# Games in extensive form - basic concepts and definitions

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## 1 Perfect information games

A perfect information game in extensive form is composed of the following elements:

- a finite set $N = \{1, \ldots, n\}$ of players,
- a non-empty set $V$ of positions with a partition $V = V_0 \cup V_1 \cup \ldots \cup V_n \cup T$, where $V_i$, $i \in N$ is the set of positions controlled by player $i$, $V_0$ is the set of chance positions and $T$ is the set of terminal positions,
- for each non-terminal position $v \in V \setminus T$ a finite non-empty set $A(v)$ of actions available at $v$,
- a transition mapping $\delta$ which for each non-terminal position $v \in V \setminus T$ and each action $a \in A(v)$ gives a position $\delta(v, a) \in V$ reachable from $v$ upon the execution of $a$,
- a mapping $p$ which for each chance position $v \in V_0$ gives a probability distribution over the set $A(v)$ of actions available at $v$. We write $p(a|v)$ to denote the probability of action $a \in A(v)$ at $v \in V_0$ and we add the requirement that $p(a|v)$ is always strictly greater than 0 for $a \in A(v)$ (we can always remove from $A(v)$ the actions for which $p(a|v) = 0$).

  Intuitively player 0 is the nature that chooses an action $a \in A(v)$ with the probability $p(a|v)$.
- a family $u_i, i \in N$ of payoff or utility mappings, $u_i : T \cup \text{Paths}^\omega \longrightarrow \mathbb{R}$, where for each terminal position $v \in T$, $u_i(v)$ gives the payoff of player $i$ if the game terminates at $v$ and for each infinite path $q \in \text{Paths}^\omega$, $u_i(q)$ is the payoff player $i$ receives for an infinite play $q$ ($\text{Paths}^\omega$ is the set of infinite paths starting at the root, a path is a sequence $q = v_1 a_1 v_2 a_2 v_3 \ldots$ such that for all $i$, $v_i \in V$ and $v_{i+1} = \delta(v_i, a_i)$).

We suppose that all this structure satisfies one additional condition. Let

$$E = \{(v, w) \in V \times V \mid v \in V \setminus T \text{ and } w = \delta(v, a) \text{ for an action } a \in V(v)\}$$

then $(V, E)$ is an oriented tree, the root of this tree is the initial game position and terminal positions $T$ are the leafs of the game tree.

**Example 1.** Figure 1 shows a two-player game tree with no chance positions. The initial position is controlled by player 1 who can execute either action $a$ or action $b$. The execution of $b$ leads to a terminal position with payoff 0 for player 1 and payoff 2 for player 2. The execution of action $a$ leads to a position controlled by player 2, where he can execute either action $A$ or action $B$.

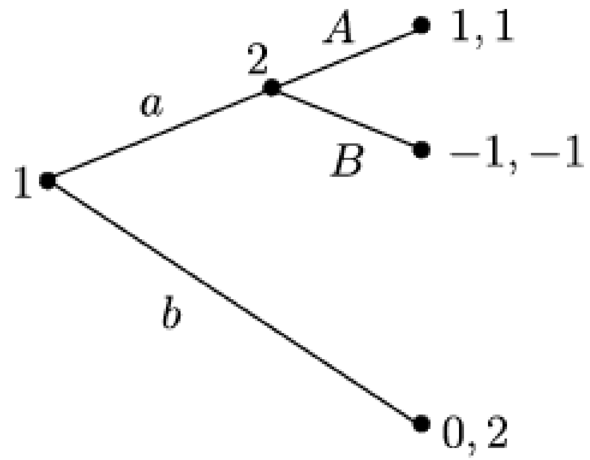


Figure 1: An extensive game two-player game. Vertices are labeled with players' names.

A *pure* strategy $\sigma_i$ for player $i \in N$ is a mapping that for each position $v \in V_i$ gives an action $\sigma_i(v) \in A(v)$ that player $i$ executes at $v$.
A *mixed* strategy for player $i \in N$ is a probability distribution over pure strategies of player $i$.
The intuition behind mixed strategies is that player $i$ uses randomization at the beginning of the game to choose one of his pure strategies and subsequently he plays using the chosen pure strategy.
However as we shall see mixed strategies are useless when we consider finite perfect information games.
There is another type of strategies, so called behavioral strategies that we will study in detail when we will examine game with imprefect information.
A *play* in the game is a path in the game tree from the root to a leaf, i.e. it is a sequence $v_0 a_0 v_1 a_1 \ldots v_k a_k v_{k+1}$ such that $v_0$ is the initial game position (the root of the game tree), $v_{k+1}$ is a terminal position (a leaf in the game tree) and for each $l$, $1 \leq l \leq k$, $v_{i+1} = \delta(v_i, a_i)$.
If the set of positions $V$ is finite then a given profile of pure strategies $\sigma = (\sigma_1, \ldots, \sigma_n)$ yields a probability distribution over terminal positions. Let $t \in T$ be a terminal position and $r$ the initial position of the game. Let $v_0 a_0 v_1 a_1 \ldots v_k a_k v_{k+1}$, $v_0 = r$, $v_{k+1} = t$ be the play from $r$ to $t$. Then the probability $P(t|\sigma)$ that $t$ is reached under

