σ). Next we show that vertices of certain types are reducible.

Lemma 10.3.3. *The signed graph (G, σ) does not contain the following types of vertices: 2_1-vertex, 3_2-vertex, 4_4-vertex, 5_5-vertex.*

Proof. Observe that (G, σ) contains no 3_2-vertex (respectively, 5_5-vertex) implies that (G, σ) contains no 2_1-vertex (respectively, 4_4-vertex). Thus it is sufficient to prove that G has no 3_2-vertex or 5_5-vertex.

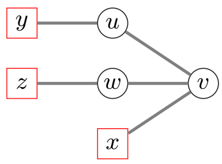


Figure 10.4. Reducible configuration 3: 3_2-vertex

Let v be a 3_2-vertex, let u and w be its 2-neighbors and let x be its third neighbor. Moreover, let y and z be the other neighbors of u and w (see Figure 10.4). We claim that any L-coloring of x, y and z could be extended to a coloring of the signed tree induced by u, v and w (rooted at v). That is because the color of x reduces the list of available colors at x to a neighbored 5-set of which at most two elements become forbidden respectively by each of the vu and vw-branches, leaving at least one admissible color at v.

The proof of 5_5-vertex is similar. Let v be a 5_5-vertex. Consider the tree T induced by v, its neighbors and their neighbors and suppose it is rooted at v. Assume all the leaves of T are colored. Thus we have a full list of colors available at v at the start, each of the five branches may forbid two from $L(v)$, leavings us with at least two colors in the admissible set of v. □

Observe that this simple argument will not work on other types of vertices. More precisely, for any other type of a vertex v, there is a coloring of its neighborhood that leaves us with an empty admissible set at v. However, in the next series of lemmas, we put some restrictions on the neighbors of vertices of type 3_1, 3_0 and 4_3.

Lemma 10.3.4. *There is no 3_1-vertex adjacent to a 4_3-vertex in (G, σ).*

Proof. Suppose to the contrary that a 3_1-vertex u is adjacent to a 4_3-vertex v. Let u_1, u_2 be the other two neighbors of u with u_2 being of degree 2, and let v_1, v_2, v_3 be the other three neighbors of v all of which are all of degree 2.

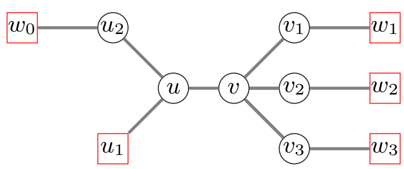


Figure 10.5. u and v do not share a common 2-neighbor

We first consider the case that u_2 is distinct from v_1, v_2, v_3. Let w_0 be the other neighbor of u_2, finally let w_1, w_2, w_3 be the other neighbors of v_1, v_2, v_3 respectively. Observe that w_i and u_1 are not necessarily distinct vertices of G, but as they are going to be precolored, for the sake of this proof we may assume that they are distinct. Let T be the tree induced by u, v, u_2, v_1, v_2, v_3 and w_0, w_1, w_2, w_3 and consider it as rooted at v. See Figure 10.5 for illustration. Let L be a DSG(K_6, M)-list assignment which assigns a list of size 1 to each of w_i's and u_1, and a full list to the other vertices of T. Observe that, by Lemma 10.2.5, the u-branch of the tree forbids at most two pairs of colors from $L(v)$ and, by Lemma 10.2.3, each other branch forbids at most one pair of colors from it, thus $L^a(v)$ contains at least one pair of admissible colors. This completes the proof of the case.

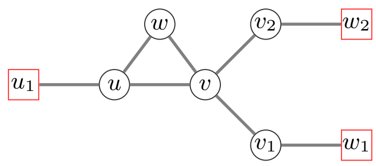


Figure 10.6. u and v share a 2-neighbor

Now we consider the case that u and v have a common 2-neighbor, say w. Let u_1 be the other neighbor of u and let v_1, v_2 be the other two neighbors of v. Furthermore let each of w_1 and w_2 be the neighbor of v_1 and v_2 (respectively) distinct from v. See Figure 10.6. As before, we may assume that u_1, w_1 and w_2 (precolored vertices) are distinct. Let T be the tree induced by $\{u_1, u, v, v_1, v_2, w_1, w_2\}$ and let L be a DSG(K_6, M)-list assignment which gives a single color to u_1, w_1 and w_2 and a full list to the other vertices. We show that T admits an L-coloring such that colors of u and v are in different layers. This completes the proof as for any such choice of colors for u and v one may find an extension for w by Observation 10.2.9. Our claim itself is the result of the fact that considering uu_1-branch of T, $L^a(u)$ is a neighbored 5-set and considering only vv_1 and vv_2-branches, $L^a(v)$ contains a paired 8-set. □

Lemma 10.3.5. *There are no adjacent 3_1-vertices who share a common 2-neighbor in (G, σ).*

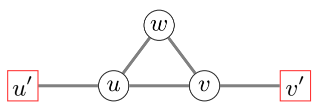


Figure 10.7. Two adjacent 3_1-vertices share a common 2-neighbor.

Proof. Assume to the contrary that u and v are two adjacent 3_1-vertices of G, and w is the common 2-neighbor of them. Let u' be the third neighbor of u and let v' be the third neighbor of v, see Figure 10.7. Thus we have a list assignment on the subgraph induced by u', u, w, v, v', where u' and v' are precolored and the other three have a full list. Our claim then follows from Observations 10.2.7 and 10.2.9 as in the proof of the previous lemma. □

Lemma 10.3.6. *There is no 3_0-vertex together with two 3_1-vertices inducing a triangle in (G, σ).*

Proof. Suppose that two adjacent 3_1-vertices u and v share a 3_0-neighbor w. Since G is 2-connected, u and v do not have a common 2-neighbor, moreover, let u' and v' be their 2-neighbors respectively. As w has no 2-neighbor, its third neighbor w' is distinct from u' and v'. Furthermore, we label the other neighbors of u' and v' as u_1 and v_1, respectively, but noting that, as we will consider them to be precolored vertices, they do not need to be distinct from each other or from w' which will also be precolored. Then the set of admissible lists on u, v, w, induced by the coloring of u_1, v_1 and w', would satisfy the conditions of Lemma 10.2.15 (1) with $k = 1$, proving that this configuration is reducible. □

Lemma 10.3.7. *There is no 3_1-vertex having two 3_1-neighbors in (G, σ).*

Proof. Suppose to the contrary that u, v, w are three 3_1-vertices and v is adjacent to both u and w. Let u_1, v_1 and w_1 each be the 2-neighbor of u, v and w respectively. By Lemma 10.3.5, we know v_1 is distinct from u_1 and w_1. Furthermore, u and w are not adjacent, as otherwise we have a sub-configuration of Lemma 10.3.6. Depending on whether u_1 and w_1 are distinct or not, we consider two cases.

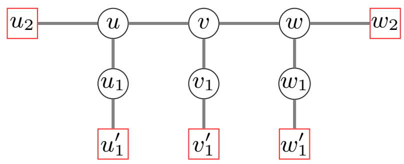


Figure 10.8. Case: $u_1 \neq w_1$

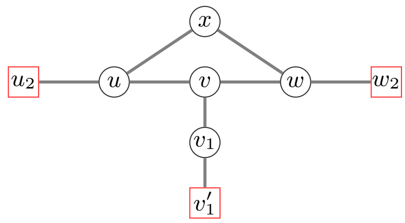


Figure 10.9. Case: $u_1 = w_1$

Case 1: $u_1 \neq w_1$. We use the labeling of vertices near u, v and w as given in Figure 10.8, noting that u_2, u'_1, v'_1, w'_1 and w_2 are distinct from u, v, w, u_1, v_1 and w_1, but they are not necessarily distinct from each other, however as they are precolored, this would not matter in our proof.

