# Geometric Views of Linear Complementarity Algorithms and Their Complexity

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# LCP - Definition

Given: $q \in \mathbf{R}^n$, $M \in \mathbf{R}^{n \times n}$

Find: $z \in \mathbf{R}^n$ so that

$$z \geq \mathbf{0} \quad \perp \quad w = q + Mz \geq \mathbf{0}$$

$\perp$ means orthogonal:

$$z^T w = 0$$

$$\Leftrightarrow \quad z_i\, w_i = 0 \quad \text{all } \mathbf{i} = 1, \ldots, n.$$

# LP in inequality form

primal: **max** $c^T x$

subject to $Ax \leq b$

$x \geq \mathbf{0}$

dual: **min** $y^T b$

subject to $y^T A \geq c^T$

$y \geq \mathbf{0}$

**Weak duality**: $x$, $y$ feasible (fulfilling constraints)

$\Rightarrow$ $c^T x \leq y^T A x \leq y^T b$

**Strong duality**: primal and dual are feasible

$\Rightarrow$ $\exists$ feasible $x$, $y$: $c^T x = y^T b$ ( $x$, $y$ optimal)

# LCP generalizes LP

LCP encodes the complementary slackness of strong duality:

$$c^T x = \quad y^T A x \quad = y^T b$$

$$\Leftrightarrow \quad \underbrace{(y^T A - c^T)}_{\geq \mathbf{0}}\underbrace{x}_{\geq \mathbf{0}} = 0, \quad \underbrace{y^T}_{\geq \mathbf{0}}\underbrace{(b - Ax)}_{\geq \mathbf{0}} = 0.$$

LP $\Leftrightarrow$ LCP

$$x \geq \mathbf{0} \quad \perp \quad -c \quad + A^T y \geq \mathbf{0}$$

$$y \geq \mathbf{0} \quad \perp \quad b \quad - Ax \geq \mathbf{0}$$

# Symmetric equilibria of symmetric games

Given: $n \times n$ **payoff matrix** $A$ for row player

$A^T$ for column player

**mixed strategy** $x$ = probability distribution on $\{1,\ldots,n\}$

$\Leftrightarrow\ x \ge \mathbf{0}\,,\ \ \mathbf{1}^T x = 1$

**equilibrium** $(x, x)$

$\Leftrightarrow$ $x$ **best response** to $x$

**Remark:** As general as $m \times n$ games $(A, B)$.

# Best responses

Given: $n \times n$ **payoff matrix** $A$,
mixed strategy $y$ of column player

$Ay$ = vector of **expected payoffs** against $y$,
components $(Ay)_i$

$x$ **best response** to $y$

$\Leftrightarrow$ $x$ maximizes expected payoff $x^T Ay$

best response condition:

$\Leftrightarrow$ $\forall \mathbf{i} : \; x_i > 0 \;\Rightarrow\; (Ay)_i = u = \max_k (Ay)_k$

# Symmetric equilibria as LCP solutions

equilibrium $(x, x)$ of game with payoff matrix $A$

$\Leftrightarrow$ $x$ **best response** to $x$

$\Leftrightarrow$ $\mathbf{1}^\top x = 1$,

$x \geq \mathbf{0} \quad \perp \quad Ax \leq \mathbf{1}u$

w.l.o.g. $A > 0 \quad \Rightarrow \quad u > 0$,

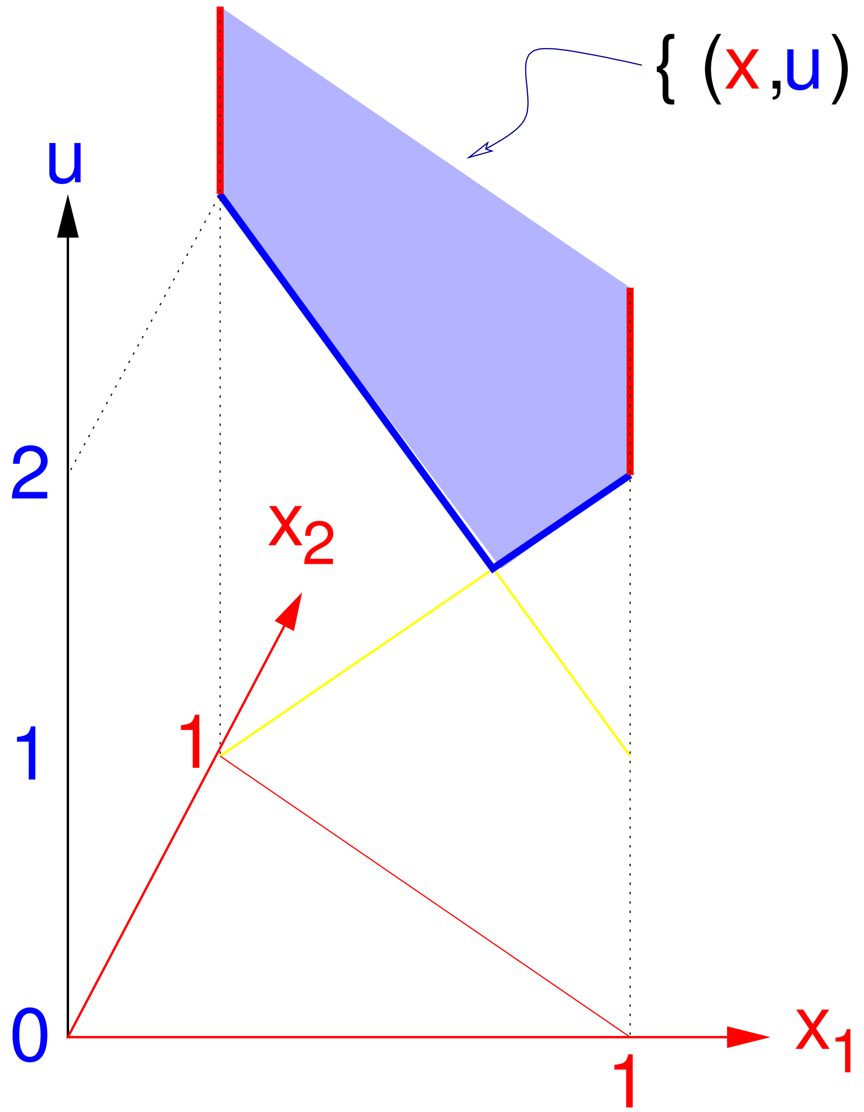
equilibrium $(x, x)$

$\Leftrightarrow$ $z = (1/u)\, x \qquad (1/u = \mathbf{1}^\top z)$,

$z \geq \mathbf{0} \quad \perp \quad Az \leq \mathbf{1}$ "equilibrium $z$"

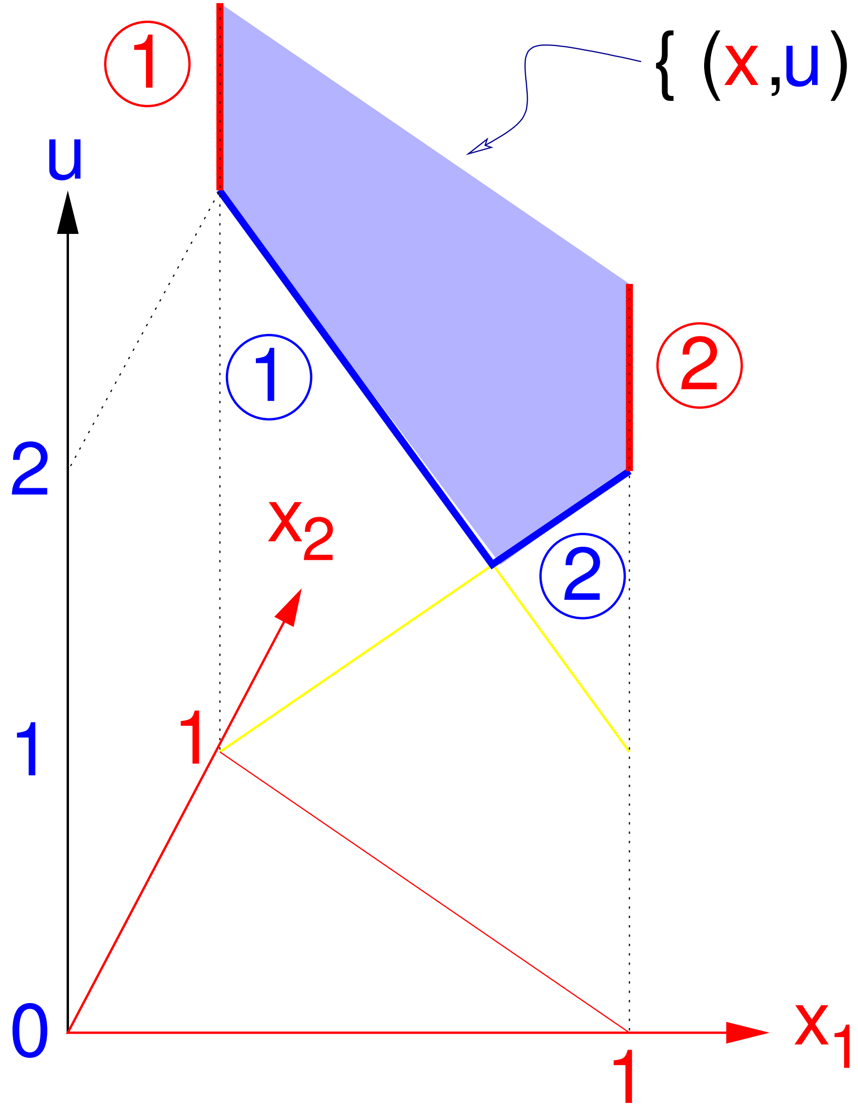
# Best response polyhedron

$\{ (x,u) \mid x \geq \mathbf{0},\ \mathbf{1}^T x = 1,\ Ax \leq \mathbf{1}u \}$




$$A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$$

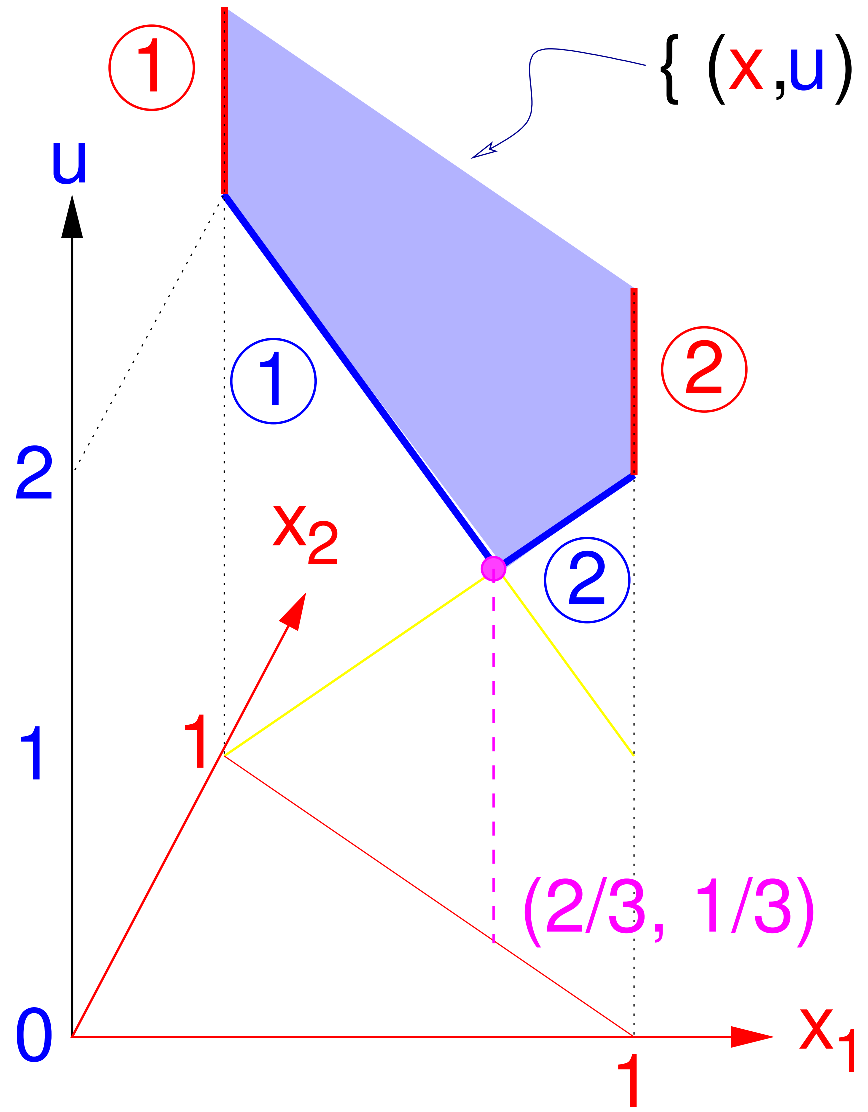
# Best response polyhedron




$$\{\ (x, u)\ |\ x \geq \mathbf{0},\ \ \mathbf{1}^T x = 1,\ Ax \leq \mathbf{1}u\ \}$$

$$A = \begin{array}{cc} 1 & 2 \\ \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} & \end{array} \begin{array}{c} \\ 1 \\ 2 \end{array}$$

# Best response polyhedron

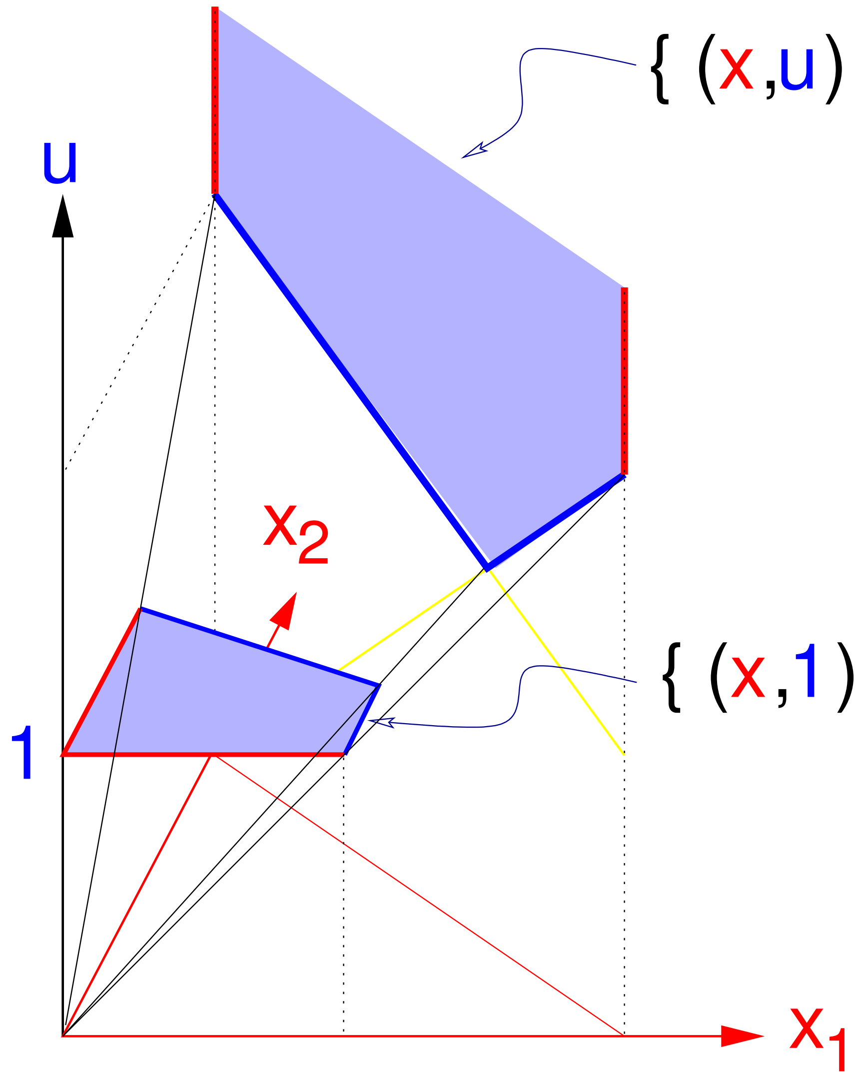



$$\{\ (x,u)\ |\ x \geq \mathbf{0},\ \ \mathbf{1}^T x = 1,\ A x \leq \mathbf{1} u\ \}$$

$$A = \begin{array}{cc} 1 & 2 \\ \boxed{\begin{array}{cc} 1 & 2 \\ 2 & 0 \end{array}} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

equilibrium
(completely labeled)