profile $\sigma$ is equal to $P(v_0, a_0) \cdot P(v_1, a_1) \cdots P(v_k, a_k)$, where $P(v_i, a_i) = p(a_i|v_i)$ if $v_i \in V_0$ is a chance position and

$$P(v_i, a_i) = \begin{cases} 1 & \text{if } a_i = \sigma_i(v_i), \\ 0 & \text{otherwise.} \end{cases}$$

Thus $P(t|\sigma)$ is the probability of the unique path from the root to $t$ if the players play using strategies of $\sigma$.
The payoff of player $i \in N$ for a strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ is defined as his payoff expectation:

$$u_i(\sigma) = \sum_{t \in T} u_i(t) \cdot P(t|\sigma). \tag{1}$$

Let us note that if the game tree has no chance positions then for a given strategy profile there is exactly one terminal position reached with probability 1 and the utility of this position gives the utility of each player for the strategy profile $\sigma$.
If we have payoffs for pure strategy profiles then, taking expectation, we can get also the payoff for mixed strategy profiles.

**Example 2.** In the game on Figure 2 the initial game position is the unique chance position. Player 1 has 8 pure strategies: $ace$, $acf$, $ade$, $adf$, $bce$, $bcf$, $bde$, $bdf$.

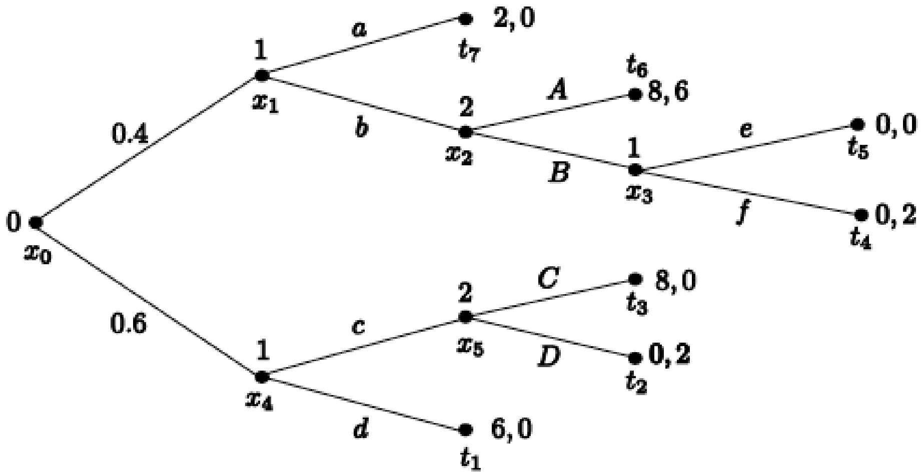


Figure 2: The unique chance position is $x_0$ from which with probability 0.4 the game moves to $x_1$ and with probability 0.6 to $x_4$.

Player 2 has 4 pure strategies: $AC$, $AD$, $BC$, $BD$ (since in this game the actions have unique names there is no need to specify the positions, for each action there is a unique position where this action is available).
A possible mixed strategy for player 1 is $\frac{5}{15}ace + \frac{3}{15}adf + \frac{7}{15}bde$, where the coefficients indicate the probabilities of playing one of the three pure strategies.
For a pure strategy profile $\sigma = (acf, AD)$ the probabilities of terminal positions are $p(t_1|\sigma) = 0$, $p(t_2|\sigma) = 0.6$, $p(t_3|\sigma) = p(t_4|\sigma) = p(t_5|\sigma) = p(t_6|\sigma) = 0$, $p(t_7|\sigma) = 0.4$,

## 1.1 Infinite perfect information games without chance positions

Let $A$ be a finite alphabet, $A^*$ the set of all finite words over $A$, $A^\omega$ the set of all infinite words over $A$, i.e. the set of all infinite sequences $a_1a_2a_3\ldots$ with $a_i \in A$.
We take $A^*$ as the set of positions of a two-player game. For each $u \in A^*$, executing action $a \in A$ takes us to a new position $ua$. The set of terminal nodes is empty. We assume that player I controls positions $\bigcup_{i=0} A^{2i}$ of even length while player II controls positions $\bigcup_{i=0} A^{2i+1}$ of odd length. Thus starting at the position $\varepsilon$ (the empty word) the players choose actions alternately, player I chooses $a_1$, player II $a_2$, player I $a_3$ etc. Let $W \subset A^\omega$ be the set of infinite paths (infinite words) winning for player I, $A^\omega \setminus W$ are infinite paths winning for player II. Thus this is a 0-sum game when player I receives payoff 1 if the resulting infinite path is in $W$ and he receives payoff $-1$ if the resulting infinite path is in $A^\omega \setminus W$.
We can define a metric $d$ on $A^\omega$, for two infinite words $u = a_1a_2\ldots$, $v = b_1b_2\ldots$ we set

$$d(u,v) = \begin{cases} 0 & \text{if } u = v, \\ 2^{-n+1} & \text{if } a_n \neq b_n \text{ and } a_i = b_i \text{ for all } i < n \end{cases}$$