We consider the rooted tree at v of Figure 10.8 whose leaves are precolored and at start all colors are available on each of the internal vertices. Then, by Lemma 10.2.5, each of the u-branch and w-branch of the tree forbids at most four colors from $L(v)$, while, by Lemma 10.2.3, the v_1-branch of the tree forbids exactly two colors. Thus there are always at least two admissible colors in $L(v)$.

Case 2: $u_1 = w_1$. We follow the labeling of Figure 10.9 where this common 2-neighbor is relabeled as x.

We note again that vertices u_2, w_2 and v'_1 of this figure are distinct from other vertices of the figure but not necessarily distinct from each other. We first assign a list to each of the vertices where u_2, w_2 and v'_1 are precolored, and other five vertices each have a full list. Then we update the lists of u, w and v according to, respectively, u_2, v'_1 and w_2. In updated lists, each of $L(u)$ and $L(w)$ is a neighbored 5-set, $L(v)$ is a paired 10-set and $L(x)$ is a full set. Thus we may apply Lemma 10.2.14 (1) with $k = 2$. □

Lemma 10.3.8. *Let u be a 3_0-vertex of (G, σ) whose neighbors x, y and z are all 3_1-vertices. If x and y have a common neighbor w, then $d(w) \geqslant 4$.*

Proof. We first observe that, by Lemma 10.3.6, $\{x, y, z\}$ is an independent set. Let w be a common neighbor of x and y. The vertex w is not a 3_1-vertex, as otherwise it would contradict Lemma 10.3.7.

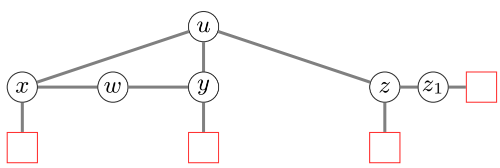


Figure 10.10. Neighborhood of a 3_0-vertex

Now we show that w cannot be a 2-vertex. Suppose to the contrary that w is a 2-vertex, see Figure 10.10. Having colored the rest vertices of (G, σ) except for the 2-neighbor of z, we are left with a list assignment on u, x, y, z, w where each of $L(x)$ and $L(y)$ is a neighbored 5-set, $L(w)$ is a full list, and by applying Lemma 10.2.5 to the uz-branch of the figure, we modify the list of u to a paired 8-set. We then apply Lemma 10.2.14 (3) on the 4-cycle $uxwy$ and we are done.

Finally we show that w cannot be a 3_0-vertex either. Depending on whether w is adjacent to z or not, we have two cases to consider:

Case 1: $w \sim z$. We follow the labelings of Figure 10.11.

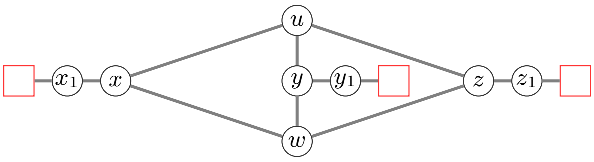


Figure 10.11. w is adjacent to z

Assuming that rest of the signed graph is precolored, to extend the coloring to this part of the graph we have a full list of colors available on u and w, and on each of x, y and z we have a paired 10-set available (equivalently, only a paired 2-set is missing). In what follows, we shall try three possible partial coloring ϕ of u and w, for each choice we either can extend ϕ to the full configuration, or give a condition on the lists for x, y and z. Then we could find fourth assignment to u and w that is extendable.

Our first coloring to consider satisfies that $\phi(u) = 1^+$ and $\phi(w) = 3^+$. This coloring can be extended to x, y and z unless for one of them, say x, one of the following holds: (1) both ux and xw are negative edges and the missing pair on x is $\{5^-, 6^-\}$ or (2) both ux and xw are positive edges and the missing pair on x is $\{5^+, 6^+\}$.

As a second choice, we try the coloring $\phi(u) = 1^+$ and $\phi(w) = 5^+$. Similarly, if this choice of colors is not extendable, for a vertex, say y, either (3) both uy and yw are negative edges and the missing two colors are $\{3^-, 4^-\}$ or (4) both uy and yw are positive edges and the missing pair on y is $\{3^+, 4^+\}$, which, in particular, justifies the choice $y \neq x$.

As a third try, on examining the coloring $\phi(u) = 1^+$ and $\phi(w) = 3^-$, we conclude that for one of the three vertices, say z, either (5) uz is positive and zw is negative with $\{5^+, 6^+\}$ as the missing pair on z or (6) uz is negative and zw is positive with $\{5^-, 6^-\}$ as the missing pair on z. Theses conditions also justify that z is distinct from both x and y.

We now observe that the choice of $\phi(u) = 1^+$ and $\phi(w) = 5^-$ is extendable on all three of x, y, z.

Case 2: $w \nsim z$. We use the labelings of Figure 10.12.

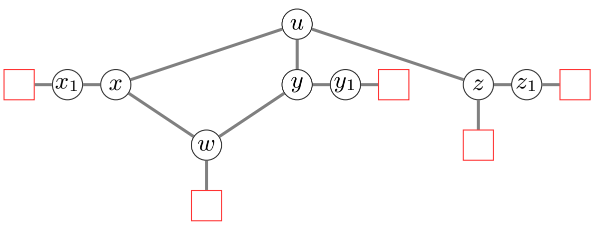


Figure 10.12. w is adjacent to z

Upon forming list of available colors on u, x, w and y using zz_1-branch for u, x_1-branch for x, and y_1-branch for y, by Lemma 10.2.5, the list of u is a paired 8-set. The list of w is a neighbored 5-set and the list of each of x and y is a paired 10-set. In particular, there is one color from each layer available at w. For one such color, say c, there must be three pairs of colors available for u each not in the same layer as c. Let c_1, c_2 and c_3 each be a color from one of these pairs. We may now proceed as in the previous lemma, assigning c to w and c_i to u would be not extendable only if x or y is of a certain type, but there are only two of these vertices and three distinct possibilities. □

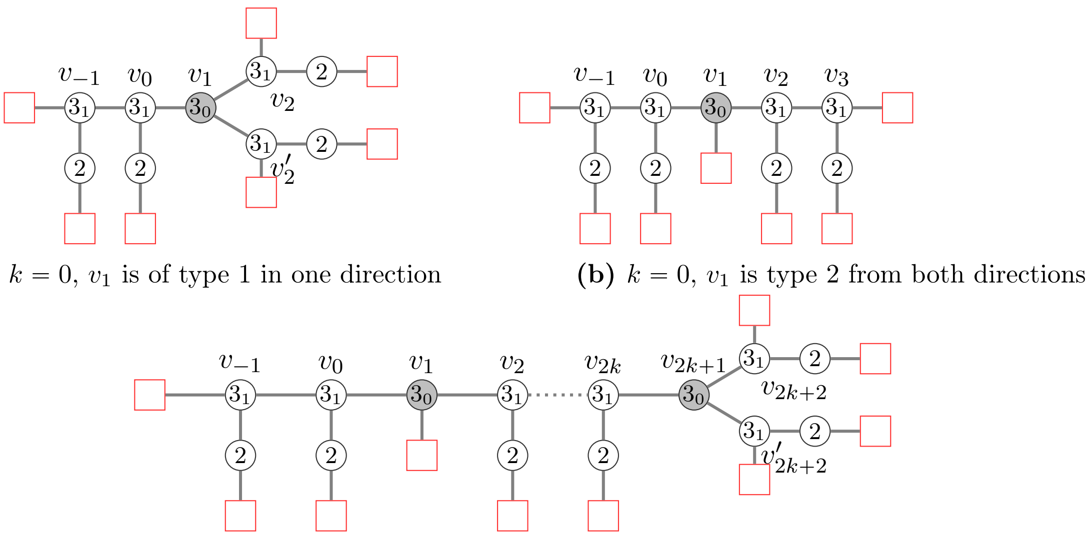
Paths in 3-subgraph

We have so far seen that the minimum counterexample (G, σ) has no 3_2-vertex and no 3_1-vertex seeing two other 3_1-vertices. To complete our proof, we need further information on the subgraph induced by 3-vertices. When applying the discharging technique in the next section, among 3_0-vertices the poorest one would be:

- (type 1) a 3_0-vertex all whose neighbors are 3_1-vertices,
- (type 2) a 3_0-vertex with two 3_1-neighbors one of which has another 3_1-neighbor.

