Combinatorial games

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1 What is a combinatorial game?

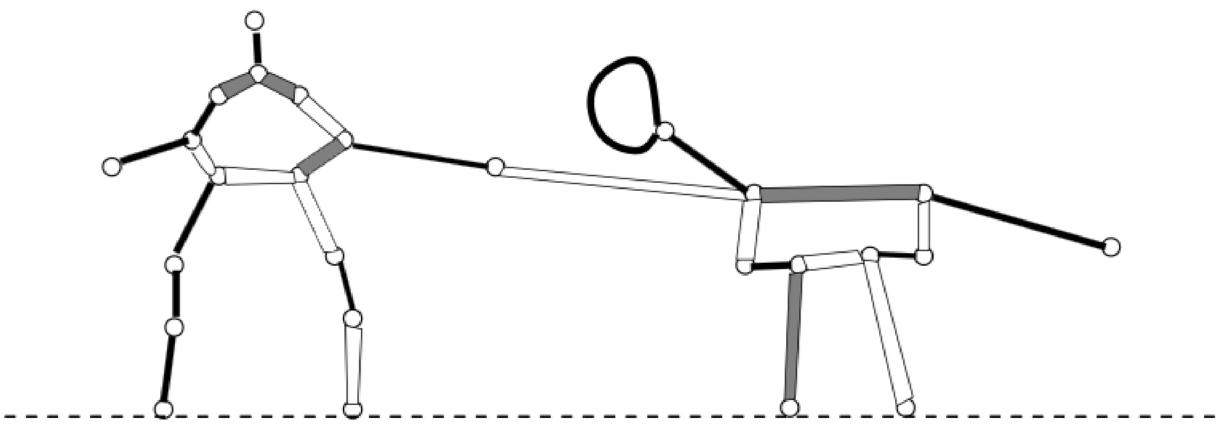
Let us consider two examples:

1.1 Hex

You can find a wealth of useful information about Hex at
`http://en.wikipedia.org/wiki/Hex_(board_game)`
Using a search engine you can even find java programs implementing hex.

1.2 Hackenbush

Hackenbush game is played on pictures composed of black/grey/white edges as below



Two players, Black (Left) and White (Right) play by alternately erasing edges from the picture, Player Black can erase any Black or any Grey edge, player White can erase any White or any Grey edge. At each round only one edge is erased by the acting player. Moreover, all edges that become not connected to the ground are immediately erased as well. If the current player has no edge to erase then he loses.

1.3 Properties of combinatorial games

1. there are two players, often called Left and Right,
2. there are several (in these notes usually finitely many) positions with a fixed start position,
3. for each position and for each player there is a set of available moves, a move specifies the next position in the game (the set of moves can be empty),
4. Left and Right move alternately,

5. both players have complete information,
6. no chance moves,
7. in the normal game, the player unable to move loses,
8. no infinite sequence of moves is possible.

1.3.1 Game as a directed acyclic graph

Vertices = positions.
Arcs are either black (Left player moves) or white (Right player moves). Sometimes we can have grey arcs the represent moves available indifferently to both players.
If the graph is infinite we assume also that there is no infinite path[1].

1.3.2 Game as a tree (1)

Each vertex labelled by a position.
For each vertex there are two types of outgoing edges:

- Left (Black) outgoing edges represent the possible moves of Left player,
- Right (White) outgoing edges represent the possible moves of Right player.

There is no infinite directed path in the tree (but the tree can have infinitely many vertices).

1.3.3 Game as a tree (2)

Vertices labelled by pairs (P, L) or (P, R) where L and R indicates the player that should move and P is the current position. Arcs (non colored this time) indicate available moves.
Vertices (P, L) with no outgoing arcs are losing for player Left, vertices (P, R) with no outgoing arc are losing for player Right.