# Projective transformation

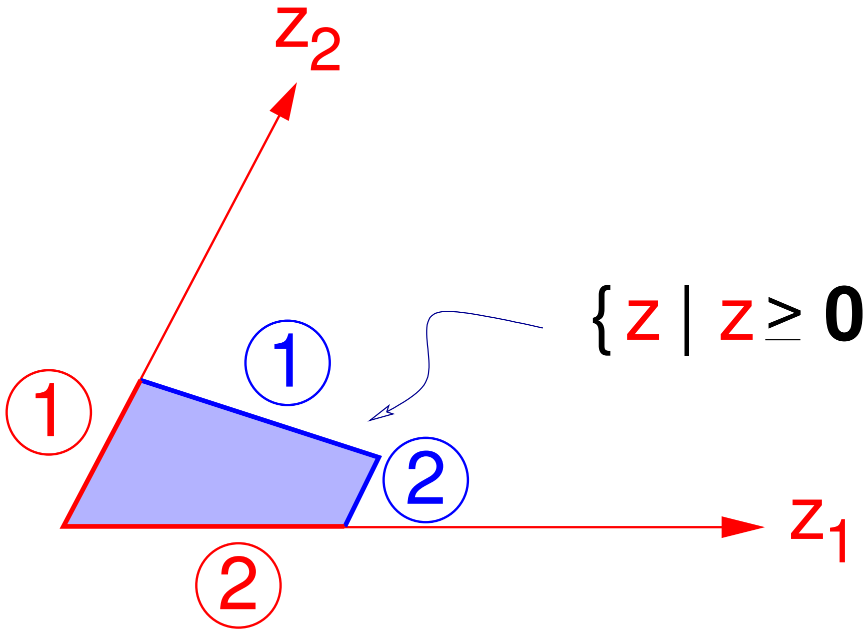



$$\{\, (x,u) \mid x \geq \mathbf{0},\ \mathbf{1}^T x = 1,\ Ax \leq \mathbf{1}u \,\}$$

$$\{\, (x,1) \mid x \geq \mathbf{0},\ Ax \leq \mathbf{1} \,\}$$

$$A = \boxed{\begin{matrix} 1 & 2 \\ 2 & 0 \end{matrix}}$$

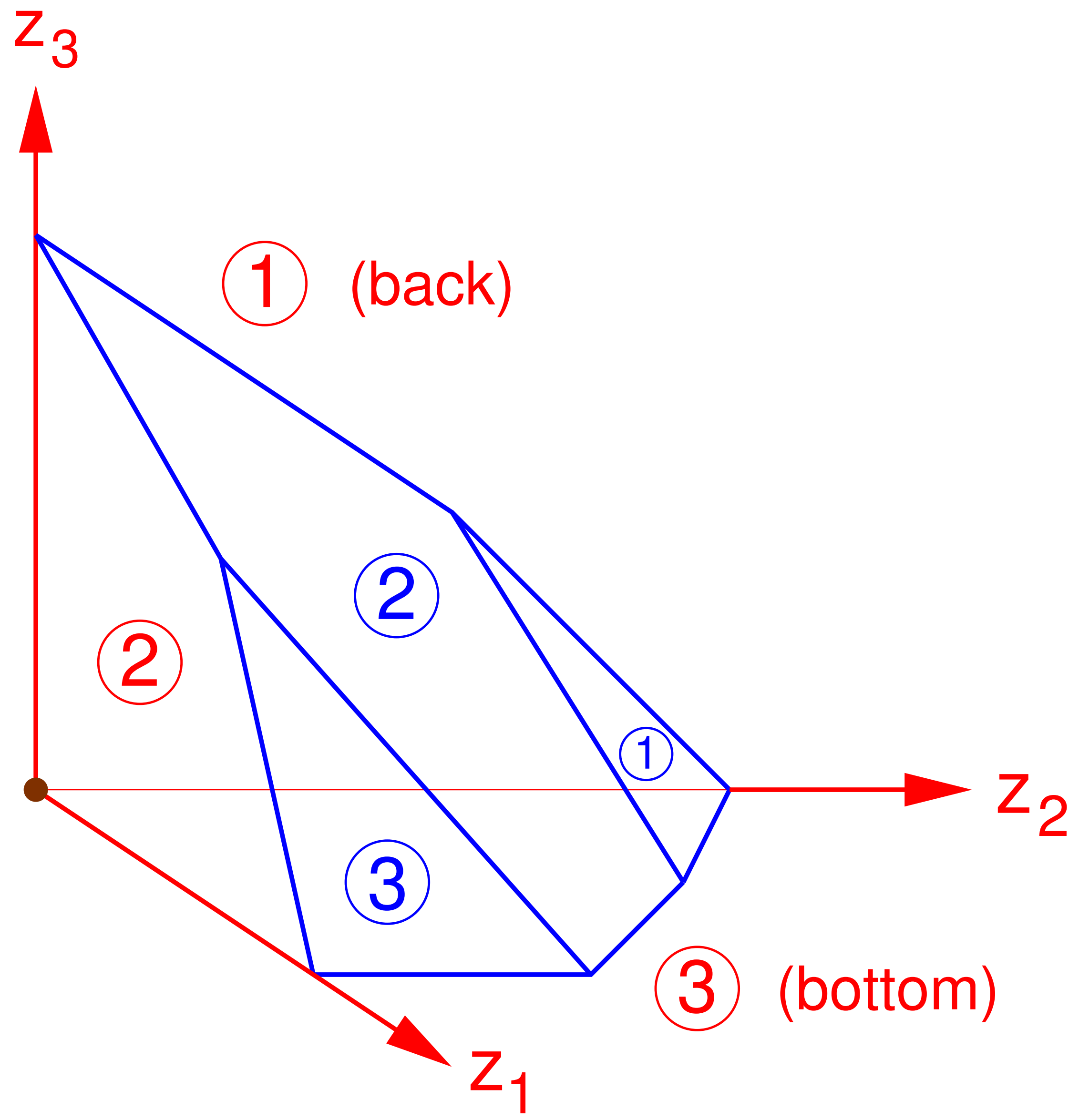
# Best response polytope



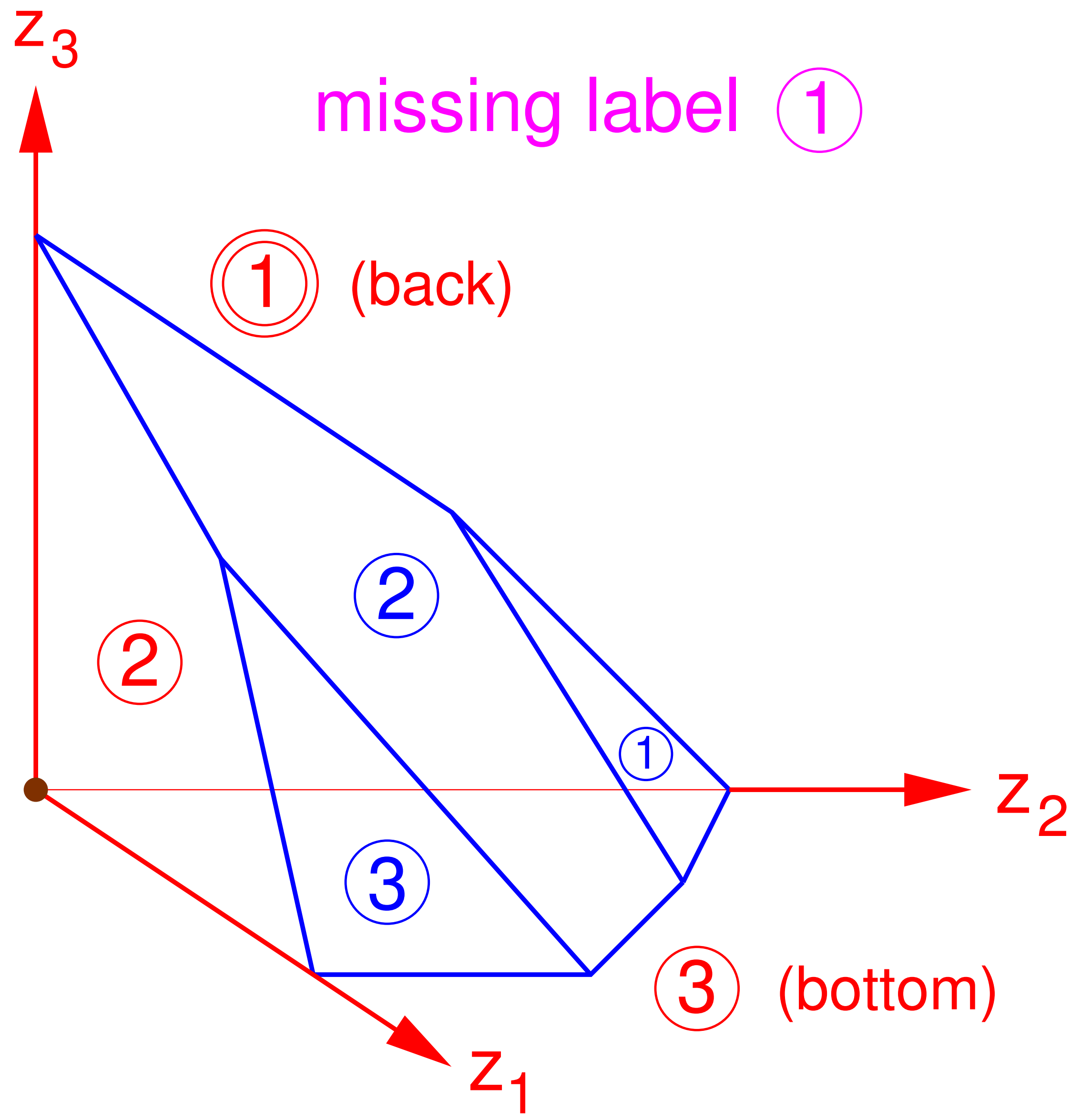

$$\{ z \mid z \geq \mathbf{0},\ A z \leq \mathbf{1} \}$$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$$