Hence, a path of (G, σ) is said to be *poor* if first of all, its vertices are alternatively of type 3_0 and 3_1, and secondly, the start and end vertices of the path are among the poorest type of 3_0-vertices.

Our goal is to show that (G, σ) does not contain a poor path. To this end, we assume that P is a minimum poor path in (G, σ) whose vertices are labeled $v_1 v_2 \cdots v_{2k+1}$. If the end vertex v_1 is of type 1, then we label its other 3_1-neighbors v_0 and v_0' and if it is of type 2, then its other 3_1-neighbor is labeled v_0 and the other 3_1-neighbor of v_0 is labeled v_{-1}. Vertices v_{2k+2}, v_{2k+2}' and v_{2k+3} are defined similarly. Observe that, by Lemma 10.3.7, vertices v_{-1} and v_{2k+3}, when exist, are two distinct vertices. Thus for $k \geqslant 1$, depending on the types of two ends of the poor path, we have three possible types of poor paths. For $k = 0$, v_1 is viewed as the end vertex from each direction, but as it is a 3-vertex, it cannot be of type 1 from each end, thus we can only have two types of poor paths in this case. In Figure 10.13, both of these two possibilities of poor paths when $k = 0$ are depicted and only one possibility for $k \geqslant 1$ is also presented.



(a) $k = 0$, v_1 is of type 1 in one direction

(b) $k = 0$, v_1 is type 2 from both directions

(c) $v_1 v_2 \cdots v_{2k+1}$ is a poor path

Figure 10.13. Poor paths

Let $I_{3_0} = \{1, 3, \ldots, 2k-1, 2k+1\}$ and $I_{3_1} = \{0, 2, \ldots, 2k, 2k+2\}$. As every 3_1-vertex has a 2-neighbor, for $i \in I_{3_1} \backslash \{0, 2k+2\}$, the vertex v_i is not adjacent to any vertex v_j for $j \in I_{3_0} \cup I_{3_1} \backslash \{i-1, i+1\}$. Moreover, by Lemma 10.3.8, vertices v_0 and v_0', when the latter exists, do not share a 2-neighbor. Furthermore, vertices v_0 and v_0' are not distinguishable in a poor path so we can switch their roles if necessary. We can similarly treat v_{2k+2} and v_{2k+2}'.

Lemma 10.3.9. *Given a minimum poor path $P = v_1v_2\cdots v_{2k+1}$, with labelings defined as above, in the minimum counterexample (G, σ), the vertex v_0 is not adjacent to v_i for $i \in \{2, 3, \ldots, 2k, 2k+1\}$ and the vertex v_{2k+2} is not adjacent to v_j for $j \in \{1, 2, \ldots, 2k\}$. Moreover, $v_0 \neq v_{2k+2}$.*

Proof. We give a proof for v_0, the argument for v_{2k+2} follows by symmetry. It is already mentioned in the paragraph proceeding the lemma that v_0 is not adjacent to v_i for even values of i. Thus, we only need to consider the odd values of i.

In the case of $k = 0$, because v_1 is of degree 3, at least one side is of type 2, thus one of v_0 or v_2 has its degree already full and both of the claims follow.

Next we consider the case $k = 1$. The claim is that v_0 is not adjacent to v_3. By contradiction suppose it is. Since v_0 is a 3_1-vertex, we already have v_1 and v_3 as the neighbors of v_0 which are not 2-vertices. This, in particular, implies that v_3 is of type 1 and that $v_0 = v_4$, and v'_4 is the other 3_1-neighbor of v_3. We may now apply Lemma 10.3.8 with $u = v_3$, and this completes the proof for $k = 1$.

For $k \geqslant 2$, and for the first part of the claim, observe that v_1 must be of type 1, as otherwise v_1 and v_{-1} are the only 3-neighbors of v_0. Assume to the contrary that v_0 is adjacent to v_j for an odd value of j. We now get a contradiction with the minimality of P by considering the shorter poor path: $v_jv_{j+1}\cdots v_{2k+1}$. It remains to show that $v_0 \neq v_{2k+2}$. Again, suppose to the contrary that $v_0 = v_{2k+2}$. Thus v_0, which is a 3_1-vertex, is adjacent to both v_1 and v_{2k+1}, and, therefore, it has no other 3-neighbor. Hence both v_1 and v_{2k+1} are of type 1. But again we get a contradiction to the minimality of P by taking the shorter poor path: $v_1v_0v_{2k+1}$. □

Lemma 10.3.10. *Given a minimum poor path $P = v_1v_2\cdots v_{2k+1}$ in (G, σ), the following statements hold:*

(a) For any $i \in I_{3_1} \cup \{-1\}$ and $j \in I_{3_1} \cup \{2k+3\}$, the vertices v_i and v_j do not have a common 2-neighbor.

(b) For any $i \in I_{3_0} \cup \{-1\}$ and $j \in I_{3_0} \cup \{2k+3\}$, the vertex v_i is not adjacent to the vertex v_j.

(c) The vertex v_0 is not adjacent to v_{2k+2}.

Proof. We prove the first two claims by contradiction. We consider all the pairs i, j for which one of the two statements do not hold. Among all such pairs then we choose one where j is the minimum possible and, based on this condition, i is the maximum possible. Then, depending on which of the statement fails for this pair of i, j, we consider two separate cases.

Case 1. The statement (a) does not hold for i and j. We consider four subcases based on i and j.

- $(i, j) = (-1, 2k+3)$. Thus, in particular, we assume v_{-1} and v_{2k+3} exists and that they are distinct from other vertices, hence, v_0 is not adjacent to v_{2k+2}. Let (H, σ) be the subgraph of (G, σ) induced by the vertices of P, the vertices $v_{-1}, v_0, v_{2k+2}, v_{2k+3}$ and all their 2-neighbors. Let u be the common 2-neighbor of v_{-1} and v_{2k+3}. Observe that, by the maximality of j and the minimality of i, expect for u, every other 2-vertex in (H, σ) sees only one vertex in (H, σ) and that there is no connection between 3_0-vertices of (H, σ). We may then color $(G \backslash H, \sigma)$ by the minimality of (G, σ), and with respect to this partial coloring, consider the list of available colors on the vertices of (H, σ). Observe that if we remove all 2-vertices but u from (H, σ), we have a subgraph (H', σ) which is a signed $(2k+6)$-cycle. Furthermore, by Lemma 10.2.3, the list coloring problem on (H, σ) can be modified to a list coloring problem on (H', σ) where $L'(u) = C$, $L'(v_{-1})$ and $L'(v_{2k+3})$ each is a neighbored 5-set, each $L'(v_i)$, $i = 0, 2, \ldots, 2k+2$, is a paired 10-set and each $L'(v_j)$, $j = 1, 3, \ldots, 2k+1$, is a neighbored 5-set. But then, by Lemma 10.2.14 (1), we do have a coloring of (H', σ) with respect to this list assignment L'.