Theorem 1. *One of the players has always a winning strategy.*

Proof. Backward induction starting from the leafs. □

2 Impartial games

A combinatorial game is said to be *impartial* if for each position both players have the same set of available moves (the same sets of options).
A game which is not impartial is called *partisan*.
An impartial game can be represented as a directed acyclic graph $G = (V, E)$, where the set V of vertices is the set of positions and for each position (vertex) $v \in V$ the arcs outgoing from v represent the moves available at v. Such a graph G represents a combinatorial game if G does not contain infinite paths. Note however that we do not assume that G is finite.

3 Wining and losing positions

When dealing with impartial games we shall speak about the first and the second player. The first player is always the player that moves at the current position, in other words when we refer to the first player we always refer to the current player. And the second player is his adversary.
Let us note that if we have an impartial game $G = (V, E)$ then the set of positions V can be partitioned into two sets W and L, where W is the set of winning positions for the first player while L is the set of losing positions for the first player. These sets satisfy the following conditions:

- each position with no available move belongs to L (the first player loses immediately),

[1] This excludes for example the infinite graph with vertices $0, 1, \ldots$ and arcs $(i, i+1), i = 0, 1, \ldots$, which is acyclic but has an infinite path.

- if v is a position such that all moves (v, w) available at v lead to a winning position $w \in W$ then v belongs to L,
- if v is a position such that the first player has a move (v, w) leading to a losing position $w \in L$ then v belongs to W.

If the game starts at a winning position $v \in W$ then the first player has an obvious strategy which consists in taking always moves (v, w) leading to a losing position $w \in L$.
If $v \in L$ is a losing position that either the first player loses immediately if there is no move available at v or each of his moves leads to a winning position $w \in W$ from which his adversary wins using the strategy described above.
In these notes we shall make an additional assumption that for each position the set of available moves (outgoing arcs) is finite. It is possible to develop the theory of impartial games without this restriction if we are prepared to work with ordinals rather than non-negative integers.
The classical Nim game plays a particularly important role in the theory of impartial games.

4 Nim

Nim is a game played with a (finite) number of stacks of chips. A legal move is to strictly decrease the number of chips in one stack (and throw away the removed chips).
The Nim-sum $a \oplus b$ of two natural numbers a and b is obtained by writing them down in binary and adding them without carrying[2] .
Example:

$$5 \oplus 9 = (101)_2 \oplus (1001)_2 = (1100)_2 = 12$$

Obviously $(\mathbb{N}, \oplus)$ is an abelian group, where $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the set of non-negative integers.

Theorem 2 (Bouton 1902)**.** *Suppose that we have p stacks of size $n_1, \ldots, n_p$ respectively. Then position $(n_1, \ldots, n_p)$ is winning for the current player in Nim iff $n_1 \oplus \ldots \oplus n_p \neq 0$.*

Proof. Let

$$V = \{(n_1, \ldots, n_p) \mid \forall i, n_i \geq 0\}$$

be the set of all positions,

$$W = \{(n_1, \ldots, n_p) \mid n_1 \oplus \ldots \oplus n_p \neq 0\}$$

the set of winning positions (for the current player),

$$L = \{(n_1, \ldots, n_p) \mid n_1 \oplus \ldots \oplus n_p = 0\}$$

the set of losing positions (losing for the current player).
Note that the only terminal position $(0, \ldots, 0)$ in Nim, with all stacks empty, belongs to L. Therefore to prove that the elements of W and L are effectively winning and losing positions respectively it is sufficient to show that

(A) for each position in W there exists a move to a position in L (the player playing in such a position can force the game to a losing position),

(B) for each non-terminal position in L each available move leads to W (player playing in such a position cannot avoid a "bad" move). .

