Congestion games

Sylvain Sorin

Equipe Combinatoire et Optimisation, Faculté de Mathématiques, UPMC-Paris 6
Laboratoire d'Econométrie, Ecole Polytechnique
****
sorin@math.jussieu.fr
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A finite set of facilities K is given.
For each $k \in K$, $c_k(x)$ (non decreasing) is its "cost" if the quantity or number of users is x.

1.1. Finite case

There are n players and each one chooses a facility. The induce profile x is thus a vector with integer components $\{x_k\}_{k \in K}$: x_k is the number of users of k, hence $\sum x_k = n$.
X denotes the set of feasible profiles. At x, the cost of player i chosing k is thus $c_k(x_k)$.

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Definition

$x \in X$ is an equilibrium profile if none of the players has an incentive to change his choice:

$$c_k(x_k) \leq c_m(x_m + 1)$$

for all k, m with $x_k > 0$.

1.2. Non atomic case

Assume that the players are represented by the [0, 1] continuum. Let x_k the fraction of them choosing facility k. Hence $\sum x_k = 1$ and X is the simplex $\Delta(K)$.

Definition

The profile $x = \{x_k\} \in X$ is an equilibrium if for any k with $x_k > 0$

$$c_k(x_k) = \min_{m \in K} c_m(x_m)$$

since one player may change his choice without affecting the overall profile $\{x_k\}$.

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1.3. Social optimum

In both cases above one could look at the best affectation over the set of feasible profiles.

Definition

A social optimum is a profile that maximizes

$$F(z) = \sum_k z_k c_k(z_k)$$

over the set of feasible profiles X.

Note that the above framework corresponds to *routing games with parallel links*.

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Simple examples where the two concepts: equilibria and social optimum differ are given by the following

Example

Finite case

Example

Non atomic case

Example

Extension

1.4. Potential approach

Finite case

Introduce the function

$$\Phi(x) = \sum_k \sum_{a=1}^{x_k} c_k(a).$$

Definition

Φ is a potential for the game in the sense that for each player i, a change of his choice leading from profile x to y induces a variation of his payoff by

$$\Phi(y) - \Phi(x)$$

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In fact if player i changes from k to ℓ his variation in payoff is

$$c_\ell(x_\ell + 1) - c_k(x_k)$$

which is the variation in Φ since $y_k = x_k - 1$, $y_\ell = x_\ell + 1$ and $y_m = x_m$ otherwise.

Theorem

Any x minima of Φ over X is an equilibrium.

This implies in particuler the existence of (pure) equilibria and of a decentralized procedure to obtain it.
There is no uniqueness in general.

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Non atomic case

The corresponding definition is

Definition

$\Psi(x) = \sum_k \int_0^{x_k} c_k(u)du$ *is a potential for the atomic game.*

Theorem

$x \in X$ *is an equilibrium iff* x *is minima of* Ψ *over* X.

Essentially unique.

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In particular

Theorem

A SO corresponds to a NE of the game with cost functions

$$\tilde{c}_k(u) = c_k(u) + uc'_k(u)$$

correponding to the derivative of $xc_k(x)$.

The general model is given by a set of nodes V and a set of directed edges S with corresponding cost function c_s.

2.1. Nonatomic case

One is given positive amounts m_i indexed by a family I of couples (origin/destination). Let R^i be the sets of roads associated to i. A flow x is feasible if, denoting by $x[r]$ the load on road r one has

$$\sum_{r \in R^i} x[r] = m_i.$$

The induced load on edge s is $x_s = \sum_{r, s \in r} x[e]$ and the cost on this edge is $c_s(x_s)$.

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The cost of road r is thus

$$C^r(x) = \sum_{s, s \in r} c_s(x_s)$$

hence the total cost of the flow x can be written as

$$C(x) = \sum_r C^r(x) x[r]$$

using a decomposition on roads, or alternatively

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Definition

A feasible flow x is a social optimum if it maximizes $C(x)$.