- $i = -1$, $j \in I_{3_1}$. Let u be the common neighbor of v_{-1} and v_j. Similar to the previous case, we consider the subgraph (H, σ) induced by $v_{-1}, v_0, \ldots, v_j$ and all their 2-neighbors, noting that, by the choice of j and i, each such a 2-neighbor is adjacent to only one vertex in (H, σ) and that no two 3_0-vertices in (H, σ) are adjacent. Thus the subgraph (H', σ) induced by u and 3-vertices of (H, σ) is a $(j + 3)$-cycle. Again, similar to the previous case, a coloring ϕ of $(G \backslash H, \sigma)$ induces a list assignment on (H', σ) which satisfies the conditions of Lemma 10.2.15 (2), therefore, ϕ can be extend to the rest of (G, σ).

- $i = 0$. By symmetry of 1 and $2k + 1$, we may assume $j \in I_{3_1}$. First we note that if $j = 2k + 2$, then v_0 is not adjacent to v_{2k+2} as otherwise we have the forbidden configuration of Lemma 10.3.5. Let u the common 2-neighbor of v_i and v_j. For this case, we consider two subcases based on the type of the vertex v_1.

  If v_1 is of type 1, then we take (H, σ) to be the subgraph induced by $v_0, v_1, v_2, \ldots, v_j$, v_0' and all their 2-neighbors. Let ϕ be a coloring of $(G \backslash H, \sigma)$. Let L be the list assignment induced on (H, σ) by the partial coloring ϕ. This L-coloring problem is reduced to an L'-coloring problem of the cycle $v_0 v_1 \cdots v_j u$ where $L'(u) = C$, $L'(v_0)$ and $L'(v_j)$ are neighbored 5-sets, $L'(v_1)$, by Lemma 10.2.5, contains at least one paired 4-set from C^+ and one paired 4-set from C^-, and the rest of $L'(v_k)$ are alternatively neighbored 5-sets and paired 10-sets. Overall this cycle with respect to L' satisfies the conditions of Lemma 10.2.14 (3), and, therefore, the coloring ϕ can be extended to the rest of (G, σ).

  If v_1 is of type 2, then, by similar arguments, the problem is reduced to the L'-coloring of the cycle $v_0 v_1 \cdots v_j u$ where the lists of v_0 and v_1 have changed the roles, with all other remaining the same as before. We may then apply Lemma 10.2.14 (2) to complete the proof.

- $i \in I_{3_1} \backslash \{0\}$. By the symmetry of $i = 0$ and $j = 2k + 2$, we may assume $j \in I_{3_1}$, $j \neq 2k + 2$. As in the previous cases, let (H, σ) be the subgraph induced by $v_0, v_1, \ldots v_j$, one of v_{-1} or v_0' depending on the type of v_1, and all the 2-neighbors of already chosen vertices. Let the common 2-neighbor of v_i and v_j be u and note that all other 2-neighbors of the vertices in (H, σ) are distinct. Furthermore, no pair of 3_0-vertices in (H, σ) are adjacent. Assume that $(G \backslash H, \sigma)$ admits a list-coloring ϕ and let L be the associated list assignment on (H, σ). As before, we reduce the L-coloring problem of (H, σ) to an L'-coloring of the cycle $u v_i v_{i+1} \cdots v_j$ which satisfies the conditions of Lemma 10.2.14 (2). To get L', if v_1 is of type 1 we apply Lemma 10.2.5 to v_1 from two directions after which we have a paired 4-set of colors available at v_1. Then using Lemma 10.2.6 (4), (5), (7) and Lemma 10.2.3, we update the lists of vertices v_l of P with $l \leqslant i$ such that we have, alternatively, lists of size 6 and 3 until v_{i-1}, and $|L'(v_i)| \geqslant 8$. The case when v_1 is of type 2 is quite similar. The only difference is that at the start v_1 would a neighbored 3-set rather than a paired 4-set.

Case 2. The statement (b) does not hold for i and j.

The proof technique is quite similar to the previous case with less subcases to consider, so we only give the general idea. The case of $i = -1, j = 2k + 3$ is not possible by Lemma 10.3.7. In all other subcases, we consider the subgraph (H, σ) induced by vertices $v_0, v_1, v_2, \ldots, v_j$, one of v_{-1} or v_0', and their 2-neighbors. The problem then is reduced to a list coloring problem on (H, σ), but as (H, σ) has a unique cycle, we may further reduce the problem to list coloring of the cycles. However, in all but one of the cases we may apply Lemma 10.2.15 (1). In the exceptional case when v_0 is adjacent to v_{2k+2} and v_1 is adjacent to v_{2k+1}, we consider the 4-cycle $v_0 v_{2k+2} v_{2k+1} v_1$ and let (H, σ) be the subgraph induced by this 4-cycle and the two 2-neighbors of v_0 and v_{2k+2}. Then a coloring of $(G \backslash H, \sigma)$ can be extended to (H, σ) by Lemma 10.2.13.

Finally, we prove the statement (c): v_0 is not adjacent to v_{2k+2}. Assume to the contrary that v_0 is adjacent to v_{2k+2}. Then $v_0v_1 \cdots v_{2k+2}$ is a cycle. In the above arguments, we have shown that first of all there is no chord in this cycle, secondly, for any two 3_1-vertices of the cycle, their 2-neighbors are distinct. As before we consider the signed subgraph (H, σ) induced by the cycle and its 2-neighbors. Then again a coloring of $(G \backslash H, \sigma)$ can be extended to (H, σ) by Lemma 10.2.15 (1). □

Lemma 10.3.11. *There is no poor path in (G, σ).*

Proof. Assume to the contrary that $P = v_1 \cdots v_{2k+1}$ is a minimum poor path in (G, σ). Let (H, σ) be the subgraph of (G, σ) induced by $v_0, v_1, \ldots, v_{2k+2}$, v'_0 or v_{-1} depending on the type of v_0, v'_{2k+2} or v_{2k+3} depending on the type of v_{2k+1} and all of their 2-neighbors. Observe that by Lemma 10.3.9 and Lemma 10.3.10, H is indeed a tree. By the minimality of (G, σ), we have a coloring of $(G \backslash H, \sigma)$ which induces a list assignment L on the signed tree (H, σ). To complete the proof, considering v_1 as the root of this tree, we show that $L^a(v_1) \neq \varnothing$.

If $k = 0$, then H is either the signed graph (a) of Figure 10.13 or the signed graph (b). In either case, to compute the number of colors forbidden on v_0, we apply Lemma 10.2.5 to the v_{-1}-branch and Lemma 10.2.3 to the 2-vertex branch, concluding that $L^f(v_0)$ is contained in a paired 6-set and, therefore, $L^a(v_0)$ contains a paired 6-set. Thus, by Lemma 10.2.6 (7), the v_0-branch will forbid at most a pair of colors from $L(v_1)$. If we have the case (a) of the figure, then each of v_2 and v'_2, by Lemma 10.2.5, will forbid at most two pairs from $L(v_1)$, altogether we have at most five pairs, thus in all case $L^a(v_1)$ contains at least one pair of color. If we have the case (b) of the figure, then by symmetry of v_0 and v_2 we have at most a pair forbidden from $L(v_1)$ by v_2. In this case, $L(v_1)$ was a neighbored 5-set, thus $L^a(v_1)$ still contains at least one element.