You can try to prove the following simple characterization of open sets: in the topology induced by the metric $d$ a set $U$ is open if and only if there exists a set $X \subseteq A^*$ of finite words such that $U = XA^\omega$ (i.e. $u \in U$ iff $u$ has a finite prefix belonging to $X$).
We say that the game with the winning set $W \subset A^\omega$ is *determined* if one of the players has a winning strategy, i.e. either player I has a strategy that allows him to wing against a the strategy that allows him to win against all strategies of the adversary).

**Theorem 1** (Gale and Stewart, 1953)**.** *If $W$ is open or closed then the game is determined.*

*Proof.* If $W$ is open then we can find the set of winning positions for player I using the attractor strategy.
Without loss of generality we can assume that the open set $W = XA^\omega$ is the winning set for player I and its complement is winning for player II. We assume that $A = \{0,1\}$ contains only two letters (the proof is similar for any finite alphabet).
We define by induction the set $Z \subset A^*$ such that player I has a winning strategy $\sigma_I$ for the games starting at $z \in Z$ (this means that if player I uses this strategy for game starting at $z$ then the resulting infinite word $zw$ belongs to $W$ whatever the strategy of player II).
We begin by setting $Z_0 = XA^*$. Certainly if we start from a position in $Z_0$ then for each possible strategy of player II the resulting infinite word will have a finite prefix in $X$. In particular we can set $\sigma_I(z)$ to be any letter (0 or 1) for $z \in Z_0$.
Suppose that $Z_i$ is already defined and $\sigma_I$ is defined for all even length words in $Z_i$. Then we set

$$\begin{aligned} Z_{i+1} = Z_i \cup \{&u \in A^* \setminus Z_i \mid u \text{ has odd length and both } u0 \text{ and } u1 \text{ belong to } Z_i\} \cup \\ \{&u \in A^* \setminus Z_i \mid u \text{ has even length and either } u0 \text{ or } u1 \text{ belongs to } Z_i\} \end{aligned}$$

We define $\sigma_I$ for even length words in $Z_{i+1} \setminus Z_i$ by setting

$$\sigma_I(u) = \begin{cases} u0 \text{ if } u0 \in Z_i, \\ u1 \text{ otherwise.} \end{cases}$$

Finally we set $Z = \bigcup_{i \geq 0} Z_i$. Clearly if we start at some word $z \in Z$ then $z \in Z_i$ for some $i$ and if player I uses the strategy described above then in at most $i$ steps we will reach the set $Z_0$ and the game will be winned by player I.
We should prove that player II has a winning strategy $\sigma_{II}$ for all games starting at $A^* \setminus Z$.
Let $u \in A^* \setminus Z$ be of odd length. Then either $u0$ or $u1$ does not belong to $Z$. Indeed if both $u0$ and $u1$ belong to $Z$ then there is $i$ such that $u0, u1 \in Z_i$ but this would imply that $u \in Z_{i+1} \subset Z$. We set $\sigma_{II}(u) = u0$ if $u0 \notin Z$ and $\sigma_{II}(u) = u1$ otherwise.
Let $u \in A^* \setminus Z$ be of even length. Then both $u0$ and $u1$ do not belong to $Z$ (otherwise both $u0$ and $u1$ belong to some $Z_i$ and then $u$ would belong to $Z_{i+1}$).
Thus using strategy $\sigma_{II}$ and starting from any word $u \in A^* \setminus Z$ player II will avoid $Z$ forever and therefore he will avoid $Z_0$ forever.
Since the initial game position is the empty word $\varepsilon$, depending on whether $\varepsilon$ belongs to $Z$ or not we can see who wins from the initial position. □

Le $\mathcal{B}$ be the family of Borel sets over $A^\omega$. This is the smallest family containing all open sets and closed under countable unions, countable intersections and complement.
The followong theorem is one of the most powerful results of game theory.

**Theorem 2** (Martin). *If $W \in \mathcal{B}$ then the game is determined.*

Note that not all games are determined:

**Theorem 3** (Gale and Stewart). *There exists $W \subset A^\omega$ such that the game with the winning set $W$ is not determined.*

The proof is inevitably non-constructive, because the sets that we can construct explicitly are Borel whence determined.