The second assertion is easy.
Let

$$v = (n_1, \ldots, n_{i-1}, n_i, n_{i+1}, \ldots, n_p) \in L,$$

and

$$w = (n_1, \ldots, n_{i-1}, m_i, n_{i+1}, \ldots, n_p)$$

[2] In other words, this is just bit-wise XOR of binary representations of a and b.

with $n_i > m_i \geq 0$, i.e. (v, w) is a valid move from a losing position v decreasing the i-th stack. Let us note

$$k = n_1 \oplus \ldots \oplus n_{i-1} \oplus n_{i+1} \oplus \ldots \oplus \oplus n_p$$

the Nim-sum of all stacks except the i-th one.
Then, by definition of L,

$$k \oplus n_i = 0,$$

which is possible if and only if $k = n_i$. But then $k \neq m_i$ and

$$n_1 \oplus \ldots \oplus n_{i-1} \oplus m_i \oplus n_{i+1} \oplus \ldots n_p = k \oplus m_i \neq 0$$

i.e. $w \in W$. This ends the proof of (B).
To prove (A) let us suppose that

$$w = (n_1, \ldots, n_p) \in W$$

i.e.

$$n_1 \oplus \ldots \oplus n_p \neq 0$$

Let

$$a = n_1 \oplus \ldots \oplus n_p.$$

For a non-negative integer x we shall note by $(x)_i$ the i-th bit in the binary development of x.
Let $l = \max\{i \mid (a)_i = 1\}$ be the most significant bit of a equal to 1.
Since $(a)_l = 1$ at least one of the numbers n_i has the l-th bit 1. This implies that $n_i \oplus a < n_i$ since

- the numbers n_i and $n_i \oplus a$ have the same bits on positions greater l-th (because all bits of a at positions higher than l-th are 0),
- the l-th bit of $n_i \oplus a$ is 0 $(1 \oplus 1 = 0)$
- while the l-th bit of n_i is 1.

Therefore the action that the player should take at v consists in reducing the size n_i of the i-th stack to $m_i = n_i \oplus a$.
Note that

$$n_1 \oplus \ldots \oplus m_i \oplus \ldots \oplus n_p = n_1 \oplus \ldots \oplus (n_i \oplus a) \oplus \ldots \oplus n_p =$$
$$n_1 \oplus \ldots \oplus n_i \oplus \ldots \oplus n_p \oplus a = a \oplus a = 0$$

i.e. the new position $(n_1, \ldots, m_i, \ldots, n_p)$ attained by this move belongs to L. □

5 Examples

Several games can be reduced to Nim.

Poker Nim

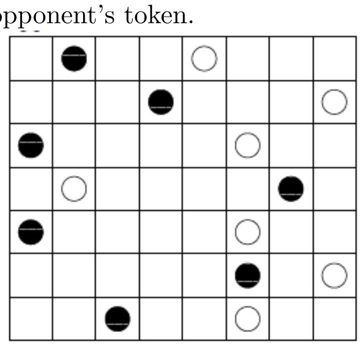
Similarly to Nim this game is played with stacks of chips. But now each player has also a private stack of chips (initially the private stack of each player can contain some finite number of chips). There are two possible valid moves:

- the player may remove any (positive) number of chips from one of the stacks like in Nim or
- the player the can add any (positive) number of chips from his private stack to one of the stacks.

This is not a combinatorial game since the corresponding game graph can have cycles.
However, it turns out that one of the players has a winning strategy assuring that the game ends after a finite number of steps with the other player unable to move (i.e. at the end it is the other player turn to move but all stacks as well as the private stack of the other player are empty).
This game can be reduced to Nim, in fact the winning and losing positions are determined exactly in the same way as in as in Nim (and the chips in the private stacks do not count when we determine winning/losing positions!).
So the only question that the winning player should answer is what to do if the adversary tries to distract him by increasing the number of chips in some stack, what is the appropriate counter-move in this case? (**Answer this question as an exercise!**).