For $k \geqslant 1$, depending on the type of v_{2k+1} and just as in the previous case, $L^a(v_{2k+1})$ contains either a paired 4-set or a neighbored 3-set. Then, by Lemma 10.2.3 and Lemma 10.2.6 ((5) or (4)), $L^a(v_{2k})$ contains a paired 6-set, which in turn implies that $L^a(v_{2k-1})$ is a neighbored 3-set. Repeating this process, $L^a(v_2)$ contains a paired 6-set, thus from this branch of the tree at most one pair of colors will be forbidden on v_1. Now if v_1 is of type 1, then the branches corresponding to v_0 and v'_0 each may forbid at most two pairs of colors, and since $L(v_1) = C$, we still have a pair of available colors. If v_1 is of type 2, then the v_0-branch forbids only one pair, and since $L(v_1)$ is a neighbored 5-set, we still have a color available at v_1. □

10.3.2 Discharging Method

Recall that (G, σ) is a minimum counterexample to Theorem 10.1.4. A 3^--*subgraph* of G is a connected component H of the subgraph induced by the set of 3-vertices and 2-vertices of G. Given a 3^--subgraph H, let $n_0(H)$ be the number of vertices of H which are 3_0-vertices in G and let $n_1(H)$ be the number of 3_1-vertices of G that are of degree 3 in H, i.e., the number of vertices in H each of which has one 2-neighbor and two 3-neighbors.

Lemma 10.3.12. *In any 3^--subgraph H of G, $n_0(H) \geqslant n_1(H)$.*

Proof. Our proof is by discharging technique. We assign an initial charge of 1 to those vertices in H that are 3_0-vertices of G and a charge of 0 to all other vertices in H. We will introduce some discharging rules and prove that, upon applying these rules, each vertex in H which is a 3_1-vertex of G receives a total charge of $\frac{1}{2}$ while no 3_0-vertex of G in H loses more than $\frac{1}{2}$. That would prove our claim. The discharging rules we use are as follows.

Rule 1. *Given a 3_0-vertex v_1 of G, assume there exists a unique path $P = v_1 \cdots v_{2k+1}$, $k \geqslant 0$ satisfying that v_i is a 3_0-vertex of G for odd i and v_i is a 3_1-vertex of G for even i. We have the following two sub-rules:*

(*i*) *If* v_{2k+1} *has two other neighbors that are* 3_1*-vertices of* G*, then for* $k \geqslant 1$*,* v_1 *gives a charge of* $\frac{1}{2}$ *to* v_2*.*

(*ii*) *If* v_{2k+1} *has one* 3_1*-neighbor,* v_{2k+2}*, which itself has a* 3_1*-neighbor in* G *(*P *can be seen as one end of a poor path), then for* $k \geqslant 0$*,* v_1 *gives a charge of* $\frac{1}{2}$ *to* v_2*.*

Rule 2. *Each* 3_1*-vertex of* G *which is of degree 3 in* H *and is of charge 0 after Rule 1, receives a charge of* $\frac{1}{4}$ *from each of its* 3_0*-neighbor.*

First observe that a 3_1-vertex x of G which is of degree 3 in H, by Lemma 10.3.7, has at least one 3_0-neighbor say y. If the other 3-neighbor z of x is a 3_1-vertex, then $P = y$ is a path described in Rule 1 (ii) where $k = 0$ and $x = v_2$, moreover, this is unique such a path as any other such a path P' together with P will form a poor path, contradicting Lemma 10.3.11. Therefore, by Rule 1, x will receive a charge of $\frac{1}{2}$ from y. If z is also a 3_0-vertex then either it receives a charge of $\frac{1}{2}$ from one of y or z when applying Rule 1 or, it will receive a charge of $\frac{1}{4}$ from each of them, thus in all cases it will have a final charge of $\frac{1}{2}$.

It remains to show that no 3_0-vertex of G in H will lose more than $\frac{1}{2}$ of its charge. Since G has no poor path, and that Rule 1 can only apply if there is a unique path P, it may only apply in one direction on a given 3_0-vertex. Thus Rule 1, on its own, will take a charge of at most $\frac{1}{2}$ from a 3_0-vertex.

Next we consider a 3_0-vertex u which has three 3_1-neighbors u_1, u_2 and u_3 each of which is a 3-vertex of H. Let u_1', u_2' and u_3' be neighbors of u_1, u_2 u_3, respectively, which are not u and not 2-vertices. Thus each of them has to be a 3_0-vertex of G as otherwise we have a poor path with $k = 0$. First we assume that $u_1'u_1uu_2u_2'$ is a part of a cycle where vertices are alternatively 3_0-vertices and 3_1-vertices of G. We claim that in this case $u_3'u_3u$ is the unique path P satisfying Rule 1. Otherwise, a second path P' starting at u_3' exists. If P' has no common vertex with P, then P and P' together form a poor path, contradicting Lemma 10.3.11. Else, P' must intersect the cycle to reach u, in which case the common part of P' and the cycle form a poor path. Thus u_3 receives a charge of $\frac{1}{2}$ from u_3' by Rule 1. When applying Rule 2, u loses only a total charge of $\frac{1}{2}$. When there is no such a cycle, then each of $u_i'u_iu$, $i = 1, 2, 3$, is a path satisfying Rule 1 with $k = 1$, furthermore, each of them satisfies the condition of being unique, as otherwise we will have a poor path. Thus u will lose no charge in this case.

It remains to show that if a 3_0-vertex u has given a charge of $\frac{1}{2}$ to a 3_1-neighbor u_1 by Rule 1 then u will not lose any charge by Rule 2. Let u_2 be another 3_1-neighbor of u which is a 3-vertex of H. Let u_2' be the other neighbor of u_2 which is not a 2-vertex. We claim that u_2' is a 3_0-vertex. Otherwise, together with the path P (of Rule 1) we have a poor path. Then, by adding u_2' and u_2 to P, we get a unique path satisfying the conditions of Rule 1, therefore u_2' must have given a charge of $\frac{1}{2}$ to u_2 and, hence, u_2 does not take any charge when applying Rule 2. □

We are now ready to prove the Theorem 10.3.1.

Proof. (of Theorem 10.3.1.) By discharging technique, the initial charge assigned to each vertex v is $c(v) = d(v) - \frac{14}{5}$. Since we have assumed that the average degree of G is strictly less than $\frac{14}{5}$, the total charge is a negative value. However, after applying the discharging rule introduced next, we will partition the vertex set so that on each part the sum of final charges is positive. This would be in contradiction with the total charge being negative and complete the proof of the theorem. The discharging rule is as follow:

Rule. *A* 4^+*-vertex gives a charge of* $\frac{2}{5}$ *to each of its* 2*-neighbors and a charge of* $\frac{1}{5}$ *to each of its* 3_1*-neighbors.*

Let $c^*(v)$ be the final charge of the vertex v after the discharging. It is immediate that if $d(v) \geqslant 5$, then $c^*(v) \geqslant \frac{1}{5}$. For a 4-vertex v, it follows from Lemma 10.3.3 and Lemma 10.3.4 that $c^*(v) \geqslant 0$. To complete the proof, we show that the total charges on each connected component of the 3^--subgraph of G is non-negative.