# 2 Nash equlibria

Let $N = \{1, \ldots, n\}$ be a finite set of players. For each player $i \in N$ let $\Sigma_i$ be his set of strategies. Fixing a strategy $\sigma_i \in \Sigma_i$ for each player $i$ we obtain a strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$.
In the abstract setting each strategy profile gives rise to a payoff $u_i(\sigma)$ for each player $i$. Intuitively, each strategy profile determines a unique outcome of the game (or, more generally, a probability distribution over outcomes).
Each player has a utility (payoff mapping) $u_i$ from the set of outcomes to $\mathbb{R}$. The role of $u_i$ is to measure the preferences of the player $i$ over outcomes, where for two outcomes $o, o'$, player $i$ prefers the outcome $o$ to $o'$ if $u_i(o) \geq u_i(o')$.
In the abstract setting we forget the outcomes and we define the utility mapping $u_i$ of each player directly as the mapping

$$u_i : \Sigma \times \cdots \times \Sigma_n \to \mathbb{R}$$

from the set of strategy profiles to $\mathbb{R}$.
The most popular solution concept for many player games is the notion of the Nash equlibrium.
A strategy profile $(\sigma_1, \ldots, \sigma_n)$ is a Nash equilibrium if for each player $i \in N$ and each strategy $\sigma'_i \in \Sigma_i$ we have

$$u_i(\sigma_1, \ldots \sigma_{i-1}\sigma_i, \sigma_{i+1}, \ldots \sigma_n) \geq u_i(\sigma_1, \ldots \sigma_{i-1}\sigma'_i, \sigma_{i+1}, \ldots \sigma_n)$$

which means that no player is better off if he changes his strategy unilaterally.

**Exercise 1** (Ice cream vendors)**.** On a sunny day $n$ ice cream vendors choose their positions on a beach. Each vendor attracts all clients that are closer to him than to any of the fellow vendors. If $k$ vendors choose the same point of the beach then they would attract the same clients as one vendor situated at this point, however now they should share these clients equitably. For which values of $n$ there exist Nash equilibria? Describe Nash equilibria in this game. (We are looking here for pure strategy equilibria, thus a strategy of each vendor is to choose deterministically the point $x$ where he puts his stand. Note that there is an infinity of strategies now.)
To present the problem mathematically we suppose that the beach is represented by an interval $[0, a]$ of length $a > 0$. The (potential) clients are distributed uniformly on the beach, i.e. the number of clients on the interval $[c, d] \subset [0, a]$ is proportional to $(d - c)/a$. Suppose that vendors are positioned at points $x_j, j \in N$, where $N$ the set of vendors. Then for each $j$, $0 \leq x_j \leq a$. For the $i$-th vendor let $l_i = \max\{0\} \cup \{x_j \mid j \in N \text{ and } x_j < x_i\}$ be the position of the closest vendor situated strictly on the left to $i$ (or 0 if the is no vendor left to $i$). Let $r_i = \min\{a\} \cup \{x_j \mid j \in N \text{ and } x_j > x_i\}$ be the position of the closest vendor on the right side of $i$ (or $a$ if there is no vendor on the right to $i$). Finally let $k_i = |\{j \in N \mid x_j = x_i\}|$ be the number of vendors sharing exactly the same position as vendor $i$. Then vendor's $i$ market share is $\frac{r_i - l_i}{2k_i}$.

**Example 3.** Figure 1 represents a perfect information extensive game with two players 1 and 2. There are two (pure) Nash equilibria $(b, B)$ and $(a, A)$. However equilibrium $(b, B)$ does not look very convincing. Suppose that player 1 plays rather $a$ than $b$. When the play is at the position controlled by player 2 would he stick to $B$ or rather would he play $A$? The threat of playing $B$ does not look credible.

This motivates the notion of the subgame perfect equlibrium introduced in the following section.

## 2.1 Subgame perfect equilibria

This example motivates the following definition.
A *subgame* of a game is defined as the game starting from some non terminal position $x$.

**Definition 4.** A strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a *subgame perfect equilibrium* in an extensive game $\Gamma$ if $\sigma$ is an equilibrium in each subgame of $\Gamma$.

Note that $(b, B)$ is not a subgame perfect equilibrium in the game examined in Example 3. In each subgame perfect equilibrium player 2 will play $A$ from the position that he controls.

For finite perfect information games we can find subgame perfect equilibria in pure strategies and payoffs of all players by backward induction. Initially the payoff is defined only for terminal positions, for each terminal position $x \in T$ we know that the utility of player $i$ is $u_i(x)$. At each step of the algorithm we extend the set $X$ of position where the payoff and the players' strategies are known by adding to $X$ a new position $v$ such that all positions $\delta(v, a), a \in A(v)$ are already in $X$. The details are given below (argmax $f(x)$ is defined as any $x$ that maximizes a real valued function $f$).

