Algorithmic Game Theory

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Game Theory + Algorithms

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* Theoretical computer science studies optimization problems, seeks to optimum, efficient computing, impossibility results, ... etc

Algorithmic Game Theory

* Research field on the interface of game theory and theoretical computer science (mostly algorithms)

* Formulating novel goals and problems, fresh looks on different issues (inspired by Internet, ...).

* The field has phenomenally exploded with many branches: computing Nash equilibrium, mechanism design, inefficiency of equilibria, ... etc

Outline

* Existence and inefficiency of pure Nash equilibrium
    * Scheduling Games in the Dark
* Online Algorithmic Mechanism Design
    * Online Auction with single-minded customers

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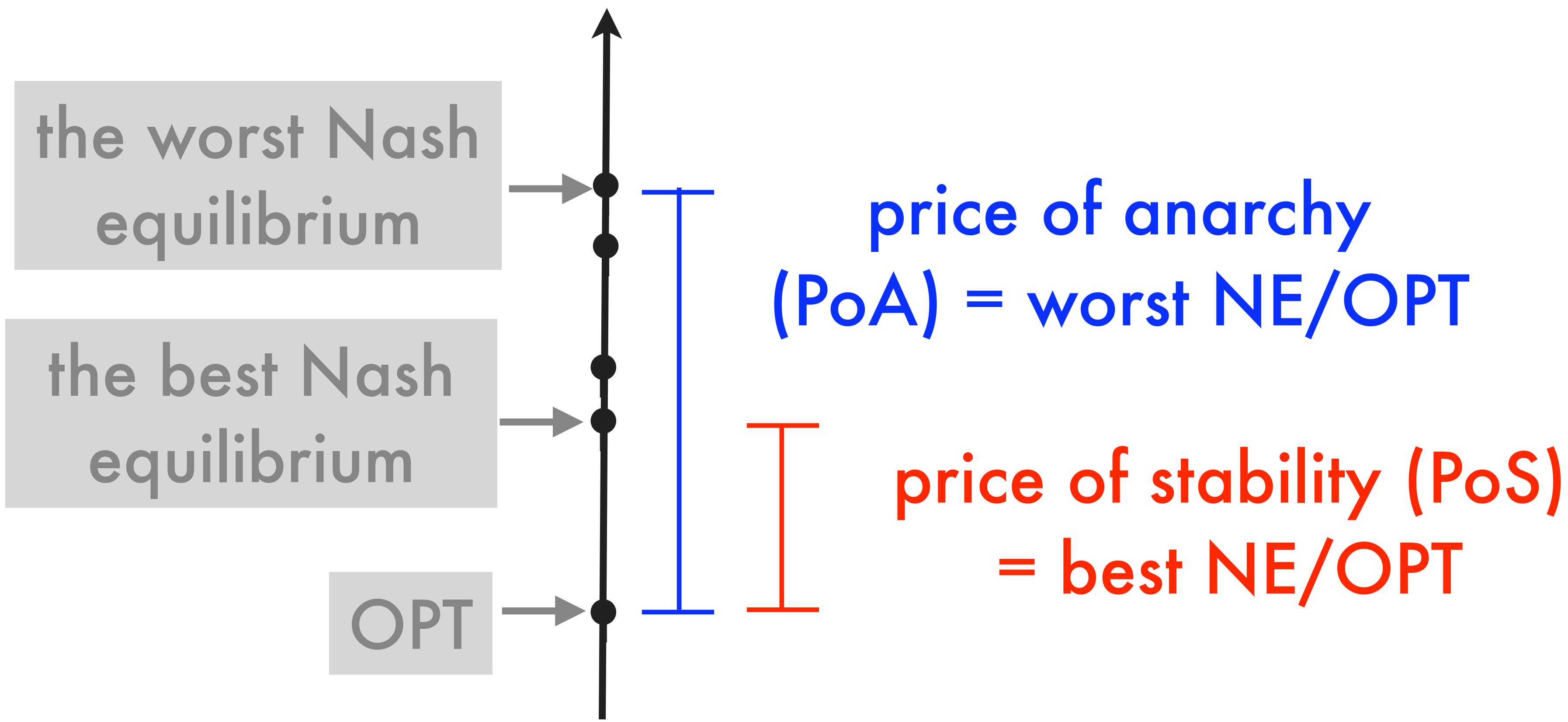
* Potential games: admit a function such that if a player change her strategy to get a better utility then the function strictly decreases.