Let H be one such a component. If H has no vertex which is a 2-vertex in G, then all vertices have positive charges. Let v be a 2-vertex of G in H. Observe that if H consists of only v, then both its neighbors are 4^+-vertices and $c^*(v) = 0$. Otherwise, either $c^*(v) = -\frac{2}{5}$ or $c^*(v) = -\frac{4}{5}$. For the former to be the case, one of the neighbors of v must be a 4^+-vertex of G, thus v has a unique neighbor in H. For the latter to be the case, both neighbors of v must be 3-vertices and thus v has two neighbors in H. Let l be the number of 2-vertices of G in H that their final charge is $-\frac{2}{5}$ and let k be the number of 2-vertices of G in H that their final charge is $-\frac{4}{5}$. By Lemma 10.3.3, the neighbors of these 2-vertices are $l + 2k$ distinct 3_1-vertices of G in H. Of these $l + 2k$ vertices in H, suppose that p of them are of degree 3 in H and that the rest are either of degree 2 or 1, the latter being possible only when H is just an edge. Observe that 3-vertices of G with at most two neighbors in H have a third neighbor that is a necessarily 4^+-vertex of G, and, therefore, such vertices have charge at least $\frac{2}{5}$. For the p vertices that are 3_1-vertices of G in H, by Lemma 10.3.12, there must be at least p other vertices in H that are 3_0-vertices of G. As all these vertices have a charge of $\frac{1}{5}$, the over-all charge in a connected component H of the 3^--subgraph of G is non-negative, proving our claim. □

10.4 Mapping sparse signed graphs to (K_8, M)

When the target signed graph is (K_8, M), we show the condition of a maximum average degree strictly smaller than 3 is sufficient for (G, σ) to admit a homomorphism to (K_8, M). Moreover, we will see that this condition is not only the best possible for (K_8, M) but that it cannot be improved for (K_{2k}, M), $k \geqslant 5$, either. As in the previous case, to prove our claim we will work with $\mathrm{DSG}(K_8, M)$. We apply the same technique developed in Section 10.2 without detailed descriptions of $\mathrm{DSG}(K_8, M)$-list assignment L and L-coloring. Note that now C represents the vertex set of $\mathrm{DSG}(K_8, M)$.

Theorem 10.4.1. *Every signed graph (G, σ) satisfying that $\mathrm{mad}(G) < 3$ admits an edge-sign preserving homomorphism to $\mathrm{DSG}(K_8, M)$.*

To prove Theorem 10.4.1, we assume that (G, σ) is a minimum counterexample. Similarly, we first study the properties of a $\mathrm{DSG}(K_8, M)$-list coloring of a signed rooted tree.

Lemma 10.4.2. *Let xy be a signed edge and let L be its $\mathrm{DSG}(K_8, M)$-list assignment. Then the following statements hold:*

(1) If $|L(x)| = 1$, then $F_x(y)$ is a paired set of size 9.

(2) If $L(x)$ contains either a neighbored 5-set or a one-sided 6-set, then $F_x(y)$ is a paired set of size at most 2.

Corollary 10.4.3. *Let (P_3, σ) be a signed path xvy and let L be a $\mathrm{DSG}(K_8, M)$-list assignment of (P_3, σ) with $L(v) = C$, $L(x) = \{c_x\}$ and $L(y) = \{c_y\}$. Then $C \backslash (F_x(v) \cup F_y(v))$ contains two colors which are in different layers unless one of the following conditions holds:*

(1) c_x and c_y are in the same layer but on different sides, and P_3 is a positive path;

(2) c_x and c_y are in the same layer and on the same side, and P_3 is a negative path.

Note that in the two special cases, (P_3, σ) admits no L-coloring. Next we list a set of forbidden configurations in the minimum counterexample (G, σ).

Lemma 10.4.4. *The signed graph (G, σ) does not contain the following vertices: 1-vertex, 2_1-vertex, 3_1-vertex, 4_3-vertex and 5_5-vertex.*

Proof. We only prove the case of a 3_1-vertex, the remaining cases being almost direct corollary of the Lemma 10.4.2. Suppose to the contrary that v is a 3_1-vertex in G. Let u be the 2-neighbor of v, let v_1, v_2 be the other two neighbors of v, and let u_1 be the second neighbor of u, see Figure 10.14.

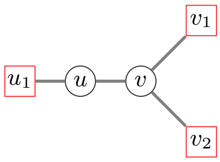


Figure 10.14. A 3_1-vertex with its neighbors.

Let G' be the graph obtained from G by removing u and let H be the subgraph induced by $\{u, v\}$. Assume that L is a DSG(K_8, M)-assignment of (G, σ) where $L(u) = C$ for each vertex $u \in V(G)$. By the minimality of (G, σ), there is L-coloring ϕ of (G', σ). Since $\phi(v)$ exists, the two exceptions of Corollary 10.4.3 cannot be the case here and, therefore, $C \backslash (F_{v_1}(v) \cup F_{v_2}(v))$ contains two colors which are in different layers. Let ϕ' be the restriction of ϕ on $(G \backslash H, \sigma)$ and let L' be an associated list assignment of (H, σ). Now we shall show that (H, σ) is L'-colorable, where $L'(u) = C \backslash F_{u_1}(u)$ is a neighbored 7-set, $L'(v) = C \backslash (F_{v_1}(v) \cup F_{v_2}(v))$. By Lemma 10.4.2 (2), $F_u(v)$ is a paired 2-set, thus $L^a(v) = L'(v) \backslash F_u(v) \neq \varnothing$, a contradiction. □

Now we are ready to prove Theorem 10.4.1.

Proof. (of Theorem 10.4.1.) By discharging method, we assign an initial charge of $c(v) = d(v) - 3$ at each vertex v of (G, σ). Then by the assumption on the average degree, we have $\sum_{v \in V(G)} c(v) < 0$. We apply the following discharging rule:

Rule. *Every 2-vertex receives $\frac{1}{2}$ from each of its two neighbors.*

It is straightforward to check that all vertices have non-negative charges after applying this rule, a contradiction with the fact that the total charge is a negative value. □

10.5 Tightness and planarity

In this section, we first give several examples to show the tightness of our theorems. These examples show that the conditions of the no-homomorphism lemma is not sufficient for mapping signed planar graphs to $(K_{3,3}, M)$, (K_6, M) and (K_8, M). We then apply our results to signed planar graphs with further structural conditions, and propose further direction of study.

The first example, given in Figure 10.15, shows that the bound of $\frac{14}{5}$ in Theorem 10.1.4 is the best possible.