1: $X := T$, $\sigma_i$
2: **while** $X \neq V$ **do**
3:     take $v \in V \setminus X$ such that $\delta(v, a) \in X$ for all $a \in A(v)$
4:     and set $X := X \cup \{v\}$
5:     **if** $v \in V_i$ for some $i \in N$ **then**
6:         $a := \text{argmax}\{u_i(\delta(v, a')) \mid a' \in A(v)\}$
7:         $\sigma_i(v) := a$
8:         **for all** $j \in N$ **do**
9:             $u_j(v) := u_j(\delta(v, a))$
10:         **end for**
11:     **else if** $v \in V_0$ **then**
12:         **for all** $j \in N$ **do**
13:             $u_j(v) := \sum_{a \in A(v)} u_j(\delta(v, a)) \cdot p(a|v)$
14:         **end for**
15:     **end if**
16: **end while**

At each iteration of this algorithm we chose a position $v$ which was not yet treated ($v \in V \setminus X$) but for which all descendants were already examined ($\delta(v, a) \in X$ for all actions $a$ available at $v$), line 3.

If $v$ is controlled by player $i$ then he will choose action $a$ that leads to a successor state $v' = \delta(v, a)$ assuring him the greatest payoff $u_i(v')$, line 6. We assume that in equlibrium player $i$ will play this action $a$ when the game is at $v$ and we set $\sigma_i(v) = a$ where $\sigma_i$ is the strategy of player $i$, lines 6 and 7. The payoffs of all players at $v$ in equilibrium are the same as their payoffs at $v'$, lines 8-1.

If $v$ is controlled by the nature then we calculate the expected payoff of each player at $v$ using transition probabilities of all action available at $v$ and the payoffs of all players in successor positions, lines 12-14.

One step of this algorithm is illustrated in Figure 3. Let us note that using backward induction we find subgame perfect equilibria in pure strategies. Since subgame perfect equilibria are also Nash equilibria we can see that extensive games with perfect information have Nash equilibria in pure strategies.

Subgame perfect equlibria seem to be intuitively more compelling than Nash equilibria. Subgame perfect equilibria allow to select a subset of Nash equilibria, a subset which seems to correspond better to our intuition of how rational players should play. Let us however consider the following game.

**Example 4.** The centipede game (Rosenthal's Centipede Game) is presented in Figure 4. The game is played by Alice and Bob. At the beginning Alice and Bob are endowed with the capital of 2$ each. At the first step Alice chooses between

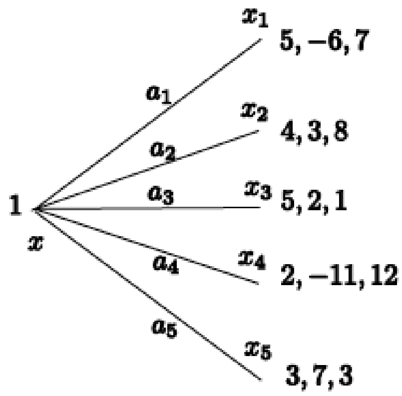


Figure 3: Suppose that $x$ is a position controlled by player 1 in a perfect information game played by three players. Suppose that we have found subgame perfect equilibria in the subgames starting at $x_1$, $x_2$, $x_3$, $x_4$ and $x_5$ and that the three numbers labeling each of these positions give the payoffs of all three players in equilibrium. Clearly the optimal move of player 1 is to play either $a_1$ or $a_3$ or, more generally, he can use any randomization $\alpha a_1 + \beta a_3$, where $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$. In equilibrium this will give him the payoff 5 while the actions $a_2$ and $a_4$ would give him only 4 and 2. If player 1 uses a strategy such that he plays $a_1$ at $x$ then then the equilibrium payoff at $x$ is the same as that at $x_1$.

two actions $S$ and $C$, $S$ means that she steals all money from Bob and the game terminates with Alice having 4$ (her own capital plus stolen money) and Bob having 0$. If Alice chooses $C$ then the game continues and she is awarded with 1$ for her honesty.
At the next step Bob chooses either $S$ and he steals all the money from Alice, the game ends with Bob having 5$ (2$ of his own capital plus 3$ stolen from Alice) and Alice having 0$.
If Bob chooses $C$ then for his honesty he obtains 1$.
And they alternate their moves choosing $S$ or $C$.
The game ends when one of the participants stoles all the money of the other player or when each player has the amount 100$ (when the generous donator awarding each honest move has no more money to distribute).
The conditions of the game are perfectly known to Alice and Bob.
Figure 4 present the game tree of the centipede game.
What is the subgame perfect equilibrium in the centipede game? Does this equlibrium seem to be reasonable as the model of a selfish player who wants to maximize his payoff?