Northcott's game

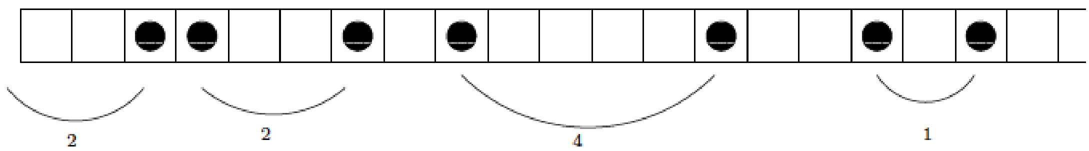
Played on the checkerboard. One white and one black token in each row. You may move a token of your color to any other empty square in the same row provided that you do not jump over your opponent's token.



If you cannot move you lose. Again this game can be easily reduced Nim (**exercise**).

A silver dollar game

Played on a right-infinite tape with a finite number of coins placed on squares (each coin on a separate square).



A legal move is to move a coin leftwards on a free square (there should be always at most one coin at a square) without passing over any other coin. The game ends when there is no legal move, i.e. when all coins are in a traffic jam at the left end of the tape.
Again this is Nim in disguise. Starting from the rightmost coin calculate the number of squares in alternate spaces between coins, take this as the size of the nim stacks. In the picture above we have 4 nim stacks of sizes $2, 2, 4, 1$.
A position is winning if and only if the corresponding configuration of nim stacks is winning.
Exercise Find the strategy for the winning player (again the crucial point is what to do if the adversary increases the size of a stack by moving the coin on the left border of the stack).

Misère Nim

What happens if we change the winning rule in the nim game from the normal one to the misère one, i.e. we declare that the player who moves *the last* loses?
Hint: Consider the following cases:
case 1: Who has a winning strategy if all stacks are of size 1? (Hint: The answer depends on the parity of the number of pils.)

case 2: Consider now the case when there is exactly one stack of size greater than 1 (and all the other stacks are of size 1).
Show that such a position is a winning position in the misère Nim (winning for the player moving in this position). What is the winning strategy for the first player in this case?
case 3: Show that if there is more than one stack of size greater than 1 then such a configuration is winning in the misère Nim if and only if it is winning in the normal Nim.

That the misère Nim can be solved relatively easily is an exception, usually misère games are much more difficult than the normal games.

6 Sprague-Grundy theory of impartial games

Grundy-Sprague theory of impartial games extends the theory of Nim developed in the previous section.
We shall code an impartial game as a couple (X, γ), where X is a set of positions and

$$\gamma : X \longrightarrow 2^X$$

is a mapping that for each position $x \in X$ gives a set $\gamma(x) \subset X$ of positions that are reachable from x in one step. As usually we assume that no infinite sequence of moves is possible.
Moreover, to avoid to tackle with ordinals we assume that for each $x \in X$, $\gamma(x)$ is finite. From the Koenig lemma this implies that X is finite as well (i.e. we deal only with short games in the terminology of Conway).
The *Grundy mapping*

$$G : X \longrightarrow \mathbb{N}$$

of an impartial game (X, γ) maps positions to non-negative integers ($\mathbb{N} = \{0, 1, 2, \ldots\}$) and is defined recursively as follows:
for each position $x \in X$, $G(x)$ is the minimal non-negative integer not appearing in the set $\{G(y)|y \in \gamma(x)\}$ of Grundy values of all successor positions.
For a proper subset X of $\mathbb{N}$ we define mex operation (mex is an abbreviation for minimal excluded)

$$\mathrm{mex}(X) = \min(\mathbb{N} \setminus X)$$

to denote the minimal non-negative integer not belonging to X.
Thus

$$G(x) = \mathrm{mex}\{G(y)|y \in \gamma(x)\} \tag{1}$$

Let us note that for each position x such that $\gamma(x)$ is empty, i.e. each terminal position $G(x) = 0$.

Theorem 3. *If G is the Grundy mapping of an impartial game then*

$$W = \{x \in X | G(x) > 0\}$$

is the set of winning positions and

$$L = \{x \in X | G(x) = 0\}$$

is the set of losing positions.