Proposition 10.5.1. *There exists a signed graph with maximum average degree $\frac{14}{5}$ which does not admit a homomorphism to (K_6, M).*

Proof. Suppose to the contrary that the signed (G, σ) of Figure 10.15 admits a homomorphism to (K_6, M) (illustrated in Figure 10.1). Equivalently, there exists a switching-equivalent signature σ' such that (G, σ') admits an edge-sign preserving homomorphism to (K_6, M). Observe that a positive triangle with two negative edges does not admit an edge-sign preserving homomorphism to (K_6, M). Thus considering the triangles uvw and uvx, all their edges must be positive in σ'. Hence, only one of xy or yw is negative. Considering the symmetry of xy and yw, we assume σ' is the signature given in the figure. Without loss of generality, we may assume xy is mapped to the edge 12. Then none of the other three vertices can map to 1 or 2. But then there is no positive triangle induced by $\{3, 4, 5, 6\}$ to map them to. □

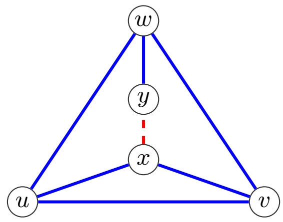


Figure 10.15. $\mathrm{mad}(G, \sigma) = \frac{14}{5}$

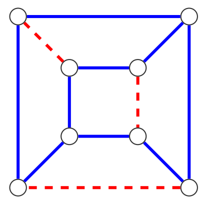


Figure 10.16. A signed cube does not map to $(K_{3,3}, M)$

With regard to mapping signed bipartite planar graphs to $(K_{3,3}, M)$ and $(K_{4,4}, M)$, the existence of a (simple) signed bipartite planar graph all whose mappings to $(K_{4,4}, M)$ are onto, follows from a general construction of [NSS16]. However, in this special case, we have a smaller example of Figure 10.16. In this example, any pair of vertices in the same part of the bipartition belongs to a negative 4-cycle, and thus identifying any such pair would create a negative 2-cycle. Hence, any mapping of this signed bipartite graph to $(K_{4,4}, M)$ is onto. Thus it does not map to its subgraph $(K_{3,3}, M)$.

Finally, noting that Theorem 10.4.1 implies that any graph of maximum average degree less than 3 maps to (K_{2k}, M) for $k \geqslant 4$, we show that the conditions of maximum average degree cannot be improved for any value of k. Our examples, depicted in Figure 10.17, are built from a negative cycle on each of whose edge we build a positive triangle.

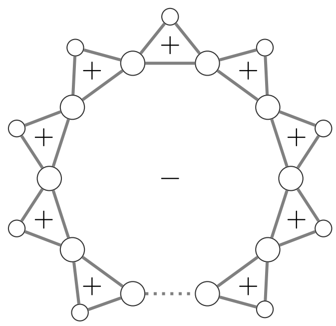


Figure 10.17. A tight example (G, σ)

Proposition 10.5.2. *The signed graph (G_ℓ, σ), built from a negative ℓ-cycle by adding a positive triangle on each edge, does not map to (K_{2k}, M).*

Proof. The proof is based on the fact that (K_{2k}, M) (for any given k) has no triangle with two negative edges. Suppose to the contrary that there exists a homomorphism of (G_ℓ, σ) to (K_{2k}, M), i.e., there exists a switching-equivalent signature σ' and an edge-sign preserving homomorphism of (G_ℓ, σ') to (K_{2k}, M). But then at least one edge of the negative ℓ-cycle is assigned a negative sign by σ' and then the triangle associated with this edge has two negative edges. But there is not such positive triangle in (K_{2k}, M), a contradiction. □

Observe that (G_ℓ, σ) is a signed planar graph that satisfies the conditions of no-homomorphism lemma with respect to (K_{2k}, M). We note that mapping signed bipartite planar graphs to (K_{2k}, M) is equivalent to mapping them to $(K_{k,k}, M)$, and that mapping to the latter is a strengthening of the 4-color theorem as stated in Theorem 2.3.13. Thus we would like to raise the following question:

Question 10.5.3. *For which values of g_{01}, g_{10}, g_{11}, the condition of $g_{ij}(G,\sigma) \geqslant g_{ij}$, $ij \in \mathbb{Z}_2^2$ would imply a mapping of signed planar graph (G,σ) to (K_8, M)?*

Applying Euler's formula to planar graphs, one concludes that any planar graph of girth at least 7 has an average degree strictly less than $\frac{14}{5}$. Since girth condition is a hereditary property, the same holds for the maximum average degree. Thus we have the following result.

Theorem 10.5.4. *Every signed planar graph of girth at least 7 admits a homomorphism to (K_6, M).*

In particular, we have the following:

Corollary 10.5.5. $\chi_c(\mathcal{SP}_7) \leqslant 3$.

We improve an early result of [CNS20] which claims that every signed planar graph of girth at least 9 admits a circular 3-coloring. However, we do not know if 7 is the best possible girth condition.

11 | Conclusion and remarks

In Part IV, we are working on different classes of signed graphs in each of the previous chapters.

Generally, we denote by $\mathcal{P}$ and $\mathcal{SP}$, respectively, the class of all planar graphs and the class of all signed planar graphs, where we allow loops and multi-edges. We denote by $\mathcal{P}_k$ (or $\mathcal{P}_k^*$) the subclass of $\mathcal{P}$ where each graph in $\mathcal{P}_k$ is of girth (or, respectively, odd-girth) at least k. Similarly, the subclass of $\mathcal{SP}$ where the shortest cycle of each member is at least k will be denoted by $\mathcal{SP}_k$. Thus, in particular, $\mathcal{SP}_2$ is the class of all loop-free signed planar graphs and $\mathcal{SP}_3$ is the class of all signed planar simple graphs.

Recall that in Section 2.2 we have mentioned three subclasses of signed graphs of special importance according to their g_{ij} conditions. They are the class $\mathcal{G}_{01} = \{(G,+) \mid G \text{ is a graph}\}$, the class $\mathcal{G}_{10} = \{(G,\sigma) \mid G \text{ is bipartite}\}$, and the class $\mathcal{G}_{11} = \{(G,-) \mid G \text{ is a graph}\}$. When restricted to planar graphs, we have the following classes.

For a given integer k, let $\mathcal{SP}_k^*$ denote the class of signed planar graphs (G,σ) such that the signed graph $(G,-\sigma)$ satisfies that for each $ij \in \mathbb{Z}_2^2$, we have $g_{ij}(G,-\sigma) \geqslant g_{ij}(C_{-k})$. There are two kinds of classes based on the parity of k:

- For an odd integer $k = 2\ell+1$, and after some suitable switching, $\mathcal{SP}_k^*$ consists of all signed planar graphs of odd-girth at least k with all edges being assigned positive signs, i.e., $\mathcal{P}_{2\ell+1}^*$. It is a subclass of $\mathcal{G}_{01}$.

- For an even value of $k = 2\ell$, the class $\mathcal{SP}_k^*$ consists of all signed planar bipartite graphs of negative-girth at least k, i.e., $\mathcal{SBP}_{2\ell}$. It is a subclass of $\mathcal{G}_{10}$.

We note that $\mathcal{SP}_k^*$ is not a subclass of $\mathcal{SP}_k$ as signed graphs in $\mathcal{SP}_k^*$ may have positive even cycles of any length while signed graphs in $\mathcal{SP}_k$ have no short cycles of length less than k. However, it is expected that the circular chromatic number of $\mathcal{SP}_k^*$ is determined by the subclass $\mathcal{SP}_k^* \cap \mathcal{SP}_k$.

The questions of determining $\chi_c(\mathcal{SP}_k^*)$ is closely related to some of the most well known theorems and conjectures in the theory of graph coloring, such as the 4-color theorem, Grötzsch's theorem, and Jaeger-Zhang conjecture and its bipartite analogue. For $k = 3$ and 4, both answers are 4, the first by the 4-color theorem, the second by the observation that 4 is the upper bound for the class of signed bipartite simple graphs in Chapter 7 showing that 4 cannot be improved. For $k = 5$, we have the Grötzsch theorem, that gives upper bound of 3 which is also shown to be the optimal value. For $k = 6$, as shown in Chapter 9, using the 4-color theorem, $\frac{14}{5} \leqslant \chi_c(\mathcal{SP}_6^*) \leqslant 3$. But we still do not know what is the exact value for $\chi_c(\mathcal{SP}_6^*)$. For $k = 8$, we study the edge-density of C_{-4}-critical signed graphs in Chapter 8. The result implies that $\chi_c(\mathcal{SP}_8^*) = \frac{8}{3}$ and the tightness of this bound is verified by a T_2-construction of a sequence of signed bipartite graphs of circular chromatic number approaching 4. Using the notion of flows of signed graphs, developed in Chapter 5, we have recently proved that $\chi_c(\mathcal{SP}_{14}^*) \leqslant \frac{12}{5}$ and $\chi_c(\mathcal{SP}_{20}^*) \leqslant \frac{16}{7}$ in [LSWW22]. For larger and more general k, we have the widely studied Jaeger-Zhang conjecture for $k = 4p+1$ and its bipartite analogue for $k = 4p$, stated in Conjectures 11.0.1 and 11.0.2. In supporting Conjecture 11.0.2, we prove in Theorem 6.3.9 that the condition of negative-girth being at least $6p-2$ is sufficient for signed bipartite planar graphs to admit a circular $\frac{4p}{2p-1}$-coloring.