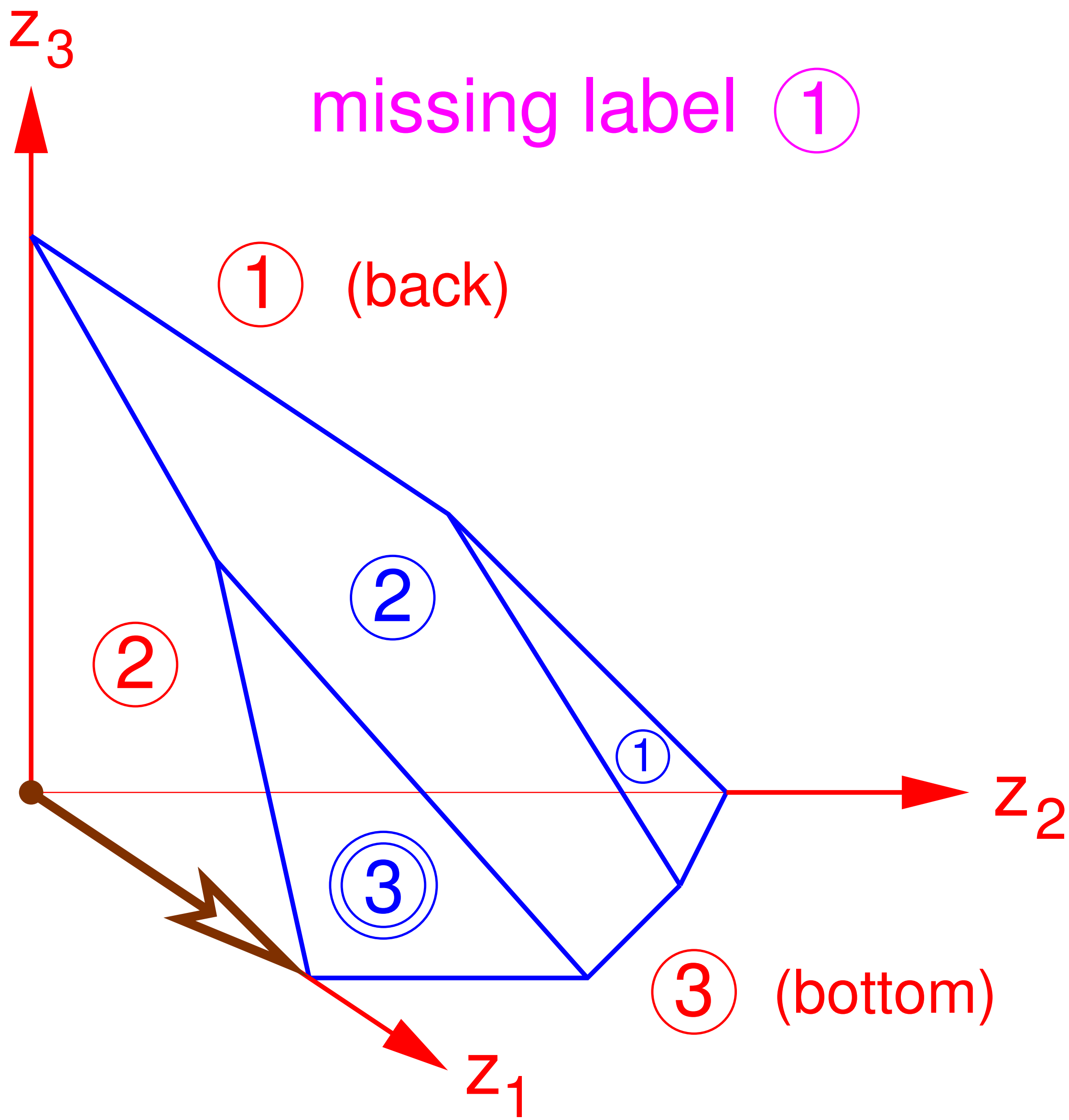
# Symmetric Lemke–Howson algorithm



# Symmetric Lemke–Howson algorithm



# Symmetric Lemke–Howson algorithm



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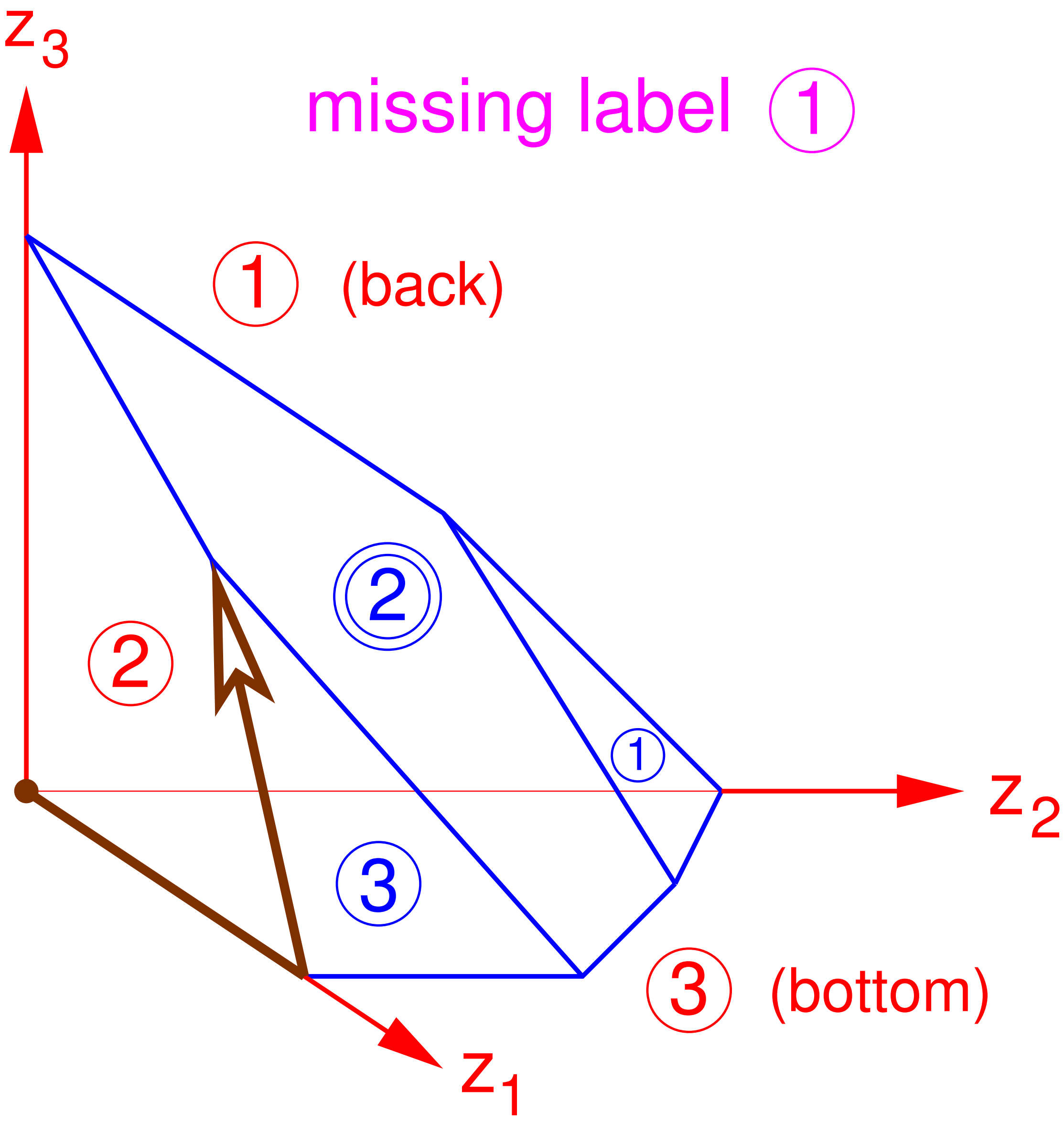
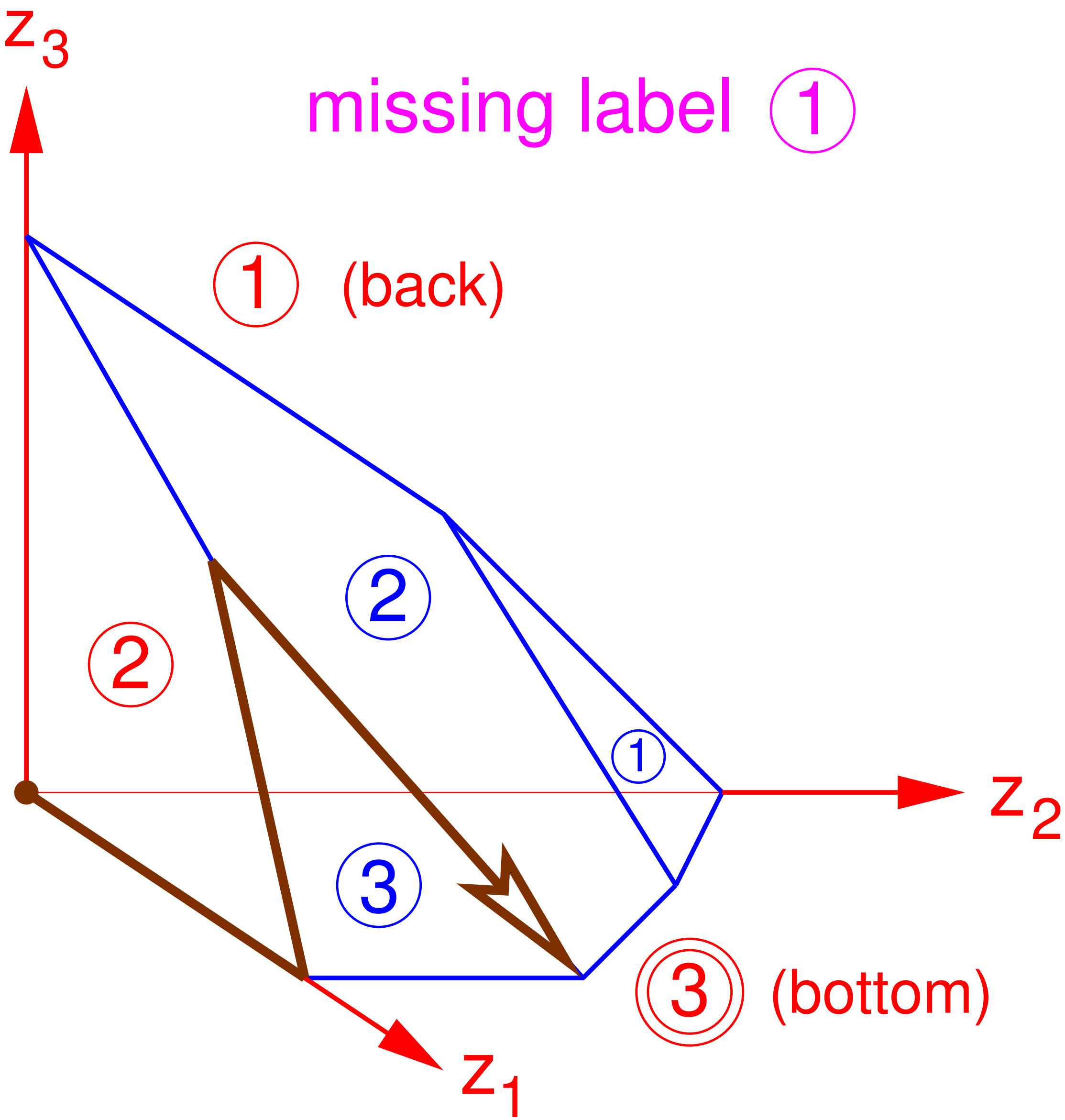


# Symmetric Lemke–Howson algorithm



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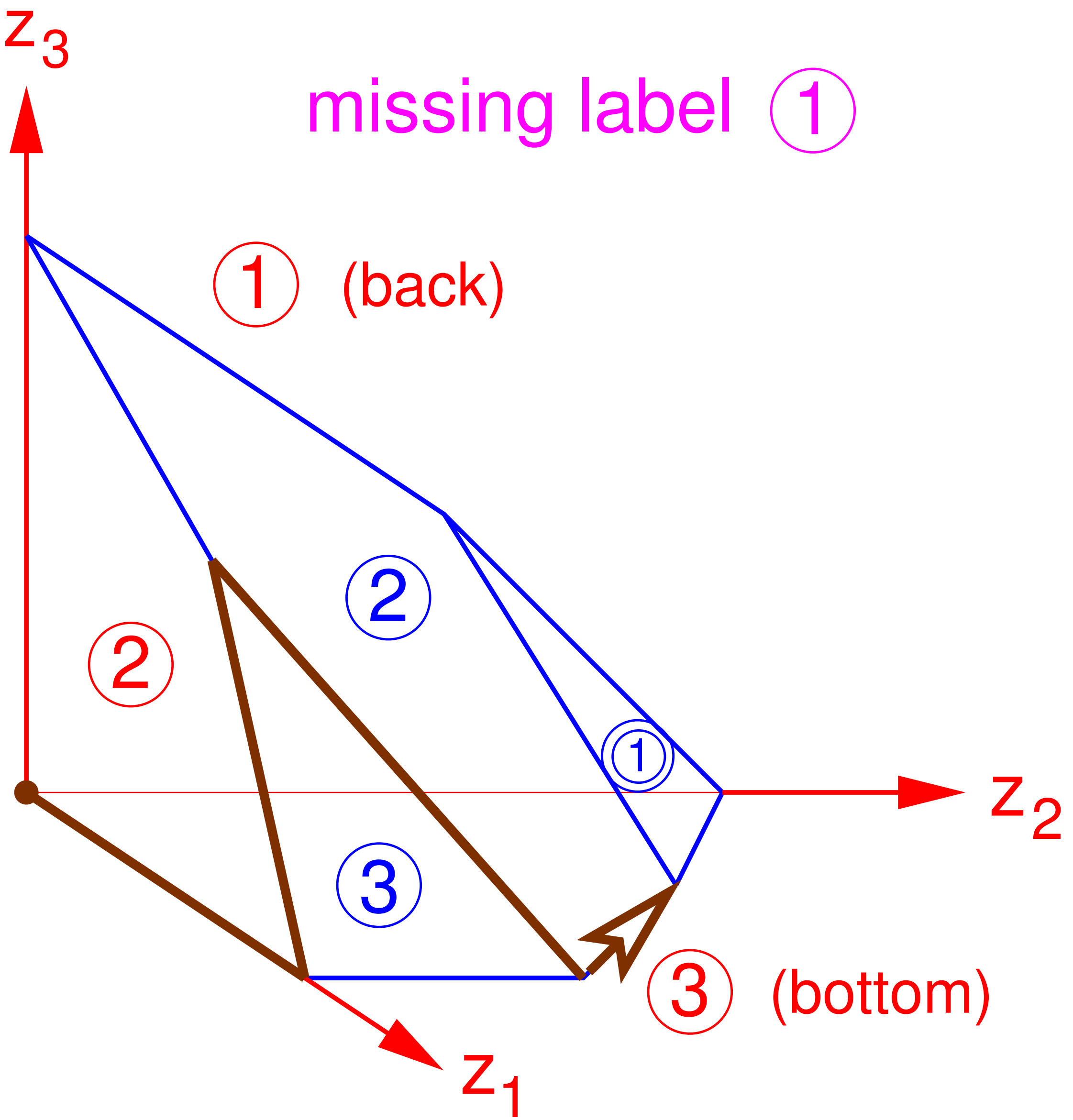
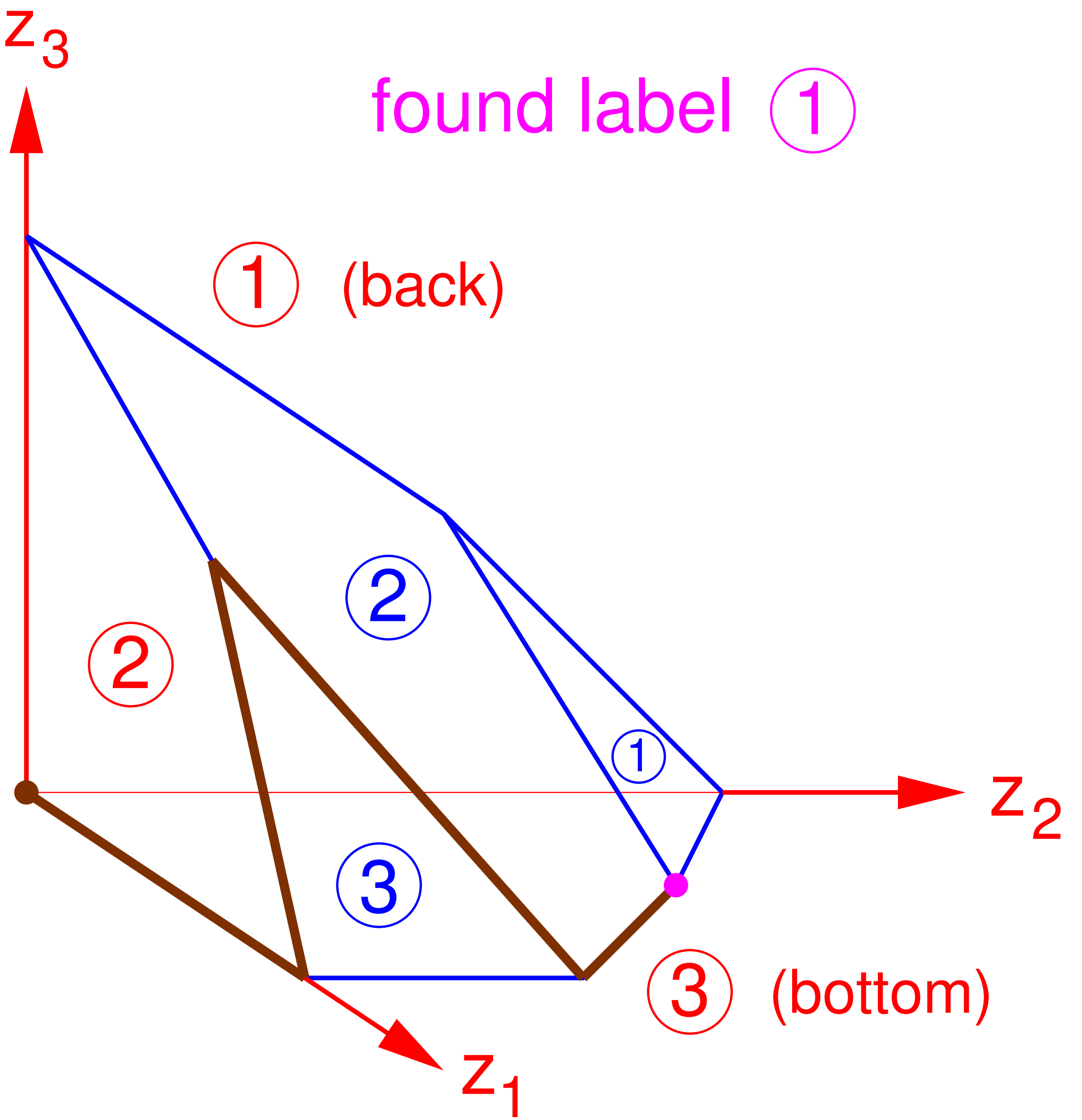


# Symmetric Lemke–Howson algorithm



# Why Lemke-Howson works

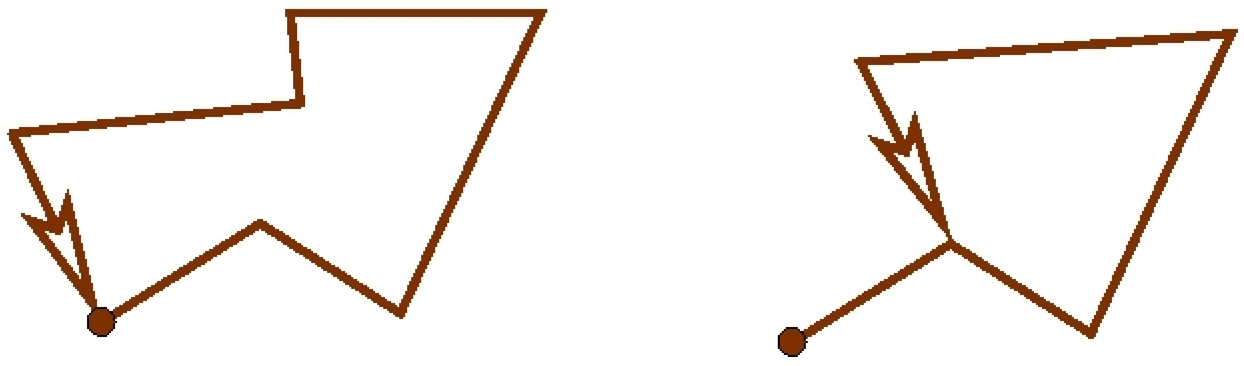
LH finds at least one Nash equilibrium because

- finitely many "vertices"

for nondegenerate (generic) games:

- **unique** starting edge given missing label
- **unique** continuation

$\Rightarrow$ precludes "coming back" like here:



# Costs instead of payoffs

| | |
|---|---|
| 1 | 2 |
| 2 | 0 |

$\rightarrow$

| | |
|---|---|
| 2 | 1 |
| 1 | 3 |

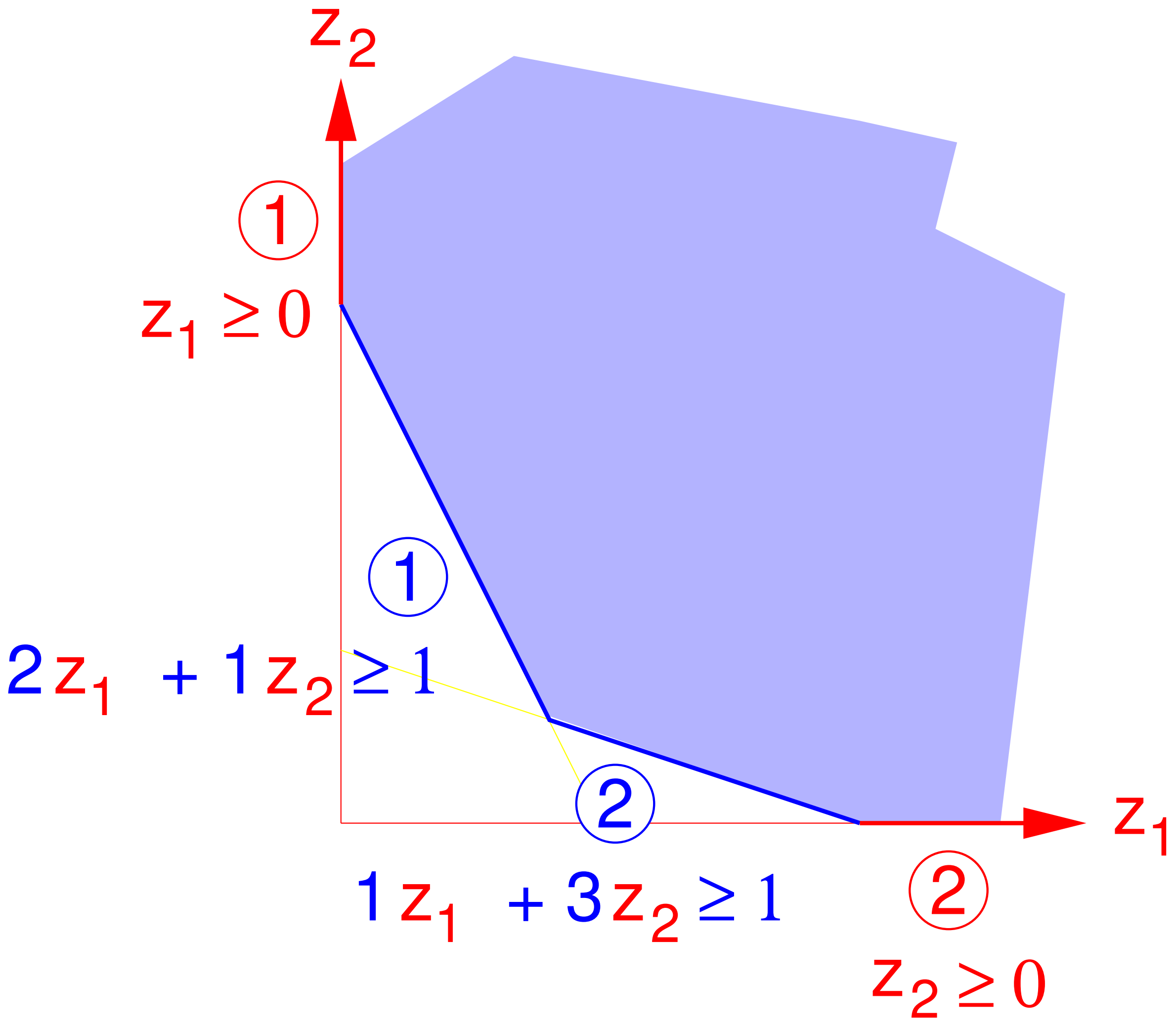
$a_{ik}$
payoff

$3 - a_{ik}$
cost

with new cost matrix $A > 0$ :

$$\text{equilibrium } z \quad \Leftrightarrow \quad z \geq \mathbf{0} \quad \perp \quad Az \geq \mathbf{1}$$

# Polyhedral view



# Lemke's algorithm

given LCP

$$z \geq \mathbf{0} \quad \perp \quad w = q + Mz \quad \geq \mathbf{0}$$

# Lemke's algorithm

**augmented** LCP

$$\mathbf{z} \geq \mathbf{0} \quad \perp \quad \mathbf{w} = \mathbf{q} + \mathbf{M}\mathbf{z} + \mathbf{d}z_0 \geq \mathbf{0}$$

$$z_0 \geq 0$$

# Lemke's algorithm

**augmented** LCP

$$z \geq \mathbf{0} \quad \perp \quad w = q + Mz + \mathbf{d}z_0 \geq \mathbf{0}$$

$$z_0 \geq 0$$

where

$\mathbf{d} > \mathbf{0}$ **covering vector**

$z_0$ extra variable

$z_0 = 0 \quad \Leftrightarrow \quad z \perp w$ solves original LCP

# Lemke's algorithm

**augmented** LCP

$$z \geq \mathbf{0} \quad \perp \quad w = q + Mz + \mathbf{d}z_0 \geq \mathbf{0}$$

$$z_0 \geq 0$$

**Initialization:**

$$\boxed{z = \mathbf{0} \quad \perp \quad w = q \qquad + \mathbf{d}z_0 \geq \mathbf{0}}$$

$z_0 \geq 0$ minimal $\Rightarrow$ $w_i = 0$ for some $\mathbf{i}$

**pivot** $z_0$ in, $w_i$ out,

$\Rightarrow$ can increase $z_i$ while maintaining $z \perp w$ .

# Lemke's algorithm for

$$M = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \quad d = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} z_1 + \begin{bmatrix} 1 \\ 3 \end{bmatrix} z_2 + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \boxed{z_0}$$

$$\begin{bmatrix} w_1 \\ z_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} z_1 + \begin{bmatrix} -5 \\ -3 \end{bmatrix} \boxed{z_2} + \begin{bmatrix} -2 \\ -1 \end{bmatrix} w_2$$

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} z_1 + \begin{bmatrix} 1 \\ 3 \end{bmatrix} z_2 + \begin{bmatrix} 2 \\ 1 \end{bmatrix} z_0$$

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$$\begin{bmatrix} z_2 \\ z_0 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} z_1 + \begin{bmatrix} -0.2 \\ 0.6 \end{bmatrix} w_1 + \begin{bmatrix} -0.4 \\ 0.2 \end{bmatrix} w_2$$

$$\begin{pmatrix} w_1 \\ z_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} z_1 + \begin{pmatrix} -5 \\ -3 \end{pmatrix} \boxed{z_2} + \begin{pmatrix} -2 \\ -1 \end{pmatrix} w_2$$

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# Polyhedral view of Lemke

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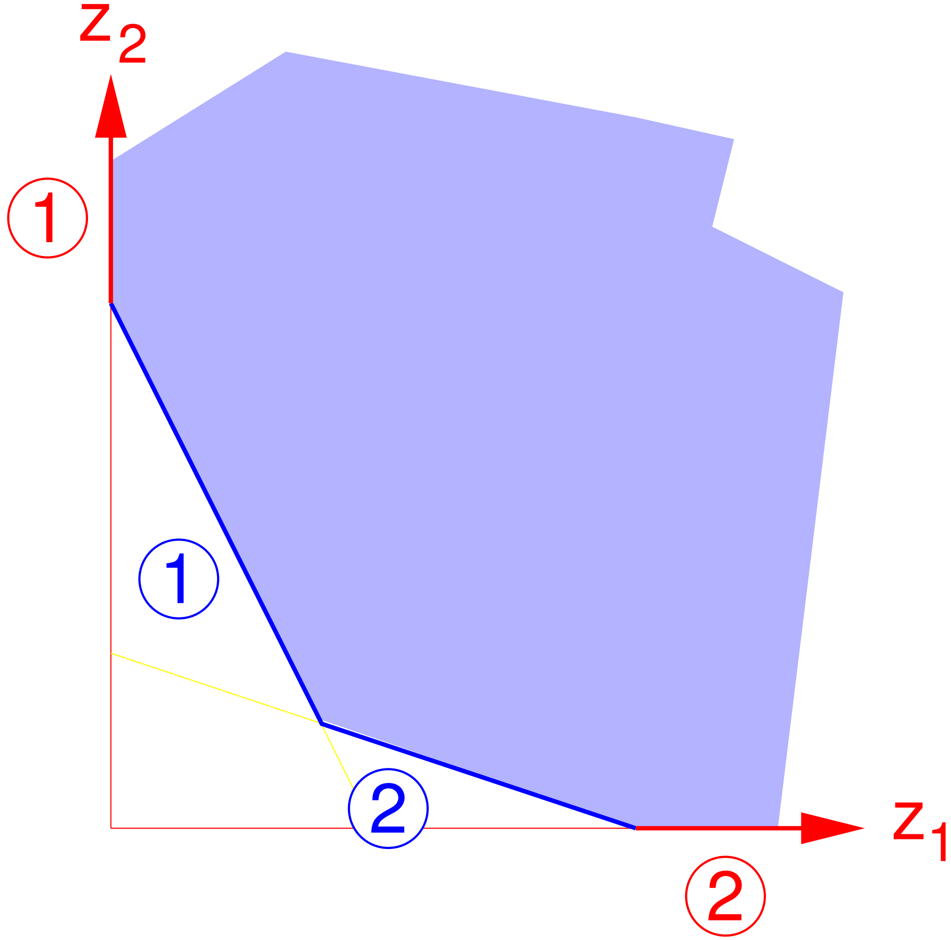

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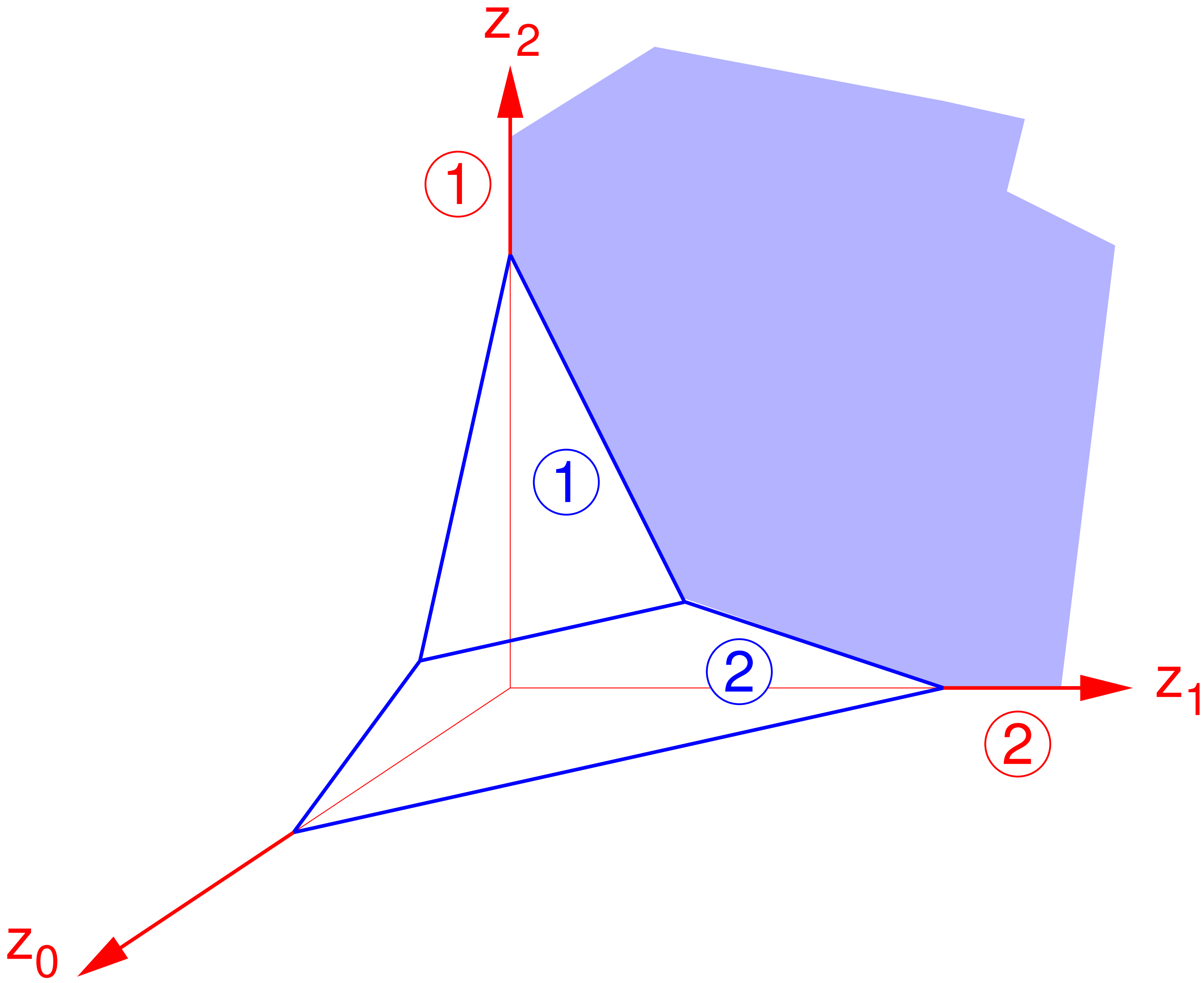

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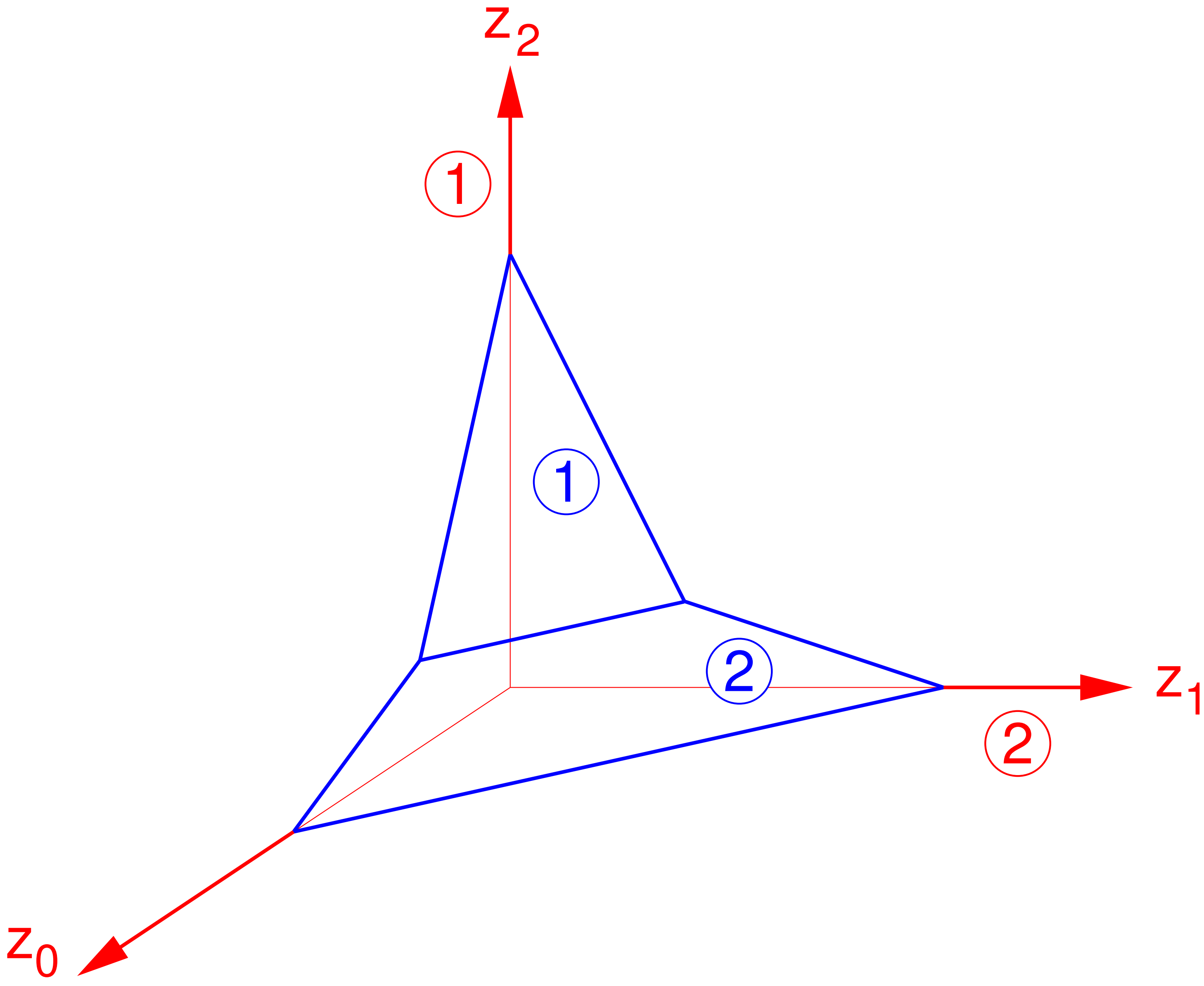