Proof. Prove this theorem as an exercise. □

6.1 Sum of impartial games

Let $\Gamma_1 = (X_1, \gamma_1)$ and $\Gamma_2 = (X_2, \gamma_2)$ be two impartial games.
Then the sum $\Gamma_1 + \Gamma_2$ of the games Γ_1 and Γ_2 is the game $(X_1 \times X_2, \gamma)$, where for $(x_1, x_2) \in X_1 \times X_2$

$$\gamma(x_1, x_2) = \{(y_1, x_2)|y_1 \in \gamma_1(x_1)\} \cup \{(x_1, y_2)|y_2 \in \gamma_2(x_2)\}.$$

In other words a valid move in $\Gamma_1 + \Gamma_2$ consists in taking a valid move either in Γ_1 or in Γ_2. In some sense you choose each time in which game you move and the other game is unaffected by such a move.

Since the games $\Gamma_1 + \Gamma_2$ and $\Gamma_2 + \Gamma_1$ are isomorphic through the bijection $(x_1, x_2) \mapsto (x_2, x_1)$ we can identify them and write the position (x_1, x_2) as a formal (commutative) sum $x_1 + x_2$. With this notation the set of positions of the sum game is $X_1 + X_2 = \{x_1 + x_2 \mid x_1 \in X_1 \quad \text{and} \quad x_2 \in X_2\}$ and the successor mapping is given by

$$\gamma(x_1 + x_2) = \{x_1 + y_2 \mid y_2 \in \gamma(x_2)\} \cup \{y_1 + x_2 \mid y_1 \in \gamma(x_1)\}$$

The basic question is if and how the Grundy value of a position $x_1 + x_2$ can be calculated from Grundy values of x_1 and x_2.

Theorem 4. *With the notation as above*

$$G(x_1 + x_2) = G(x_1) \oplus G(x_2),$$

where $G(x_1 + x_2)$ is the Grundy value of the position $(x_1 + x_2)$ in the sum $\Gamma_1 + \Gamma_2$ of Γ_1 and Γ_2.

Let us note that Theorem 4 generalizes Bouton's Theorem 2.
Indeed let (n) be Nim game with just one stack of size n. What is the Grundy value of (n)? By induction it is n.
(Since (0) has no successors $G(0) = 0$, the stack (n) of size n has as possible successors all stacks (k) with $k < n$ thus $G((n)) = \text{mex}\{G((0)), \ldots, G((n-1))\} = \text{mex}\{0, \ldots, n-1\} = n$.)
Since Nim game with the initial position $(n_1, \ldots, n_p)$ is the sum of one-stack games (n_1),...,(n_p) Theorem 4 asserts that $G(n_1, \ldots, n_p) = n_1 \oplus \ldots \oplus n_p$. Thus in Bouton's solution to the Nim game in fact we just calculate the Grundy value of the game.

6.2 Recursive definition of the Nim addition

Let n be the position of the Nim game with just one stack of size n. Obviously $G(0) = 0$ and by induction on n we get that $G(n) = n$ (since from the stack n we can go in one of the stacks $\{0, 1, \ldots, n-1\}$ and mex of this set is n).
Since

$$(n_1, \ldots, n_p) = (n_1) + \ldots + (n_p)$$

where $(n_1, \ldots, n_p)$ is a position of the nim game with p stacks and the sum on the right hand side is the sum of one-stack Nim games, the preceding theorem implies that

$$G(n_1, \ldots, n_p) = n_1 \oplus \cdots \oplus n_p$$

i.e. the Grundy value of the Nim game is the Nim sum of the stack sizes.
These remarks and Theorem 4 imply that for all $a, b \in \mathbb{N}$,

$$a \oplus b = \text{mex}(\{a' \oplus b | a' < a\} \cup \{a \oplus b' | b' < b\}) \tag{2}$$

The direct proof of (**??**) of course the same as the proof of Theorem 4. As an (not so trivial) exercise you can try to prove (**??**) and deduce Theorem 4. Note that (2) gives in fact a recursive definition of the Nim-addition.