Definition

A feasible flow x is an equilibrium if for any i and any roads $r, u \in R^i$ with $x[r] > 0$

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Using the above formulation in terms of edges one obtains, for any feasible flows x and y:

$$\sum_s c_s(x_s)y_s = \sum_i \sum_{r\in R^i} C^r(x)y[r]$$

which leads to the following variational characterization:

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As above one can also introduce a potential:
$\Psi(x) = \sum_s \int_0^{x_s} c_s(u)du$ and one has

Theorem

x is an equilibrium iff x is a minima of Ψ over X.

2.2. Braess paradox

Example

This example shows another consequence of strategic interaction (in addition to the loss of optimality at equilibrium): to increase the strategic choices may harm.

2.3. Price of anarchy

Definition

The cost of anarchy is the ratio between the cost of the worst equilibrium and that of a social optimum.

Theorem

Assume

$$uc_s(u) \leq M \int_0^u c_s(v)dv$$

for all edges s and $u \geq 0$. Then the price of anarchy is at most M.

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Proof

Note that $uc_s(u) \geq \int_0^u c_s(v)dv$ since c_s is increasing hence if x in an equilibrium and y is feasible

$$C(x) \leq M\Psi(x) \leq M\Psi(y) \leq MC(y)$$

in particular if y is a SO. ∎

Definition

Given a family C of cost functions, define the bound

$$\alpha(C) = \sup_{c \in C} \sup_{x,r \geq 0} \frac{rc(r)}{xc(x) + (r - x)c(r)}$$

Theorem

If all c_s belong to the family C the price of anarchy is less than $\alpha(C)$.

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Proof Let y be feasible and x an equilibrium flow.

$$C(y) = \sum_s c_s(y_s) y_s$$

$$\geq [\frac{1}{\alpha(C)} \sum_s c_s(x_s) x_s] + \sum_s (y_s - x_s) c_s(x_s)$$

$$\geq \frac{C(x)}{\alpha(C)}$$

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Corollary

For affine cost functions the price of anarchy is at most 4/3.

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Corollary

For affine cost functions the price of anarchy is at most 4/3.

2.4. Finite game

In the finite framework each player i has a couple (origin/destination) and choose a road $r \in R^i$ accordingly. The overall induced traffic on edge s is then x_s (number of roads chosen by the players and going through s) and the cost of his choice for player i is

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Example

Price of anarchy 5/2 with linear costs

Example

Non existence in the weighted case

Proposition

Existence in the weighted case for affine cost functions through a potential.

3.1. Atomic games: splitting case

This corresponds to the case where there are non negligible players but they can split the load they have to transport through an OD pair.

Given the other players behavior, inducing a flow x, player i is in the position of of chosing a SO, hence is facing a Wardrop equilibrium problem with the new cost function

$$\tilde{c}_s(u) = uc'_s(x_s + u) + c_s(x_s + u).$$

However the results will differ.

Example

Non monotonicity.

Example

Price of anarchy greater than 4/3 with linear costs.

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3.2. General case

We keep the framework of a finite set of locations (or of a finite network).

Set of players : N (atomic part) + $NA = [0, M]$ (non atomic part). Let T be the set of OD pairs. The non atomic players are splitted on T.

For each $i \in N$ and each $t \in T$ there is an atomic part given by a vector $\alpha^i_t = \{\alpha^i_{tm}\}$ and a non atomic part $[0, v^i_t]$. The total size of player i is thus $w^i = \sum_t(\|\alpha^i_t\| + v^i_t)$. Player i has to choose for each t a road for each indivisible package of size α^i_{tm} and splits the amount v^i_t among the feasible roads for t.

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Existence of equilibria in pure strategies
Uniqueness
Price of anarchy
Gain from collusion
Connection with “Strong equilibrium”
Paths from NE to SO where all agents are better off.
Real procedure: intensity, delay
Comparison via replica : NE of Γ versus SO of $t\Gamma$.
Algorithms, given the players structure; on the formation of coalitions
Adapted pricing
Networks with capacities
Network formation

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