Inefficiency of equilibria

social objective function

Good equilibria ?



Scheduling Game

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Non-clairvoyant policies

existence
Nash equilibrium

small PoA

Natural policies

* RANDOM: schedules jobs in a random order.

  In the strategy profile σ, i is assigned to j:

$$c_i = p_{ij} + \frac{1}{2} \sum_{i' : \sigma(i') = j, i' \neq i} p_{i'j}$$

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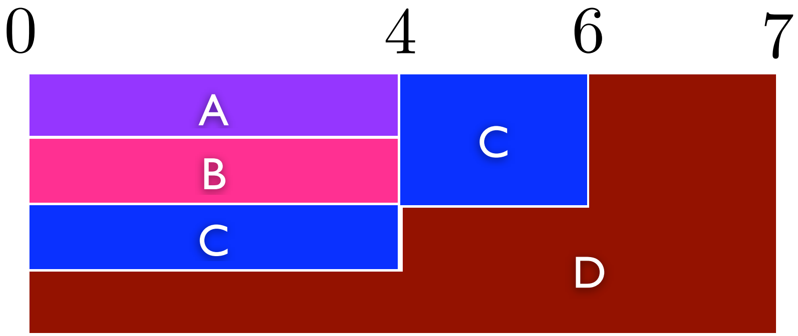
$$p_A = 1 \quad p_B = 1 \quad p_C = 2 \quad p_D = 3$$



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  If there are k jobs on machine j s.t: $p_{1j} \leq \ldots \leq p_{kj}$

$$c_i = p_{1j} + \ldots + p_{i-1,j} + (k - i + 1)p_{ij}$$

Models

* Def: A job i is balanced if $\max p_{ij} / \min p_{ij} \leq 2$

* Def of models:
  - Identical machines: $p_{ij} = p_i \ \forall j$ for some length p_i
  - Uniform machines: $p_{ij} = p_i / s_j$ for some speed s_j
  - Unrelated machines: p_{ij} arbitrary

Existence of equilibrium

- Theorem:
  - The game under EQUI policy is a potential game.
  - The game under RANDOM policy is a potential game for 2 unrelated machines but it is not for more than 3 machines. For uniform machines, balanced jobs, there always exists equilibrium.

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* Jobs have length $p_1 \leq p_2 \leq \ldots \leq p_n$
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$$p_{ij} = p_i / s_j$$

* Lemma: Consider a job i making a best move from a to b. If there is a new unhappy job with index greater than i, then $s_a > s_b$

Potential function

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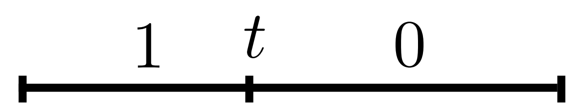
- For any strategy profile σ, let t be the unhappy job with greatest index.

$$f_\sigma(i) = \begin{cases} 1 & \text{if } 1 \le i \le t, \\ 0 & \text{otherwise.} \end{cases}$$



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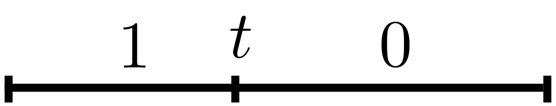

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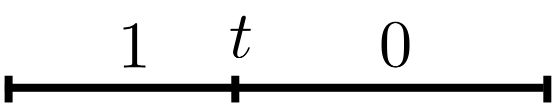

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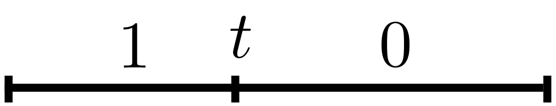

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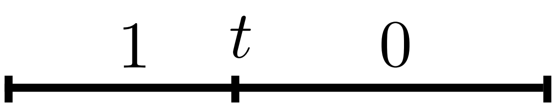