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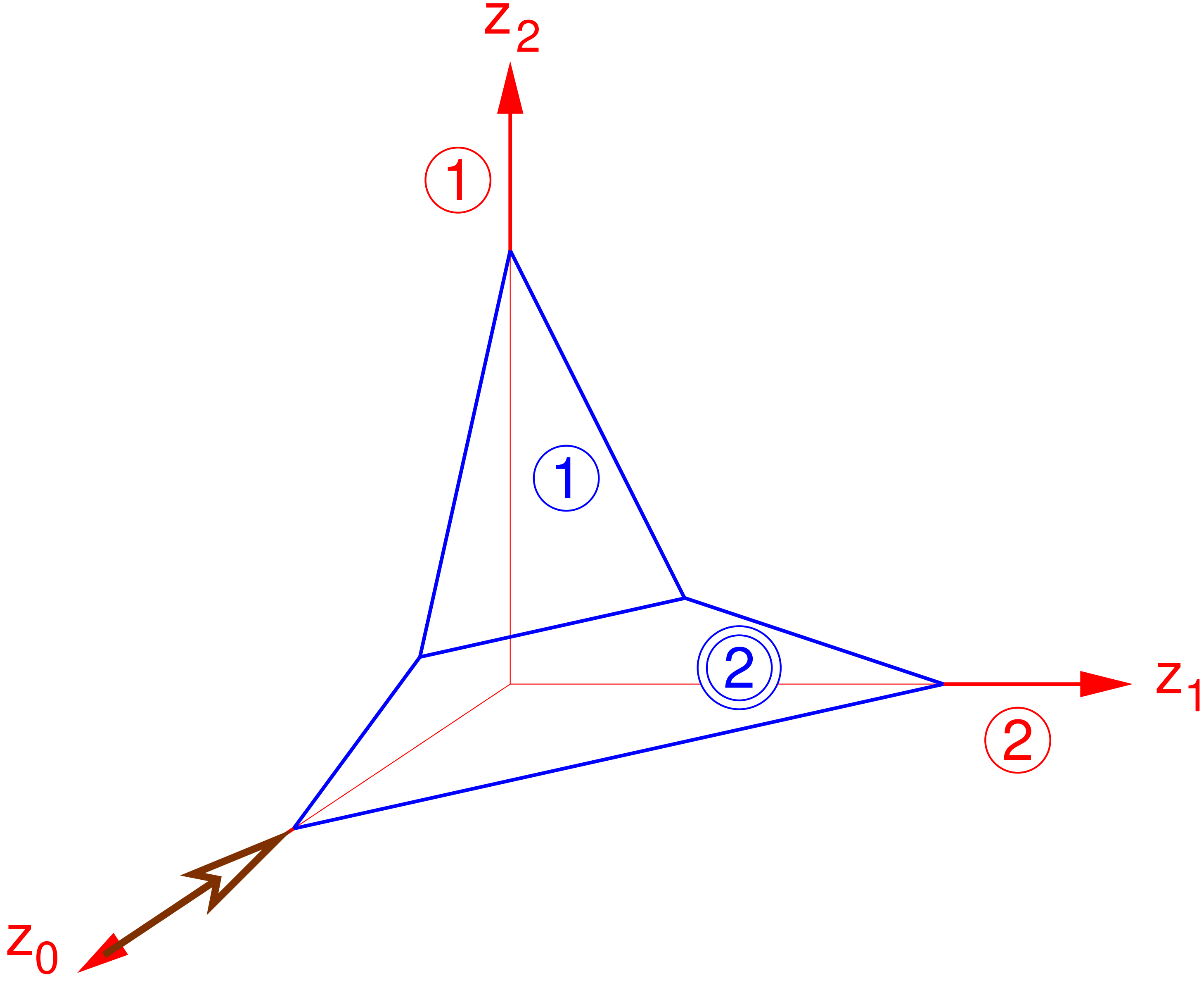
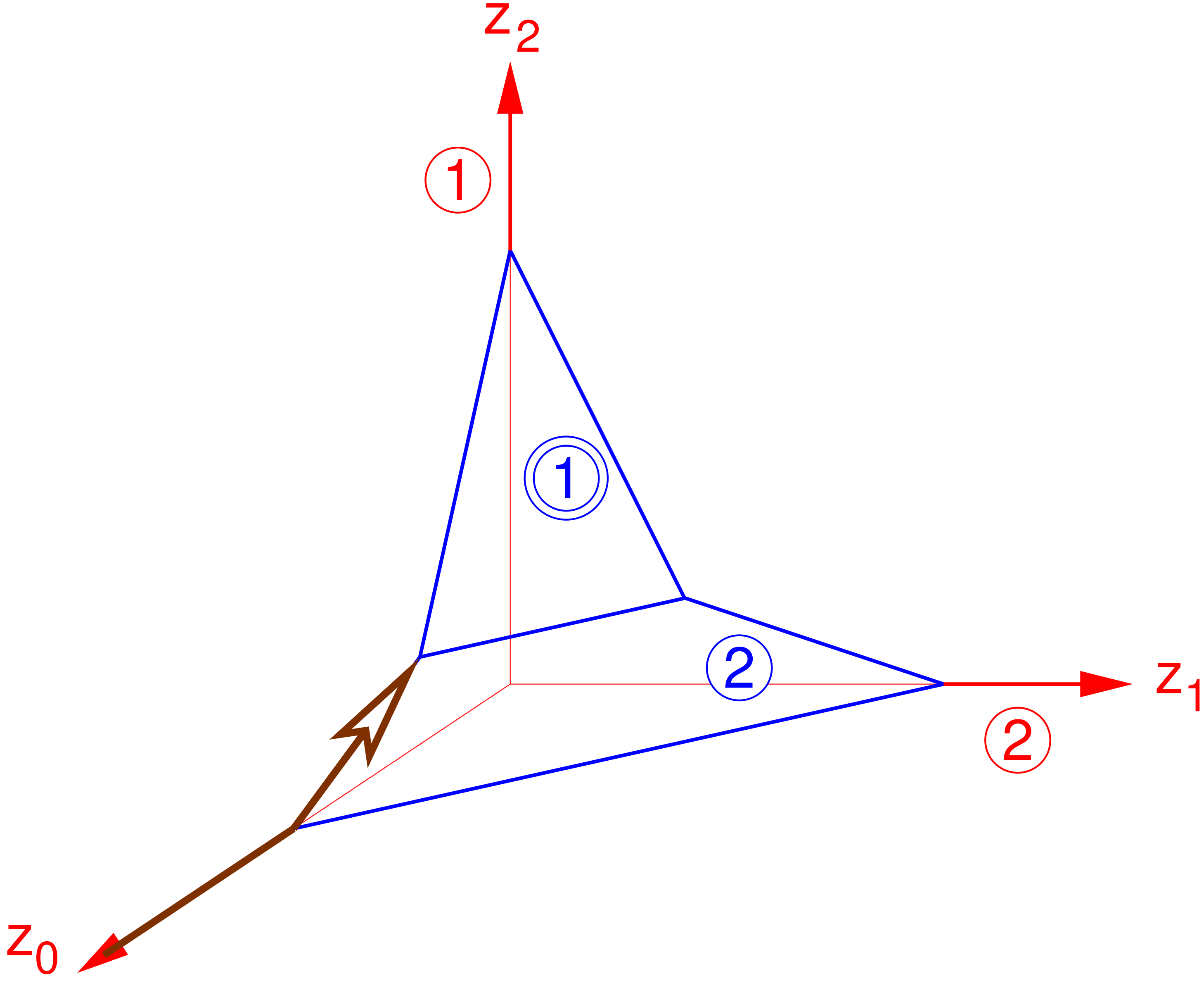
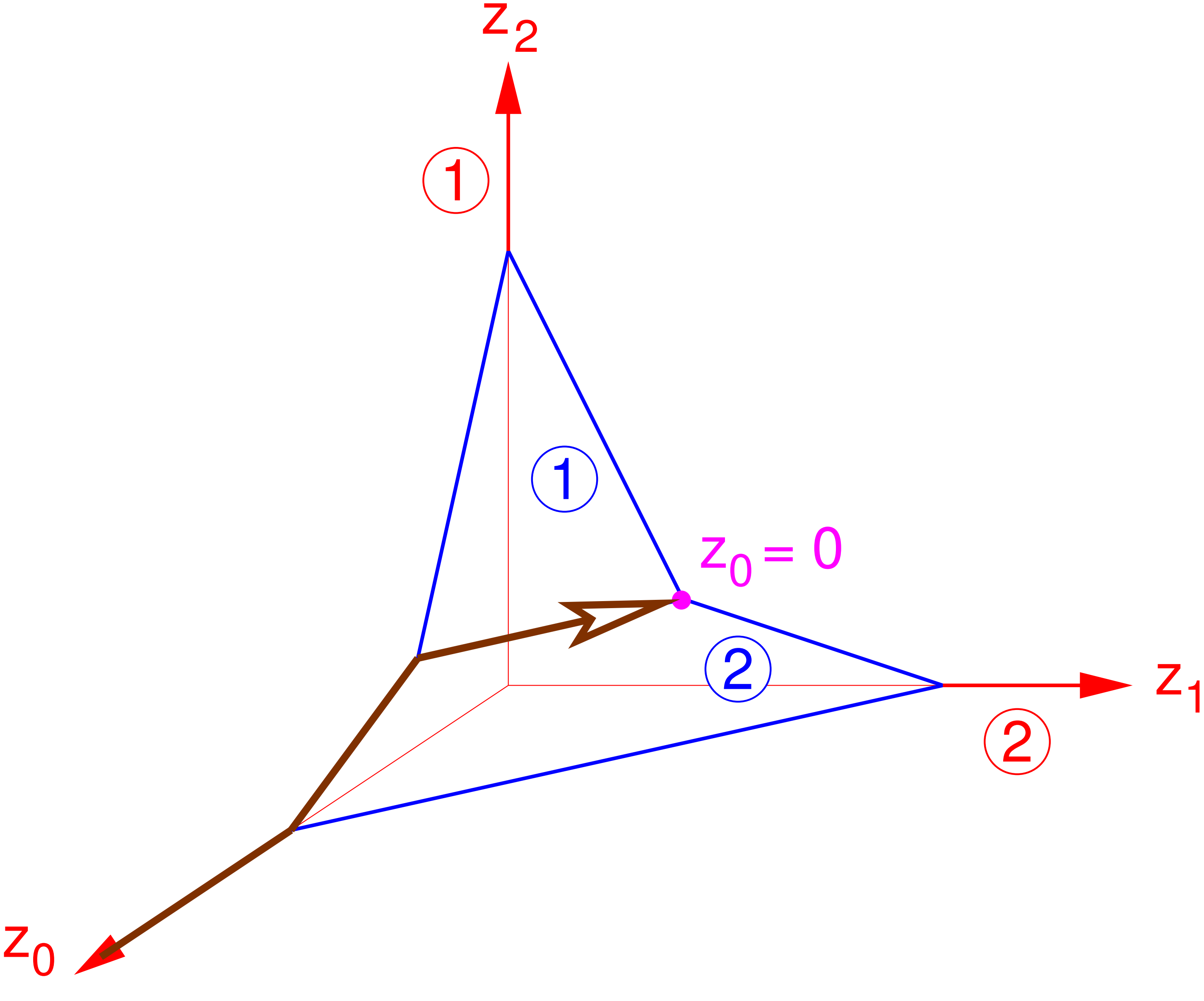


# Polyhedral view of Lemke



# Polyhedral view of Lemke



# Complementary cones

LCP $\quad z \geq \mathbf{0} \quad \perp \quad w = q + Mz \geq \mathbf{0}$

$\Leftrightarrow \quad z \geq \mathbf{0} \quad \perp \quad w \geq \mathbf{0}, \quad \boxed{-q = Mz - w}$

$\Leftrightarrow \quad -q$ belongs to a **complementary cone**:

$$\boxed{-q \in \mathbf{C}(\alpha) = \mathbf{cone}\,\{\, M_{\mathbf{i}}\,,\ -e_{\mathbf{j}} \mid \mathbf{i} \in \alpha,\ \mathbf{j} \notin \alpha \,\}}$$