This also leads to the importance of the question of determining $\chi_c(\mathcal{SP}_k)$. The circular chromatic number bound for the class of signed planar simple graphs is discussed in Section 4.4 of Chapter 4, we show that $\frac{14}{3} \leqslant \chi_c(\mathcal{SP}_3) \leqslant 6$ and the precise value is still an open problem. Also, in Chapter 10, our result of mapping sparse signed graphs to (K_6, M) implies that $\chi_c(\mathcal{SP}_7) \leqslant 3$ but we do not know whether this bound is tight or not. In a recent work [LSWW22], we show that $\chi_c(\mathcal{SP}_{10}) \leqslant \frac{8}{3}$. Following from some earlier flow results in Chapter 6, we provide some bounds on $\chi_c(\mathcal{SP}_{6p+i})$ for $i \in \{1, 2, \dots, 6\}$ in Theorem 6.3.5. In the next table, we summarize the best known results for these questions for some value of k.

Circular chromatic number of $\mathcal{SP}^*_k$ and $\mathcal{SP}_k$

k	Bounds on $\chi_c(\mathcal{SP}^*_k)$	Reference	Bounds on $\chi_c(\mathcal{SP}_k)$	Reference
2	$\chi_c(\mathcal{SP}^*_2) = 4$	Prop. 3.1.7	$\chi_c(\mathcal{SP}_2) = 8$	the 4-color thm
3	$\chi_c(\mathcal{SP}^*_3) = 4$	the 4-color thm	$\frac{14}{3} \leqslant \chi_c(\mathcal{SP}_3) \leqslant 6$	Thm. 4.4.1
4	$\chi_c(\mathcal{SP}^*_4) \cong 4$	Thm. 7.2.1	$\chi_c(\mathcal{SP}_4) \leqslant 4$	[MRŠ16]
5	$\chi_c(\mathcal{SP}^*_5) = 3$	[Grö58], [SY89]	$*$	
6	$\frac{14}{5} \leqslant \chi_c(\mathcal{SP}^*_6) \leqslant 3$	Thm. 9.2.4	$*$	
7	$*$		$\chi_c(\mathcal{SP}_7) \leqslant 3$	Cor. 10.5.5
8	$\chi_c(\mathcal{SP}^*_8) \cong \frac{8}{3}$	Thm. 8.5.3	$*$	
10	$*$		$\chi_c(\mathcal{SP}_{10}) \leqslant \frac{8}{3}$	[LSWW22]
11	$\chi_c(\mathcal{SP}^*_{11}) \leqslant \frac{5}{2}$	[DP17], [CL20]	$*$	
14	$\chi_c(\mathcal{SP}^*_{14}) \leqslant \frac{12}{5}$	[LSWW22]	$*$	
17	$\chi_c(\mathcal{SP}^*_{17}) \leqslant \frac{7}{3}$	[CL20], [PS22]	$*$	
20	$\chi_c(\mathcal{SP}^*_{20}) \leqslant \frac{16}{7}$	[LSWW22]	$*$	
...	...	...	...	
$6p-2$	$\chi_c(\mathcal{SP}^*_{6p-2}) \leqslant \frac{4p}{2p-1}$	Thm. 6.3.9	$\chi_c(\mathcal{SP}_{6p-2}) \leqslant \frac{8p-2}{4p-3}$	Thm. 6.3.5
$6p-1$	$\chi_c(\mathcal{SP}^*_{6p-1}) \leqslant \frac{4p}{2p-1}$	[LWZ20]	$\chi_c(\mathcal{SP}_{6p-1}) \leqslant \frac{4p}{2p-1}$	Thm. 6.3.5
$6p$	$*$		$*$	
$6p+1$	$\chi_c(\mathcal{SP}^*_{6p+1}) \leqslant \frac{2p+1}{p}$	[LTWZ13]	$\chi_c(\mathcal{SP}_{6p+1}) \leqslant \frac{8p+2}{4p-1}$	Thm. 6.3.5
$6p+2$	$*$		$\chi_c(\mathcal{SP}_{6p+2}) \leqslant \frac{2p+1}{p}$	Thm. 6.3.5

In this table, when we write $\chi_c(\mathcal{C}) = r$, it means that $\chi_c(\hat{G}) \leqslant r$ for each member $\hat{G}$ of the class $\mathcal{C}$ and that the equality is known to hold for at least one member of the class. When we write $\chi_c(\mathcal{C}) \cong r$, we mean that $\chi_c(\mathcal{C}) \leqslant r$ and there is a sequence of signed graphs of $\mathcal{C}$ whose limit of the circular chromatic number is r. In such cases, sometimes it is verified that the r is never reached by a single member of $\mathcal{C}$. For example, this is indeed the case for the class $\mathcal{SP}^*_4$. In other cases, it is not known if the equality holds for some members or the inequality is always strict. In particular, for $\mathcal{P}^*_8$ the sequence that gives the limit of $\frac{8}{3}$ is $\{T^*_2(\Gamma_i)\}$ where Γ_i is the sequence reaching the limit of 4 for $\mathcal{SP}^*_4$. It remains an open problem whether the equality can be reached in this case.

There are some trivial inclusions among the classes considered here: $\mathcal{SP}^*_{k+2} \subseteq \mathcal{SP}^*_k$ and $\mathcal{SP}_{k+1} \subseteq \mathcal{SP}_k$. In such cases, any upper bound for the larger class works also on the smaller one and any lower bound for the smaller one works on the larger one as well. In the entries of the table where we write "$*$" the best known bounds come from the other entries of the table based on these inclusions.

To tighten the gap in the bounds or, more ambitiously, to determine the exact values, is the subject of some of main work in the theory of coloring planar graphs. A notable conjecture is that of Jaeger-Zhang which can be restated as:

Conjecture 11.0.1. [Jaeger-Zhang conjecture] *Given a positive integer p, we have that*

$$\chi_c(\mathcal{SP}^*_{4p+1}) = \frac{2p+1}{p}.$$

A bipartite analogue of this conjecture was first proposed in [NRS15], but considering Theorems 8.5.1 and 8.5.2, based on this work, we propose the following modification.

Conjecture 11.0.2. [Bipartite analogue of Jaeger-Zhang conjecture] *Given a positive integer p, we have that*

$$\chi_c(\mathcal{SP}^*_{4p}) = \frac{4p}{2p-1}.$$

Note that the value of Conjecture 11.0.2 seems to what one may expect for $\chi_c(\mathcal{SP}^*_{4p-1})$ and that of Conjecture 11.0.1 seems to be what one may expect for $\chi_c(\mathcal{SP}^*_{4p+2})$ as well.

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