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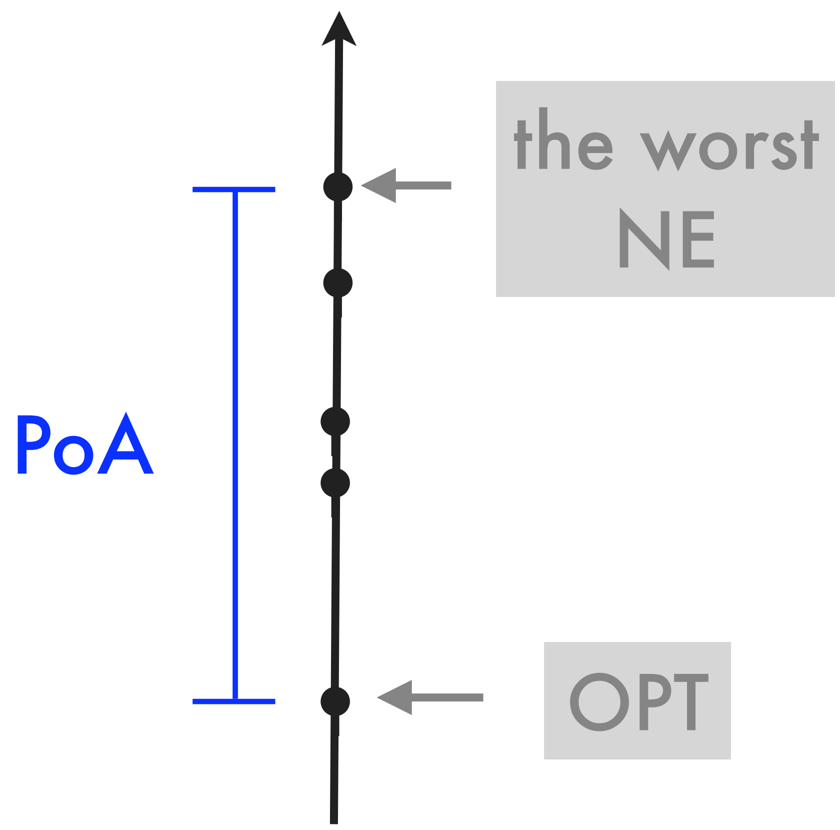
by Lemma: $s_{\sigma(t)} > s_{\sigma'(t)}$

Inefficiency

* Theorem: For unrelated machines, the PoA of policy EQUI is at most 2m – interestingly, that matches the best clairvoyant policy.

* PoA is not increased when processing times are unknown to the machines.



Mechanism Design

Define the game

Goal: self-interested behavior yields desired outcomes.

Online Auction

* A company produces one perishable item per time unit (items have to be immediately delivered to customers, e.g. electricity, ice-cream, ...)
* Single-minded customers arrive online: a customer arrives at r_i, pays w_i if he receives k_i items before deadline d_i, otherwise he pays nothing.
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* Mechanism design:
  - w_i are private
  - Customers may misreport their value. They bid b_i

Mechanism Design (MD)

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receives all bids

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allocation algorithm:
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Goal: self-interested behavior yields **truthfulness**, $b_i = w_i$

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critical payment:
the smallest bid that a winner needs to bid in order to win.

Truthful MD

- Theorem: for single-parameter domain, a mechanism is truthful iff its allocation algo is monotone and it uses the critical payment scheme.

- Our problem:
  - design a monotone algorithm
  - verify whether the critical payment scheme can be computed efficiently.

Online Algorithm

* Maximizing the welfare $\sum_i w_i$ is hard.

* Def: an online algorithm ALG is c-competitive if for any instance I, the outcome $c \cdot ALG(I) \geq OPT(I)$

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- Theorem: if all $k_i = k$ then there exists a 7-competitive truthful mechanism.