for some $\alpha \subseteq \{1,\ldots,n\}$, $\quad M = [M_1\, M_2 \cdots M_n]$

$\alpha = \{\, \mathbf{i} \mid z_{\mathbf{i}} > 0 \,\}$

# Polyhedra versus cones

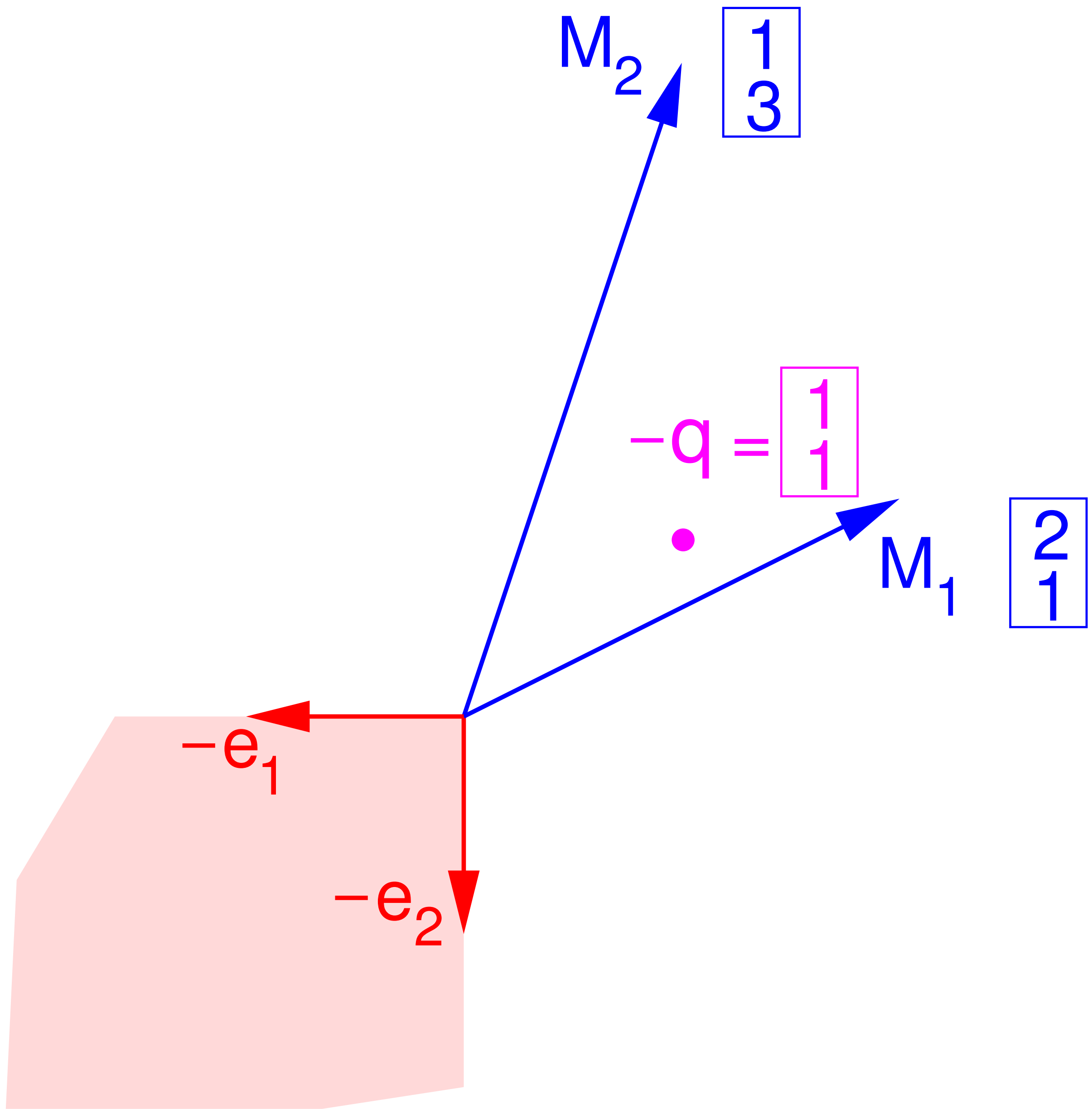
**polyhedral view :**

– gives feasibility,
want complementary **vertex**

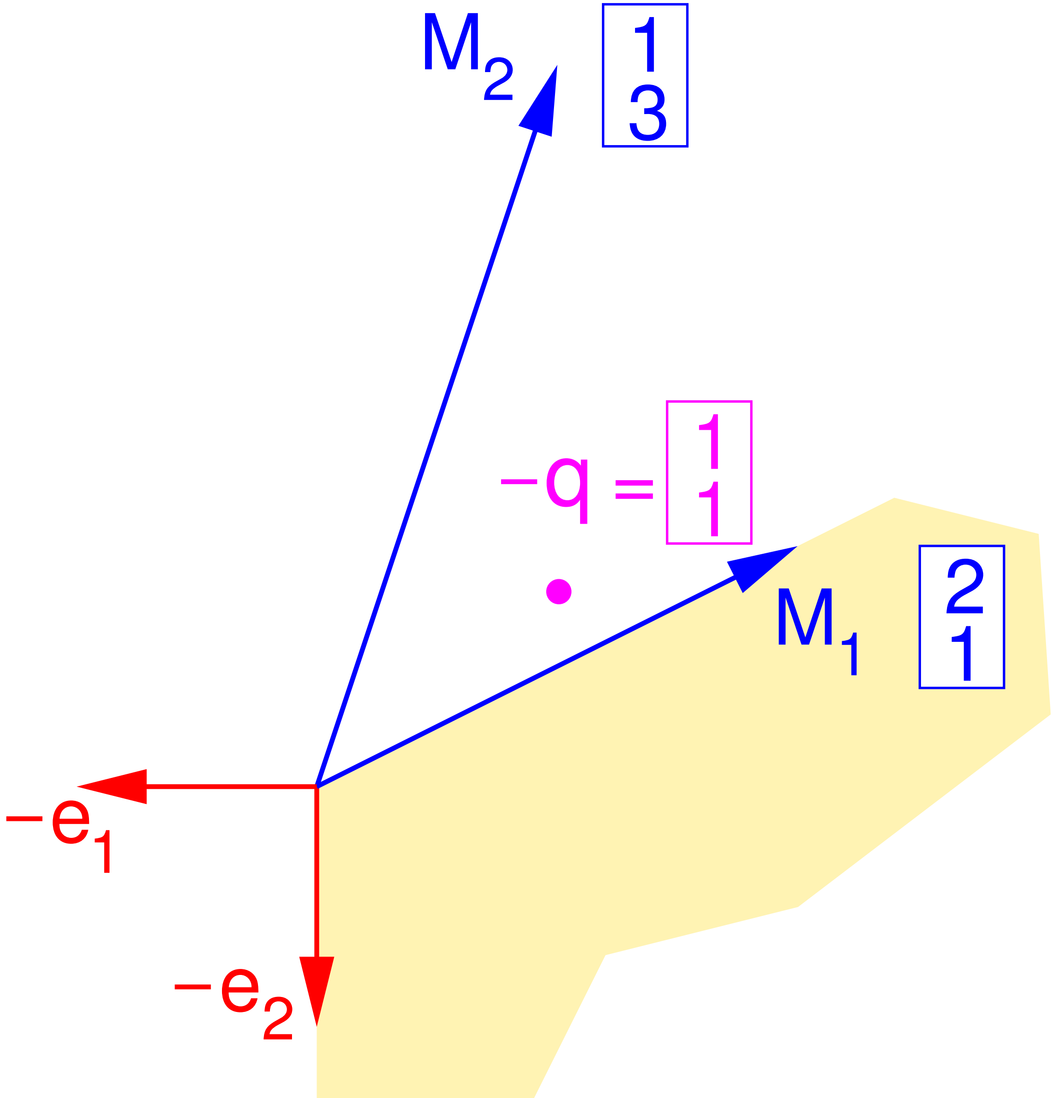
**complementary cones :**

– gives complementarity and feasibility,
want $\alpha$ giving **cone** **C**($\alpha$) containing $-q$

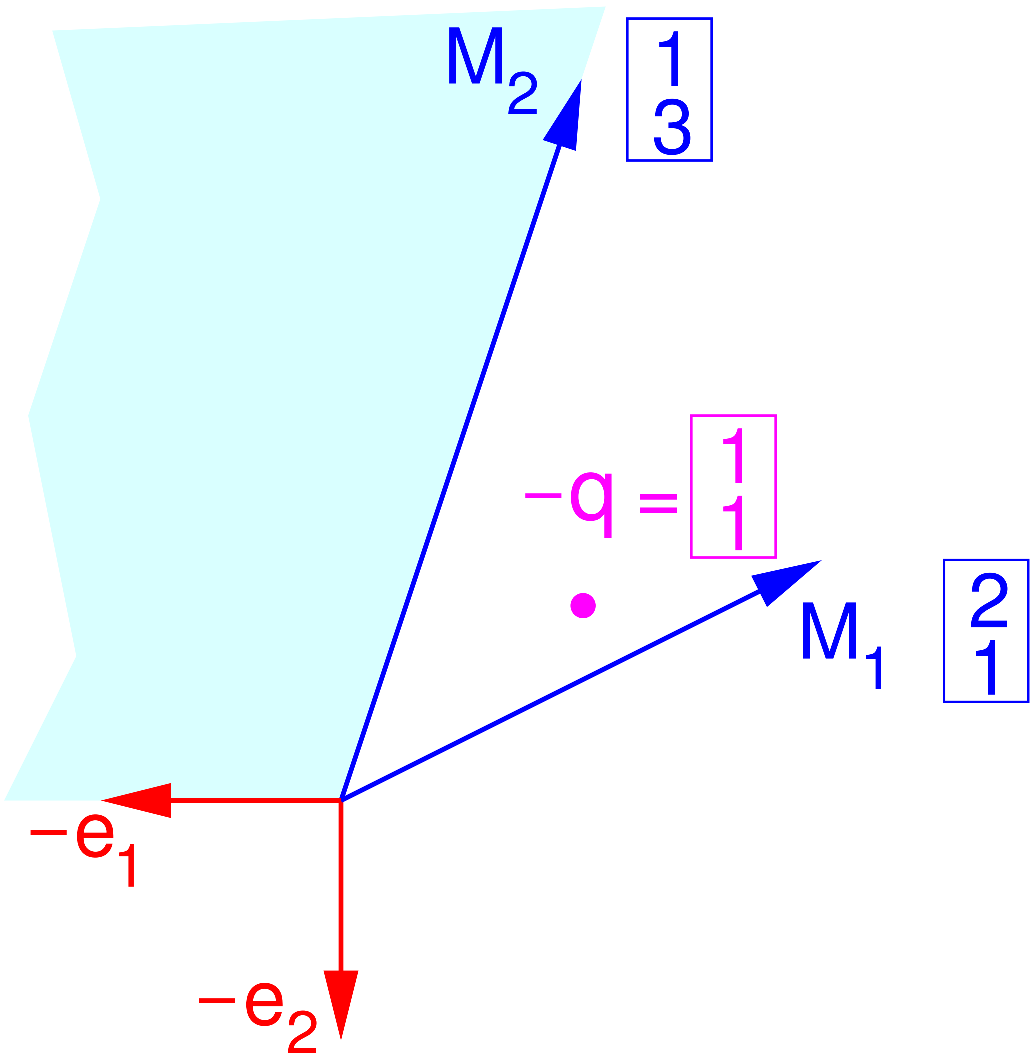
# Complementary cone C({})



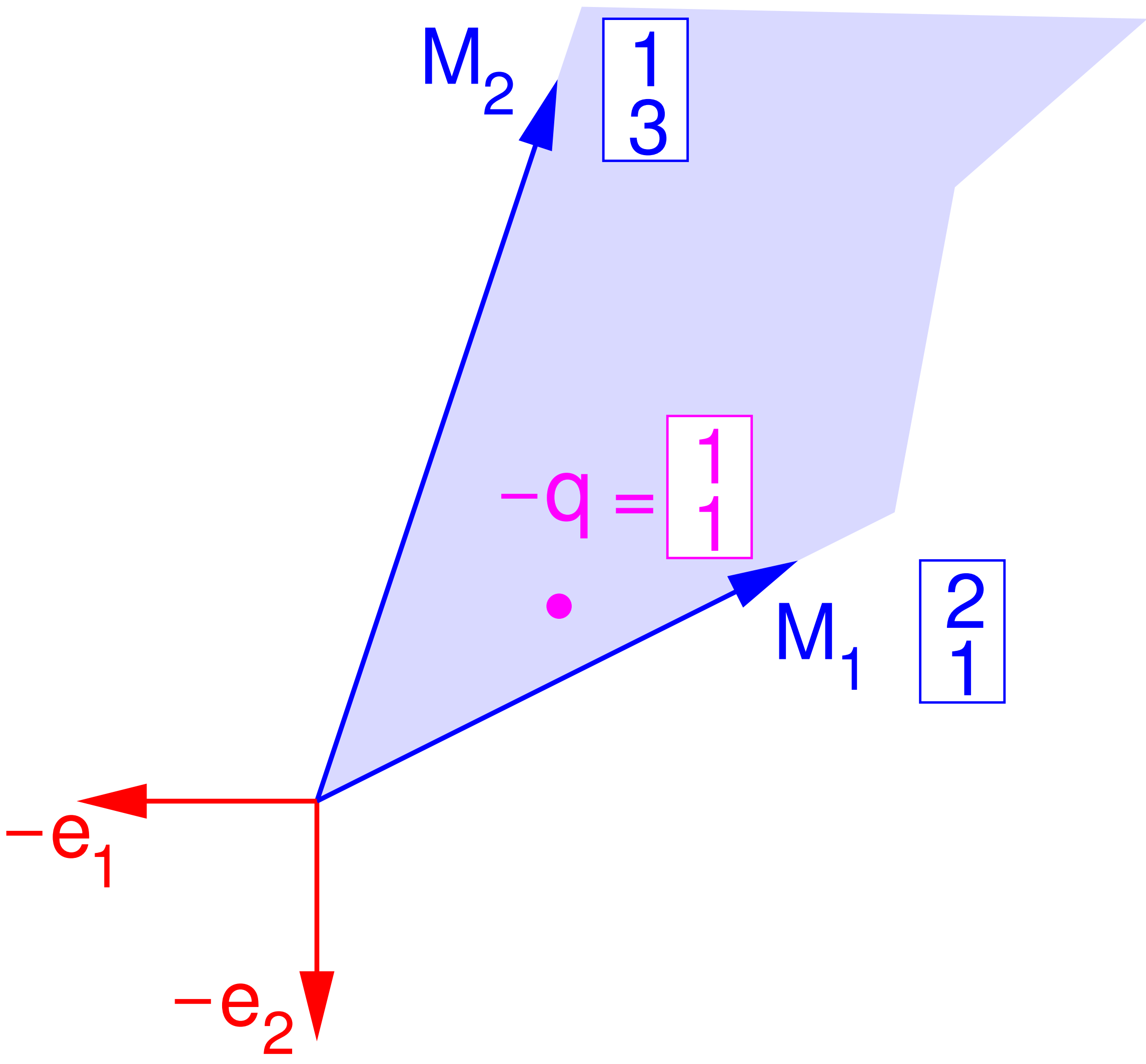
# Complementary cone C({1})



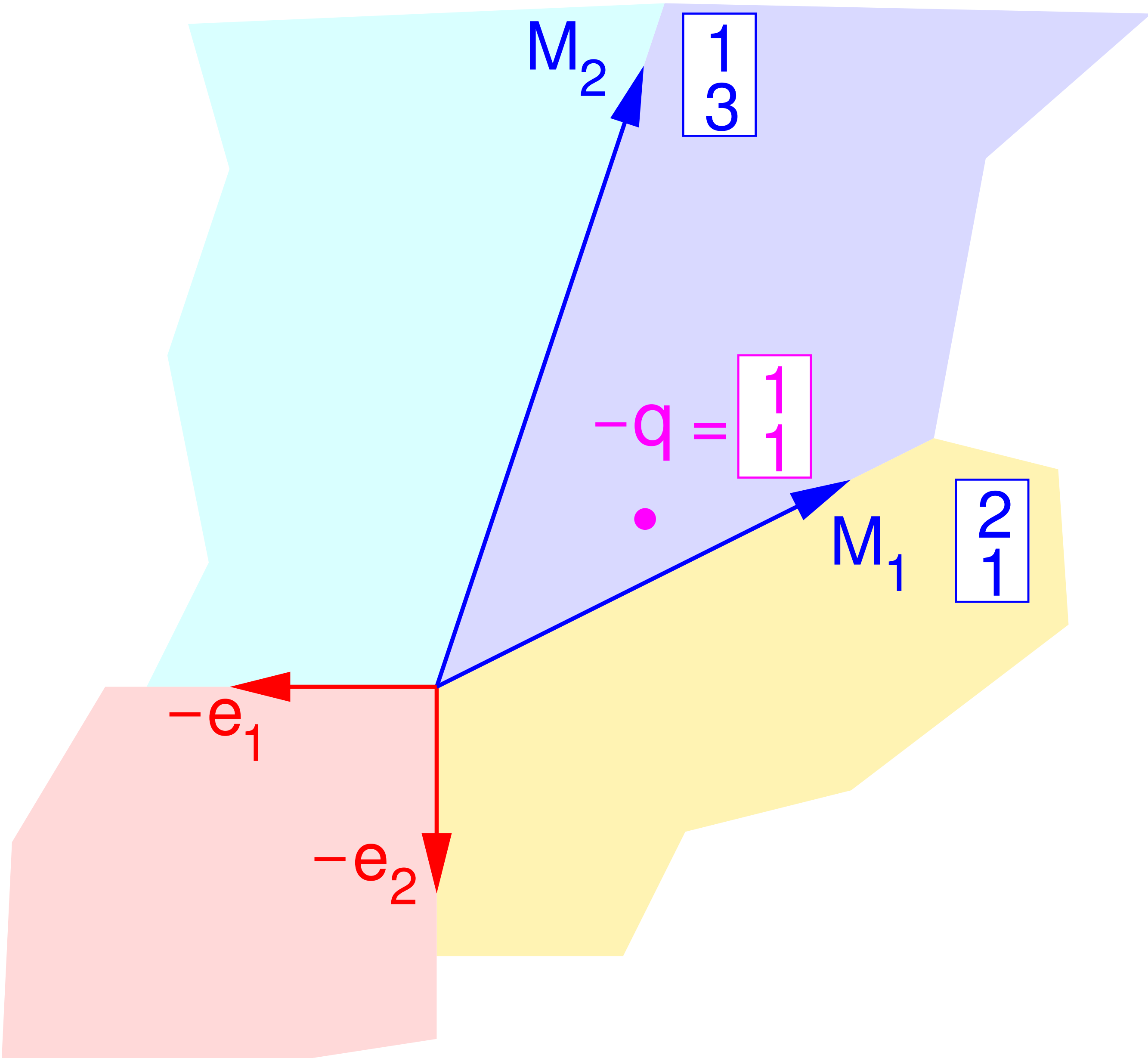
# Complementary cone C({2})