**Example 5.** The bargaining game of alternating offers. Two players bargain over the division of a pie of size $\pi$ (we suppose that the pie is infinitely divisible). The play has a potentially unbounded number of rounds.
At first round player $A$ offers to player $B$ a share $0 \leq x_1^B \leq 1$. If player $B$ accepts then player $A$ receives the share $x_1^A = 1 - x_1^B$ and player $B$ receives the proposed

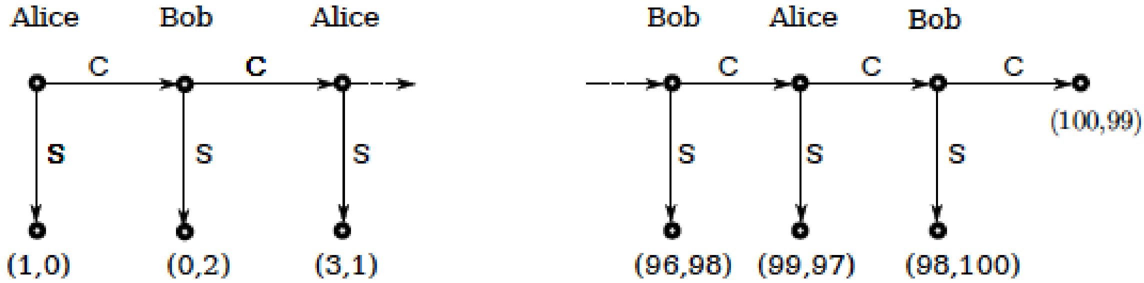


Figure 4: Centipede game tree. Each nonterminal position is labelled with the name of the player playing at this position and with a pair $(a, b)$ where $a$ is Alice's capital and $b$ is Bob's capital at this position. Each terminal position gives the utility for both players.

share $x_1^B$ (the actual payoff is obtained by multiplying the share by the size $\pi$ of the pie).
If player $B$ rejects then with probability $1-\delta_B$ the game terminates with payoff 0 for both players (the pie is declared improper for the consumption and thrown away). With probability $\delta_B$ we pass to the next round where player $B$ makes an offer of $x_2^A$ to player $A$. Either player $A$ accepts and then he receives the proposed share $x_2^A$ while player $B$ receives the remaining part $x_2^B = 1 - x_2^A$ or player $B$ refuses.
If player $A$ refuses then with probability $1 - \delta_A$ the game terminates with payoffs $(0, 0)$ and with probability $\delta_A$ we pass to the next round where player $A$ will make an offer $x_3^B$ to player $B$. This game continues until either one of the players accepts the offer or the game terminates by a chance move. Figure 5 presents first rounds of the game. Note that at $a_1$ we have in fact infinitely many available actions, one action for each $x_1^B \in [0, 1]$. The same remark holds for positions $a_2$ and $a_3$ etc.
We assume that $0 < \delta_A, \delta_B < 1$.
For each $x \in [0, 1]$ there is a Nash equilibrium with payoffs $(x\pi, (1 - x)\pi)$ for both players. Indeed the pair of strategies where player $A$ never offers more than $1 - x$ and never accepts less then $x$ while player $B$ never offers more then $x$ and never accepts less than $1 - x$ is a Nash equilibrium.
However, what we want it to find subgame perfect equilibria.

**Assumption 1.** We assume that at an equilibrium player $A$ makes always the same offer $x_B$ and player $B$ always makes the same offer $x_A$.

**Assumption 2.** Let us suppose also that the payoff of player $A$ for the games starting at positions $a_1, a_3, a_5, \ldots$, i.e. starting at positions where he makes an offer is $u_A^*\pi$. Thus the payoff of player $B$ for the games starting at these positions is at most $\pi - u_A^*\pi$. Symmetrically, let us suppose that the payoff of player $B$ for the games starting at $a_2, a_4, a_6, \ldots$, i.e. starting at positions where he makes an offer is $u_B^*\pi$. Thus the payoff of player $A$ for the games starting at these positions is at most $\pi - u_B^*\pi$, $0 \le u_A^*, u_B^* \le 1$.