Algorithm

- The CONSERVATIVE algo:
    - if there is no currently running job, serve the pending one with highest value
    - still schedule the current customer except there is a new one with value at least 2 that of the current customer

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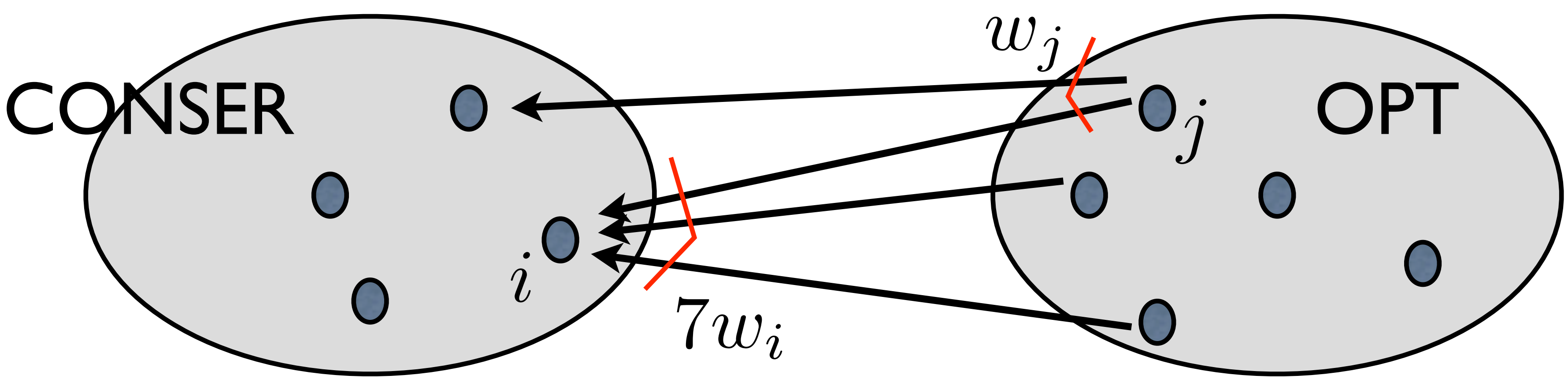
  - if there is no currently running job, serve the pending one with highest value
  - still schedule the current customer except there is a new one with value at least 2 that of the current customer
- Proof:
  - the algorithm is monotone
  - 7-competitive by a charging scheme



Proof (sketch)

CONSER

OPT

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$$\begin{array}{llll}
\text{CONSER} & & & \\
 & ? & ? & ? \\
 & \Big\uparrow w_j/k & \Big\uparrow w_j/k & \Big\uparrow w_j/k \\
\text{OPT} & \boxed{(j,k)} & \boxed{(j,b)} & \boxed{(j,1)}
\end{array}$$