## Complementary cone C({1,2})



# All complementary cones



# LCP map

Let $\alpha \subseteq \{1,\ldots,n\}$ ,

**$\alpha$-orthant** $= \mathbf{cone}\ \{ e_{\mathbf{i}} ,\ -e_{\mathbf{j}} \mid \mathbf{i} \in \alpha,\ \mathbf{j} \notin \alpha \}$ ,

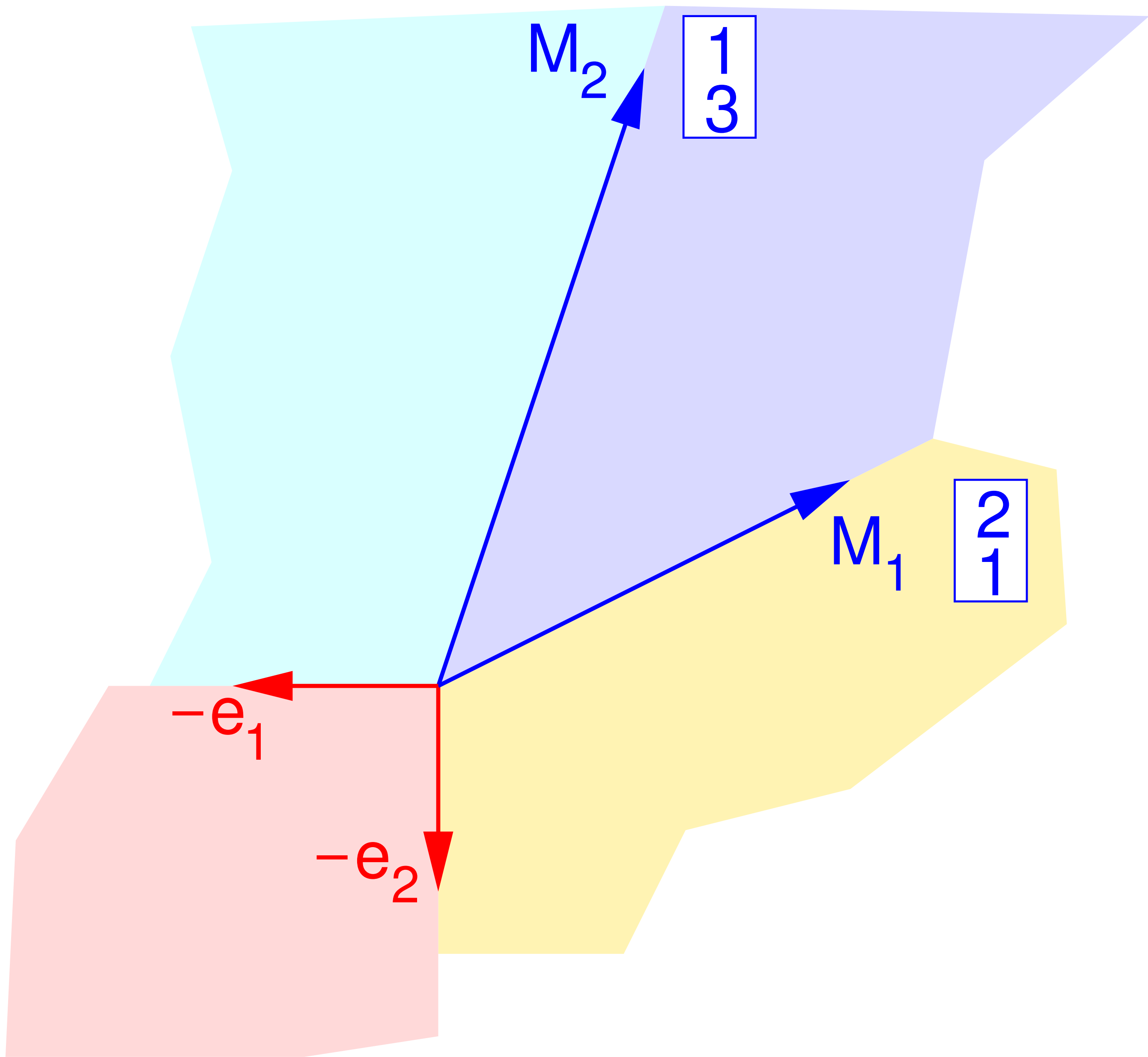
**C($\alpha$)** $= \mathbf{cone}\ \{ M_{\mathbf{i}} ,\ -e_{\mathbf{j}} \mid \mathbf{i} \in \alpha,\ \mathbf{j} \notin \alpha \}$ ,

$x_{\mathbf{i}}^{+} = \max ( x_{\mathbf{i}} , 0)$, $x_{\mathbf{i}}^{-} = \min ( x_{\mathbf{i}} , 0)$

**LCP map:** $\boxed{F(x) = Mx^{+} + x^{-}}$

so $\boxed{F(\alpha\text{-orthant}) = \mathbf{C}(\alpha)}$

## Bijective LCP map F



# P-matrix

**P-matrix**

$\Leftrightarrow$ every **principal minor** is positive:

$\det(M_{\alpha\alpha}) > 0$ for all $\alpha \subseteq \{1,...,n\}$

e.g.

$$\begin{array}{|cc|} 2 & 1 \\ 1 & 3 \end{array}$$

$\det(M_{1,1}) = 2 > 0$

$\det(M_{2,2}) = 3 > 0$

$\det(M_{12,12}) = \det(M) = 5 > 0$

# P-matrix

**P-matrix**

$\Leftrightarrow$ every **principal minor** is positive:

$\det(M_{\alpha\alpha}) > 0$ for all $\alpha \subseteq \{1,\dots,n\}$

**P-matrix**

$\Leftrightarrow$ **F** bijective

$\Leftrightarrow$ $\forall\, q \in \mathbf{R}^n \quad \exists!\, z \quad$ s.t. $\quad z \geq \mathbf{0} \quad \perp \quad Mz \geq -q$

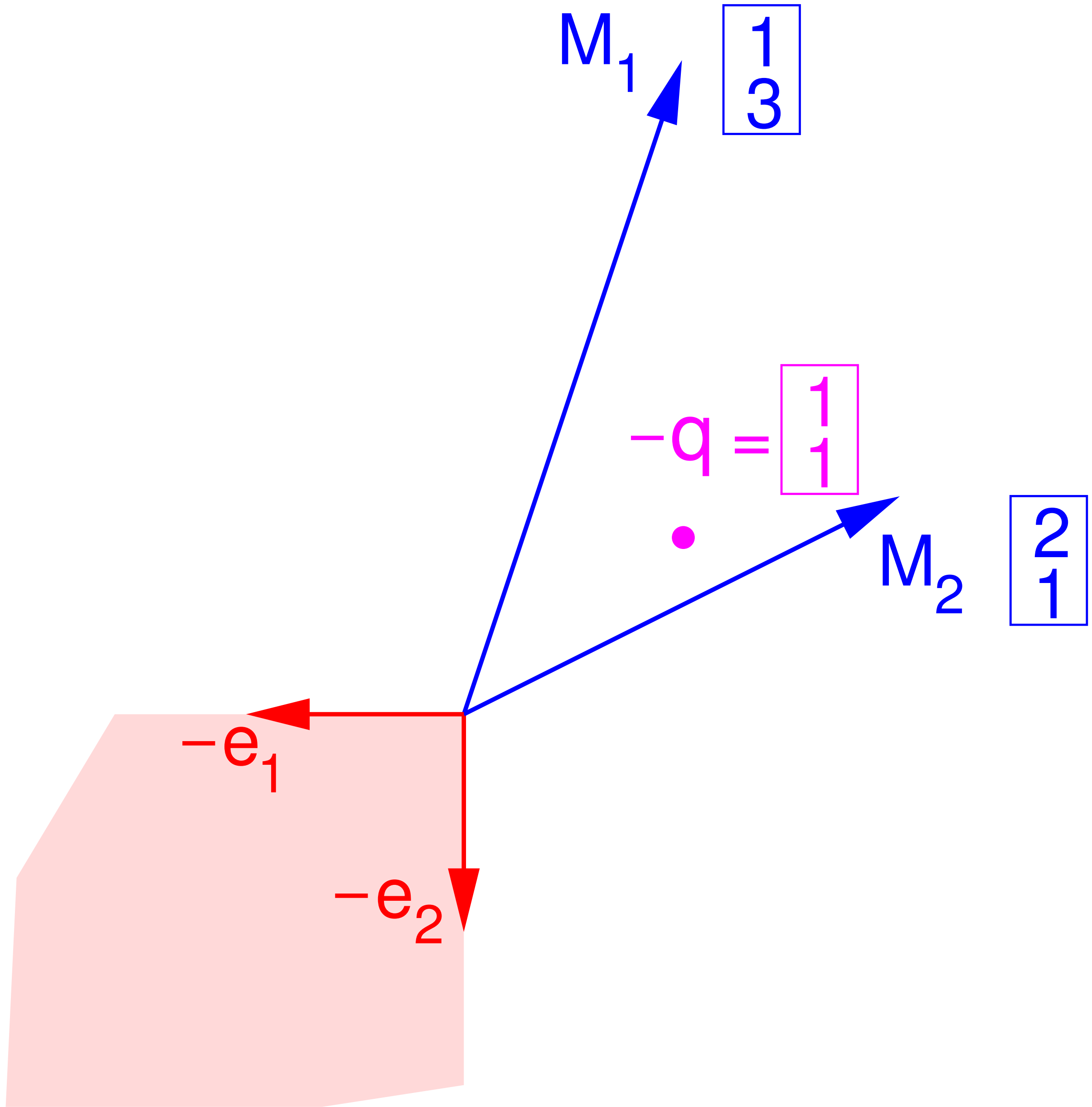
# Not a P-matrix

**Example:**

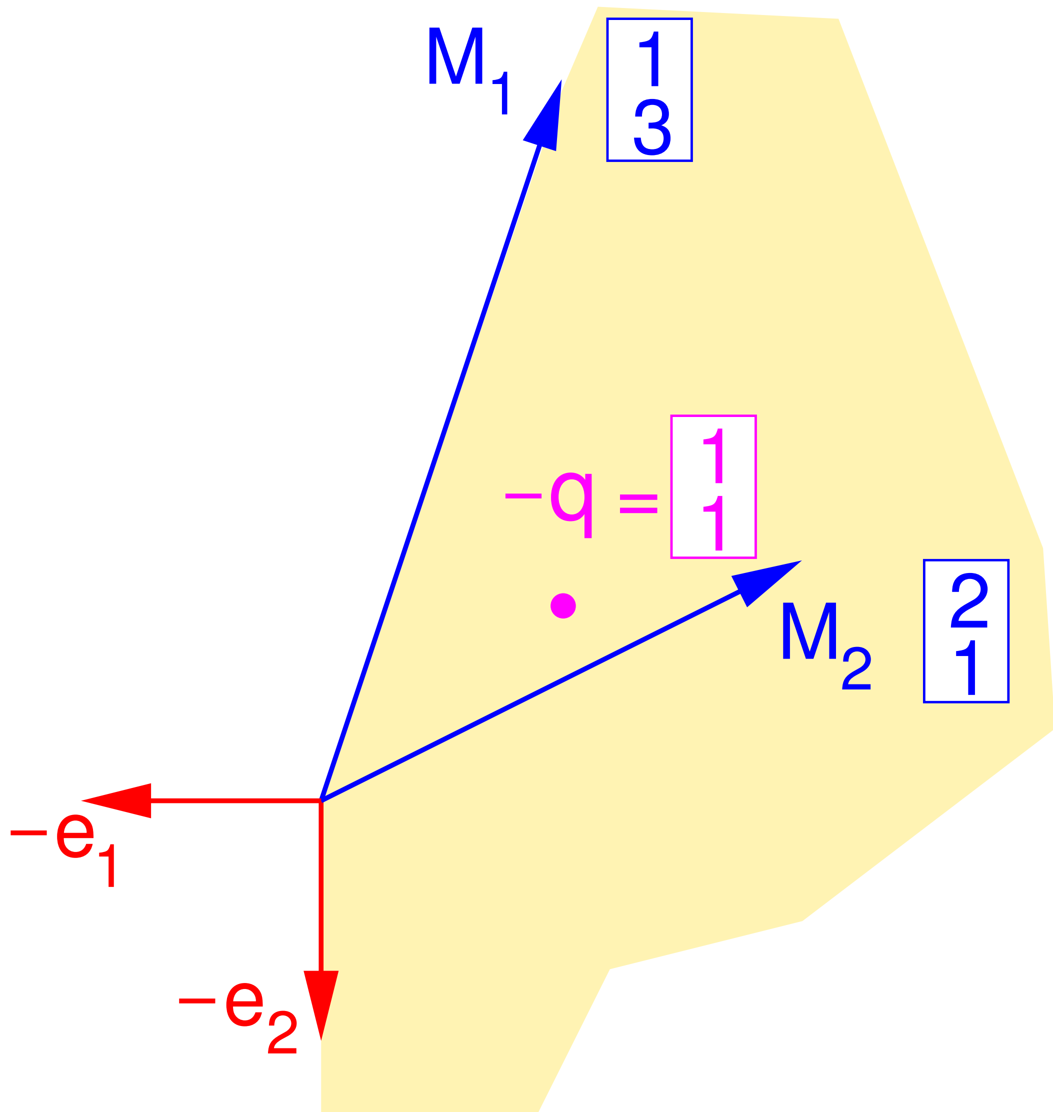
| | |
|---|---|
| 1 | 2 |
| 3 | 1 |

$\det(M_{12,12}) = \det(M) = -5 < 0$

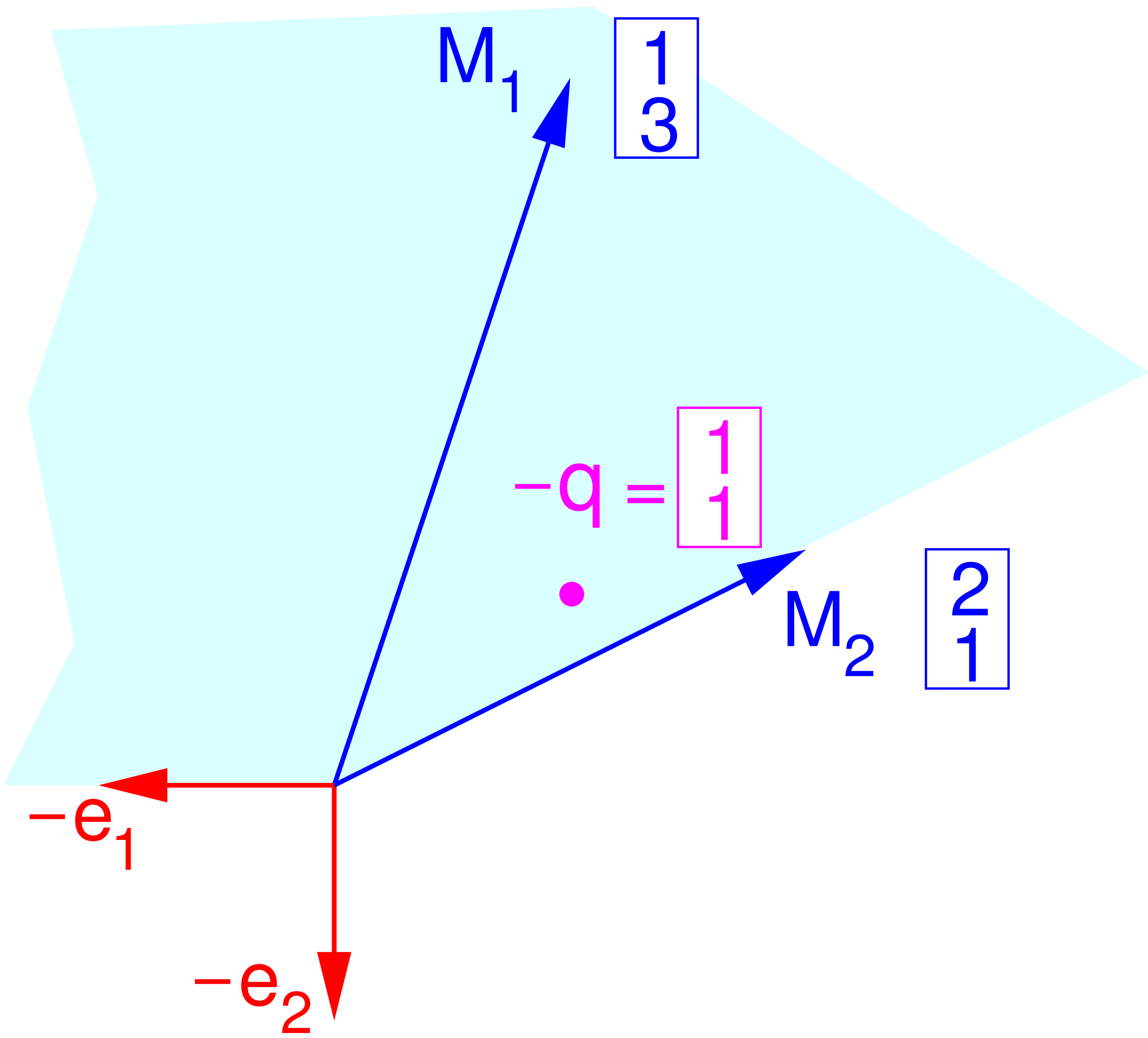
# Complementary cone C({})