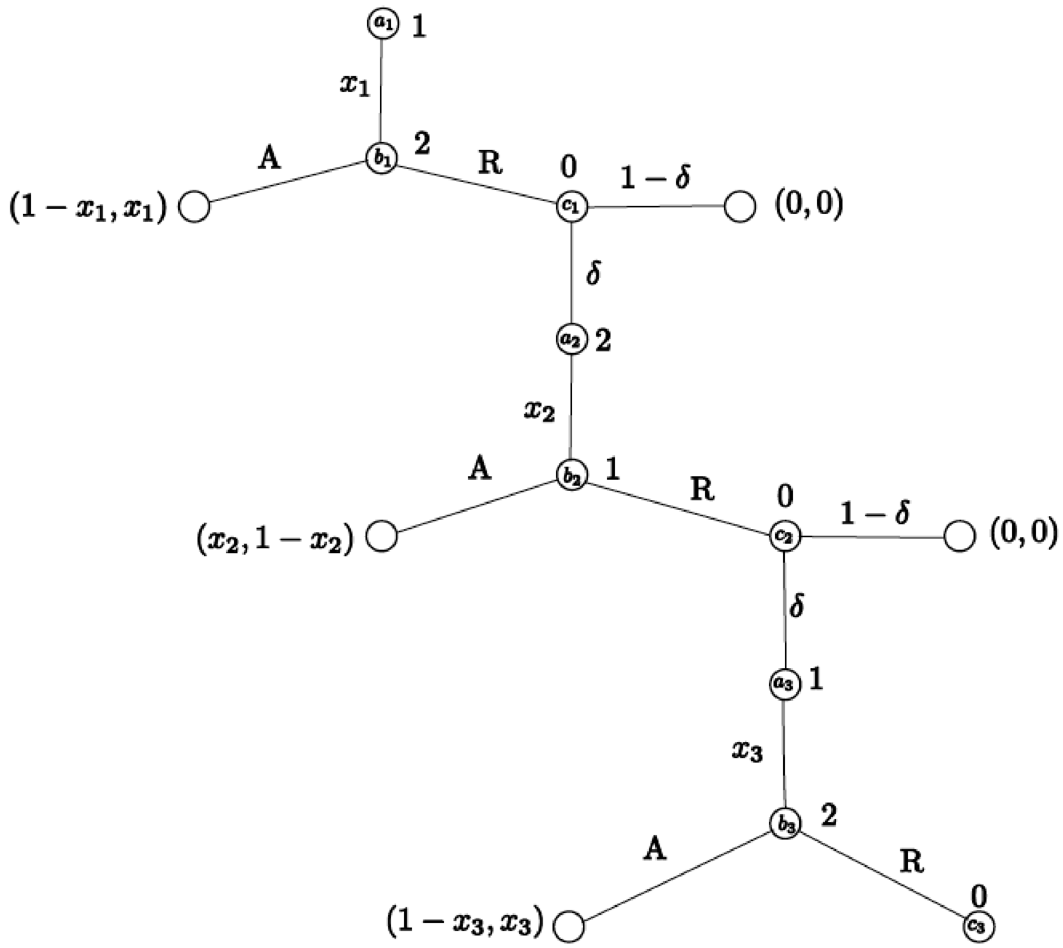


Figure 5: A part of the game tree in the bargaining game

**Assumption 3** Finally we assume that at the player making the decision to accept or reject will always accept if accepting or rejecting gives him the same payoff[1].

Let us consider the moment when player $B$ makes the offer $x_A$ (for example at position $a_2$).
If the offer is accepted then the players obtain respectively $(x_A\pi, (1-x_A)\pi)$.
If the offer is rejected by player $A$ then we go to a position $c_i, i = 2, 4, 6 \ldots$, where the nature stops the game with probability $1-\delta_A$ and payoff $(0,0)$ and with probability $\delta_A$ we go to a position where player $A$ will make an offer and by Assumption 2 the payoff obtained from this position is $(u_A^*\pi, \pi - u_A^*\pi)$. Thus the payoff at position $c_i, i = 2, 4, 6 \ldots$ is $(\delta_A u_A^*\pi, \delta_A\pi(1-u_A^*))$.
The are three possibilities concerning the amount of the offer $x_A$:

(1) $x_A = \delta_A u_A^*$, then by Assumption 3 player $A$ will accept and players get the

[1]Of course if accepting gives him a strictly better payoff than rejecting then he will accept and similarly if rejecting gives him a better payoff than accepting then he will reject. This follows directly from the definition of subgame perfect equilibria. However the definition does not say which action to take when both accept and reject yield the same payoff.

payoff $(x_A\pi, (1-x_A)\pi) = (\delta_A u_A^*\pi, (1-\delta_A u_A^*)\pi)$.

(2) $x_A > \delta_A u_A^*$, then player $A$ will accept since this will give him a better payoff then reject. However as we have seen, player $A$ will accept also a smaller offer equal to $\delta_A u_A^*$. And smaller offer is better for player $B$ since this leaves a greater part of the pie for him. Thus in subgame perfect equilibrium player $B$ will never make such an offer.

(3) $x_A < \delta_A u_A^*$, then player $B$ will reject since this gives him a strictly greater payoff than accept. And as we have seem before rejecting yields the payoff $(\delta_A u_A^*\pi, \delta_A\pi(1-u_A^*))$. If we compare the payoff of player $B$ in cases (1) and (3) then we can see that this payoff is strictly greater in case (1), since $(1-\delta_A u_A^*)\pi) > \delta_A\pi(1-u_A^*)$ for $0 < \delta_A < 1$.

    Thus in a subgame perfect equilibrium player $B$ will never give an offer smaller than $\delta_A u_A^*$.