7 Take-and-Break games

Nim is just an example of "take-and-break" games[3]. In such games we have a number of stacks of pieces and in one step the current player acts on one of the stacks by

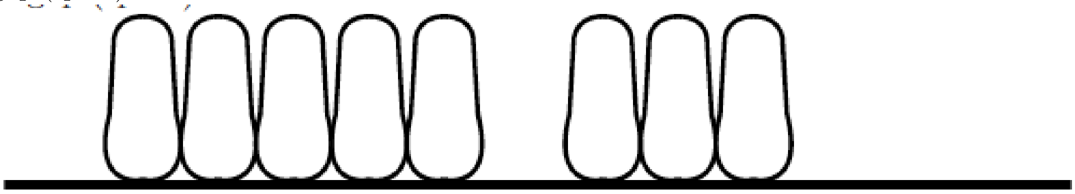
- taking away some pieces and
- breaking the remaining stack onto several smaller stacks.

Another example is given by Keyles.

[3]But there is no "breaking" in nim.

Keyles

There is a row of pins. Two consecutive pins can either touch each other or can be separated by a gap (space).



A player can

- either remove one pin or
- remove two pins that touch each other (he cannot remove two adjacent but not touching pins).

A configuration as on the picture above consisting of 5 touching pins, a gap, 4 touching pins, a gap and 6 touching pins can be coded as $K_5 + K_4 + K_6$. Obviously from the game description it follows that the Grundy mapping satisfies $G(K_5 + K_4 + K_6) = G(K_5) \oplus G(K_4) \oplus G(K_6)$. And, in general, if we can find for each n the value $G(K_n)$, where K_n is a configuration consisting of n adjacent touching pins, then we can calculate the Nim value of any Keyles game through the Nim addition.
Let us try to calculate some values.
$G(K_0) = 0$ trivially.
Since $\gamma(K_1) = \{K_0\}$, we get $G(K_1) = \text{mex}\{G(K_0)\} = 1$.
Now $\gamma(K_2) = \{K_0, K_1\}$ and therefore $G(K_2) = \text{mex}\{0, 1\} = 2$.
And we continue.
$\gamma(K_3) = \{K_1, K_2, K_1 + K_1\}$ where the last option is obtained by removing the pin in the middle and creating a gap between two remaining pins. Thus $G(K_3) = \text{mex}\{1, 2, 1 \oplus 1\} = 3$.
$\gamma(K_4) = \{K_2, K_3, K_1 + K_1, K_1 + K_2\}$ and $G(K_4) = \text{mex}\{2, 3, 1 \oplus 1, 1 \oplus 2\} = \text{mex}\{2, 3, 0\} = 1$.
$\gamma(K_5) = \{K_3, K_4, K_1 + K_3, K_1 + K_2, K_2 + K_2\}$, i.e. $G(K_5) = \text{mex}\{3, 1, 1 \oplus 3, 1 \oplus 2, 2 \oplus 2\} = \text{mex}\{3, 1, 2, 0\} = 4$.
In general,

$$G(K_n) = \text{mex}\{G(K_a) \oplus G(K_b) | a + b = n - 1 \quad \text{or} \quad a + b = n - 2\}.$$

The big question for this and for other "take and break" games is if the infinite Nim-sequence $G(K_0), G(K_1), G(K_2), G(K_3), \ldots$ is ultimately periodic.
It turns out that for Keyles the answer is positive, the last exceptional value is $G(K_{70}) = 6$ and next we have the period 12.
For some games the answer is not known as for example for the Grundy game where a move consists in dividing a stack of chips on two stacks of unequal size. For this game the Grundy values were calculated up to 2^{35} and no periodicity was found (neither it was proved that the sequence of Grundy values is not ultimately periodic).

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