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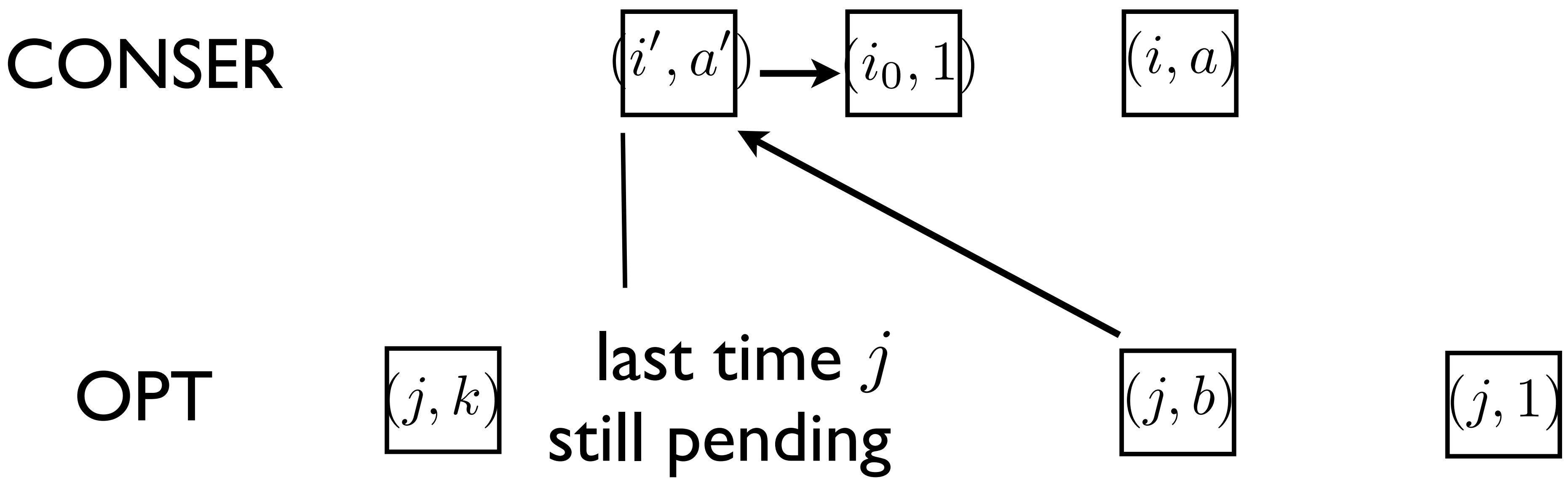
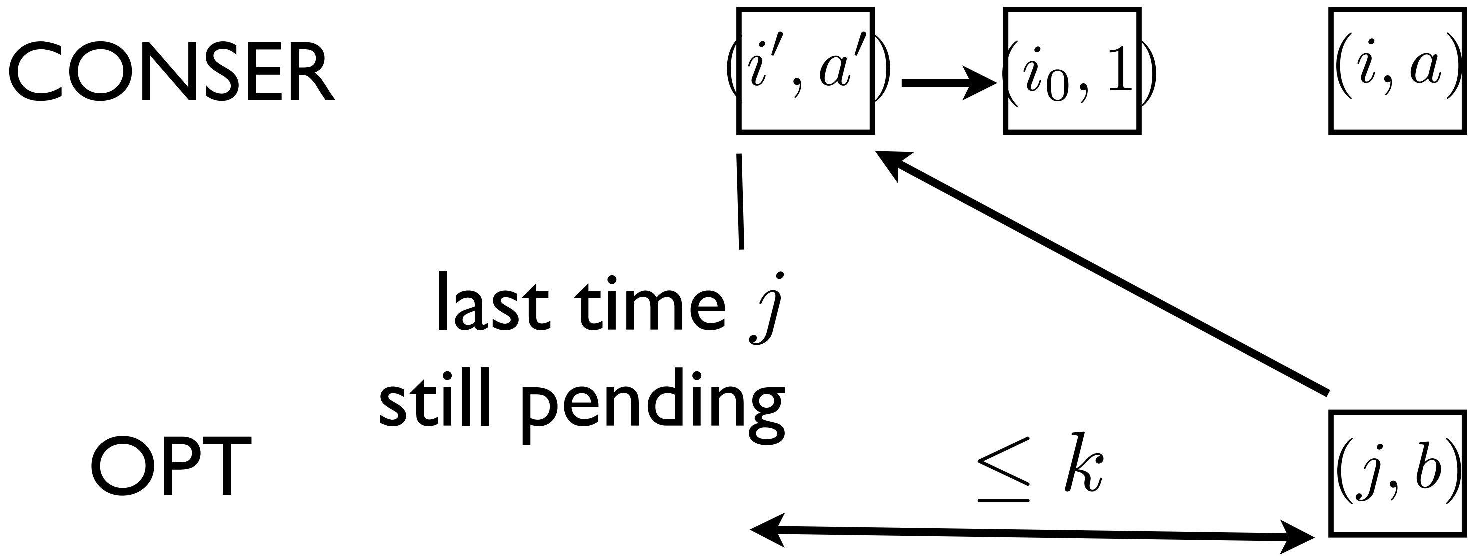


- type 1: if j is completed by CONSER
- type 2: if $2w_i > w_j$
- type 3: otherwise $2w_i \le w_j$, j is not pending.

  $2w_{i'} > w_j$ then $2w_{i_0}/k > w_j/k$

Proof (sketch)



- Observation: $(i_0, 1)$ receives at most k charges of type 3.

- Summing up all charges, we get 7-competitive.

General case

* Theorem: if all $k_i \leq k$ then there exists a $O(k/\log k)$-competitive truthful mechanism. This mechanism is optimal.

* Proof: more elaborated but the idea is similar.

Conclusion

- ☑ Motivation through two problems.
  - theoretically beautiful
  - real problems, practical importance.
- ☑ Inspired by Game Theory, using technique of Computer Science
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