In conclusion only case (1) can hold, i.e. player $B$ makes an offer

$$x_A = \delta_A u_A^* \tag{2}$$

and this offer is immediately accepted by player $A$. This gives the payoffs $(\delta_A u_A^*\pi, (1-\delta_A u_A^*)\pi)$ at each position $a_{2i}$ where player $B$ makes an offer. By Assumption 2,

$$(1-\delta_A u_A^*)\pi = u_B^*\pi.$$

A symmetric reasoning yields

$$x_B = \delta_B u_B^* \quad \text{and} \quad (1-\delta_B u_B^*)\pi = u_A^*\pi. \tag{3}$$

Solving this system of equation we obtain

$$u_A^* = \frac{1-\delta_A}{1-\delta_A\delta_B}, \quad u_B^* = \frac{1-\delta_B}{1-\delta_A\delta_B}$$

and by (2) and (3)

$$x_A = \delta_A\frac{1-\delta_A}{1-\delta_A\delta_B}, \quad x_B = \delta_B\frac{1-\delta_B}{1-\delta_A\delta_B} \tag{4}$$

In the equilibrium players make offers $x_A$ and $x_B$ respectively given by (4) and player $A$ accepts any offer greater equal $x_A$ while player $B$ accepts any offer greater equal $x_B$. All smaller offers are rejected.
Let us note that we rather guessed that these strategies form a subgame perfect equilibrium. It is better to verify this fact formally.
Let $(\sigma_A, \sigma_B)$ be the strategy profile described above.
Suppose that player $B$ changed his strategy, i.e. he uses some another strategy $\sigma_B'$ while player $A$ uses $\sigma_A$.
Suppose that at some stage player $B$ accepts an offer of player $A$ or player $A$ accepts an offer of player $B$. Then by backward induction we can show[2] for all previous stages

[2]Try to do it yourself.

- player $B$ cannot win more than $u_B^* \pi$ for the games starting at positions where he makes an offer,
- he cannot win more than $\pi(1-u_A^*)$ for games starting at positions where player $A$ makes an offer.

Suppose that from some moment onward both players always refuse. Then from this point onward player B's payoff is 0.
Thus we can see that changing the strategy is not profitable for player $B$. The reasoning for player $A$ is symmetrical.
Instead of supposing that the game ends with probability $1-\delta$ (we assume here that $\delta = \delta_A = \delta_B$) at the end of each round in the case of refusal we can assume that the pie shrinks by factor $\delta$ at the beginning of each round, i.e. at the first round the players have a pie of size $\pi$, at the second round a pie of size $\delta\pi$, at the third round a pie of size $\delta^2\pi$ etc. and after each refusal we pass directly to the next round, there is no chance move that can stop the game. The game analysis and the result are the same for this variant of the game[3].

**Exercise 2** (From Herbert Gintis, Game theory evolving)**.** Miss Muffet is eating an ice cream and is confronted with a wasp which apparently likes to take its share.
Miss Muffet proposes the following bargaining game. She proposes a certain share $x$. If the wasp accepts the offer then each player goes away with her/its share. If the offer is rejected then in the next round the wasp makes a counter offer under the same conditions as above. However the day is hot, by the time the second offer is accepted or rejected the ice have melted down to the half of the original size. Both Miss Muffet and the wasp know perfectly all these conditions, are fully rational and they refuse the offer only if they can gain something in this way. We assume that after the second rejection Miss Muffet and the wasp get 0 and there is no third round in this game, by this time the ice cream will melt down completely.
How much Miss Muffet offers at the first round? Is this offer accepted?

For the sequel suppose that the whole ice cream is of size 1.
Consider the case where there are even number $n$ of rounds, Miss Muffet and the wasp make the offers alternately, with Miss Muffet making the first offer, and after each refusal the ice cream shrinks by $\frac{1}{n}$, i.e. after $k$ refusal the size of the ice cream at the beginning of the $(k+1)$st round is $\frac{n-k}{n}$ (in particular after $n$ refusals the ice cream melts down completely). Show that Miss Muffet will offer $\frac{1}{2}$ of the ice cream and this offer will be immediately accepted.
Now suppose that with the same conditions as above the number of rounds $n$ is odd. Show that Miss Muffet will offer $\frac{1}{2} - \frac{1}{2n}$ and this offer will be accepted.

**Exercise 3.** *Warning: This is a rather difficult exercise, still you should try to solve it.* Jacques, Pierre and Magali want to split a pie. They proceed as in Example 5, i.e. at first round Jacques proposes to split the pie giving $p$ to Pierre, $m$ to Magali and keeping $j$ to himself. If Pierre and Magali accept then the pie is shared as proposed. If either Pierre or Magali refuse then with probability $1-\delta$ the game stops and with probability $\delta$ we pass to the next round where Pierre makes a proposal. If

[3]Since the chance cannot stop this variant of the bargaining game there exist strategy profiles with infinite plays, for such plays we assume the payoffs $(0, 0)$.

this proposal is rejected either by Magali or by Jacques then in the next round it is Magali's turn to make a proposal. The players turn this way unless the game is stopped by the nature or a proposal of one player is accepted by the other two players. Show that there is a unique symmetric and stationary[4] subgame perfect equilibrium.
There exist also non symmetric subgame perfect equilibria, but they are difficult to find and to describe.

[4]Symmetric stationary means that all players make the same offer at each round.