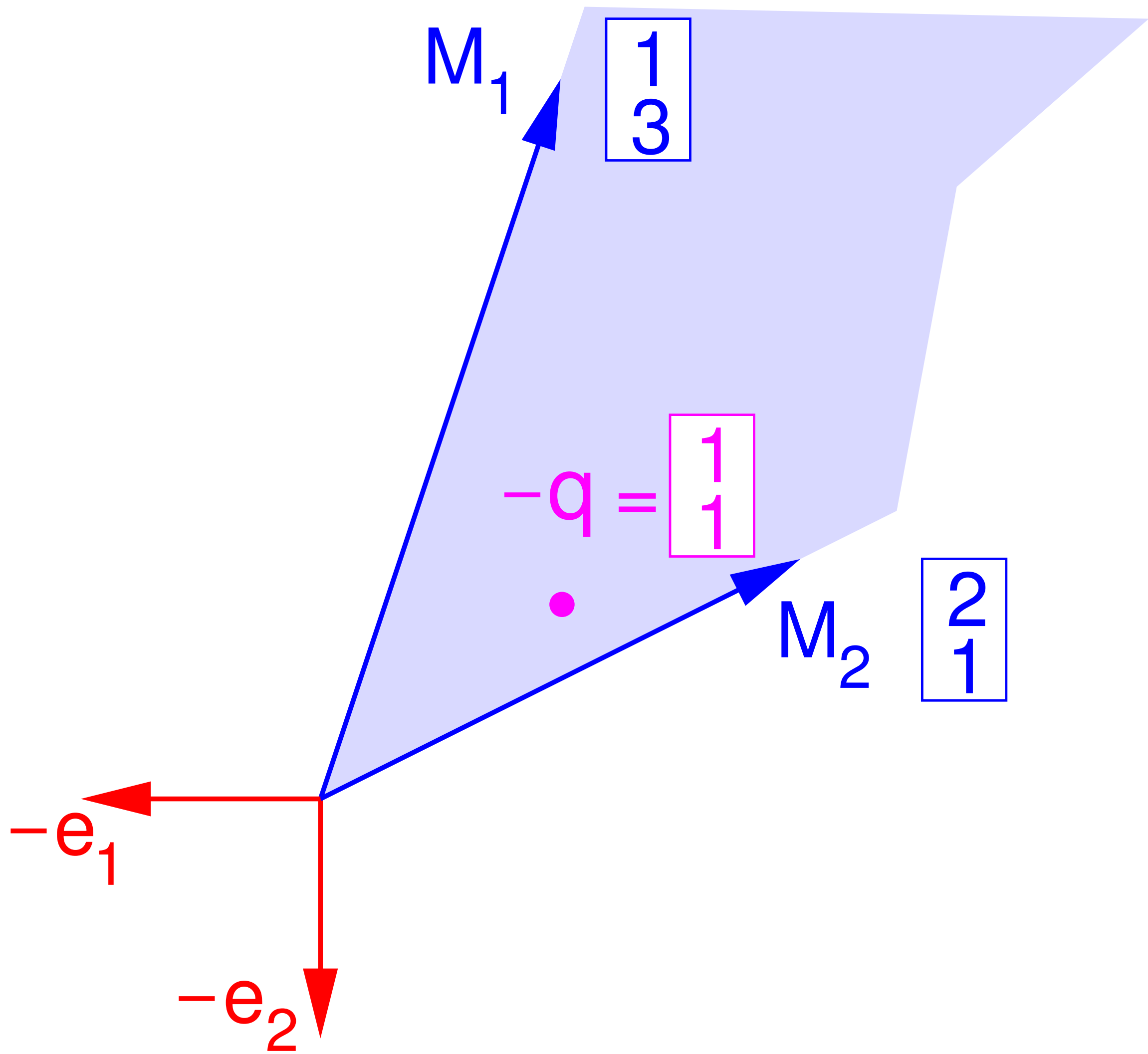
# Complementary cone C({1})



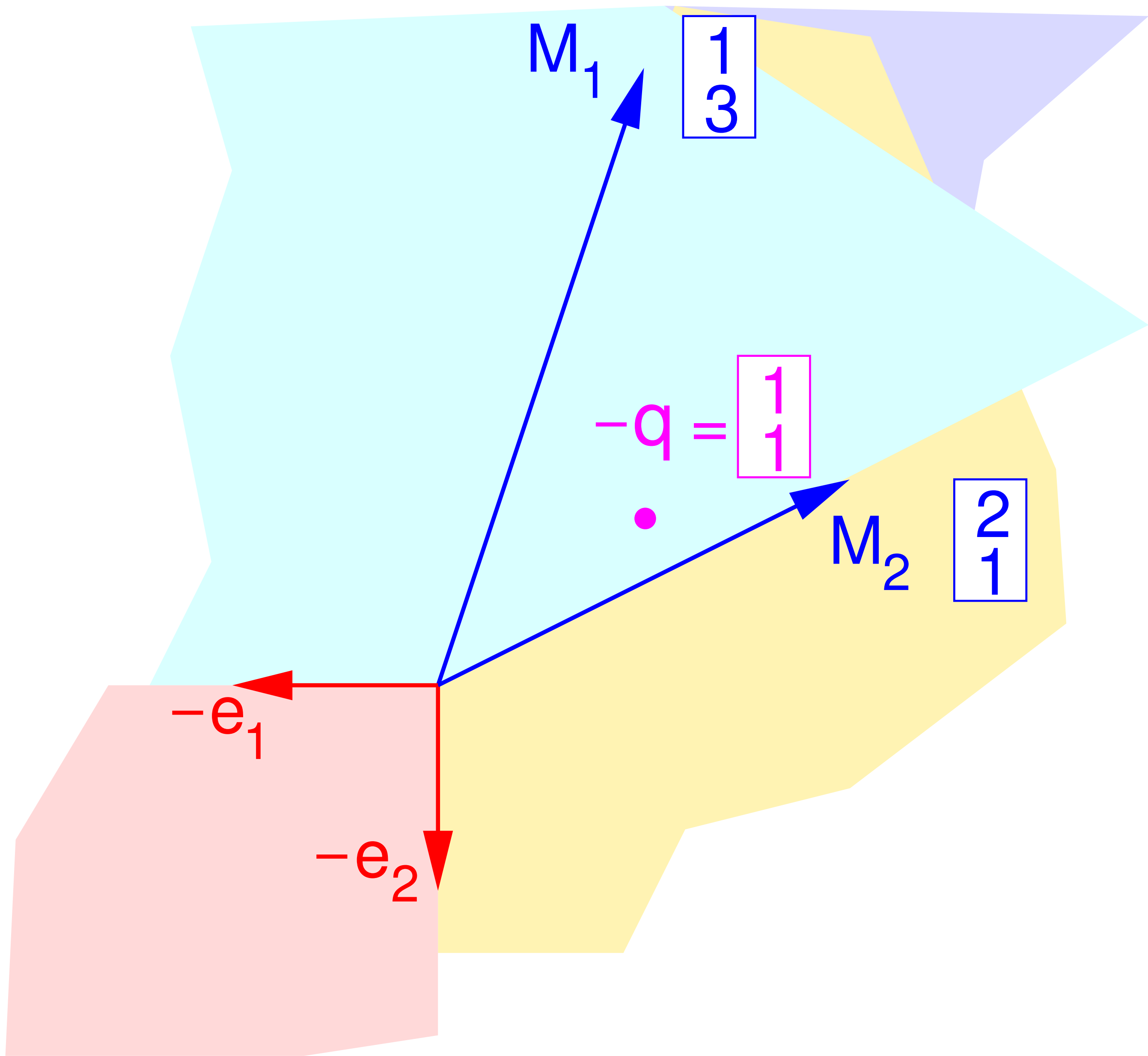
# Complementary cone C({2})



# Complementary cone C({1,2})



# Non-injective LCP map F



# F is surjective for M > 0

Given: $p \in \mathbf{R}^n$.

Claim: $$\boxed{\exists\, x : \; F(x) = Mx^+ + x^- = p}$$

# Proof (solving $F(x) = p$ )

Let $p \in \mathbf{R}^n$, $\alpha = \{ i \mid p_i > 0 \}$.

**Step 1.** Consider only rows $i \in \alpha$. Solution $x^+$ to

$$\forall\, i \in \alpha \qquad x_i \quad \perp \quad \sum_{j \in \alpha} m_{ij}\, x_j \geq p_i$$

# Proof (solving $F(x) = p$ )

Let $p \in \mathbf{R}^n$, $\alpha = \{ i \mid p_i > 0 \}$.

**Step 1.** Consider only rows $i \in \alpha$. Solution $x^+$ to

$$\forall\, i \in \alpha \qquad x_i \quad \perp \quad \sum_{j \in \alpha} (m_{ij} / p_i)\, x_j \geq 1$$

exists as Nash equilibrium (game matrix $m_{ij} / p_i$ ).

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**Step 1.** Consider only rows $i \in \alpha$. Solution $x^+$ to

$$\forall\, i \in \alpha \qquad x_i \quad \perp \quad \sum_{j \in \alpha} (m_{ij} / p_i)\, x_j \geq 1$$

exists as Nash equilibrium (game matrix $m_{ij} / p_i$ ).

**Step 2.** $\forall\, k \notin \alpha$ choose $-x_k^- = w_k \geq 0$ so that

$$\sum_{j \in \alpha} m_{kj}\, x_j^+ - w_k = p_k \ (\leq 0).$$

Gives $\boxed{F(x) = p.}$

# Lemke via complementary cones

Invert the piecewise linear map F($x$) along the line segment $[\,-d, -q\,]$ :

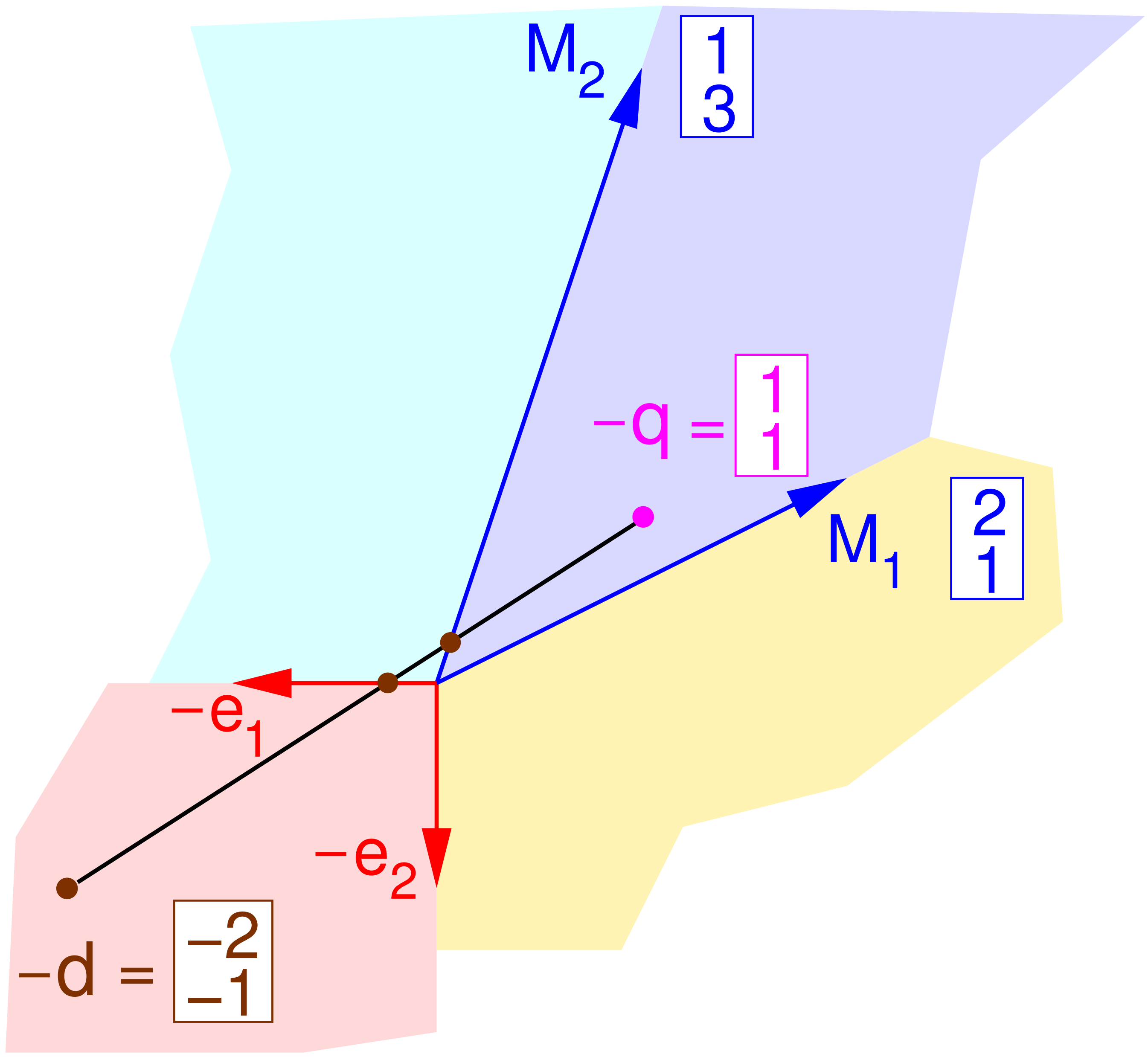
$$\boxed{F(x) = Mx^+ + x^- = (-d)(1-\mathbf{t}) + (-q)\mathbf{t}} \qquad (0 \le \mathbf{t} \le 1)$$

$\mathbf{t} > 0$:

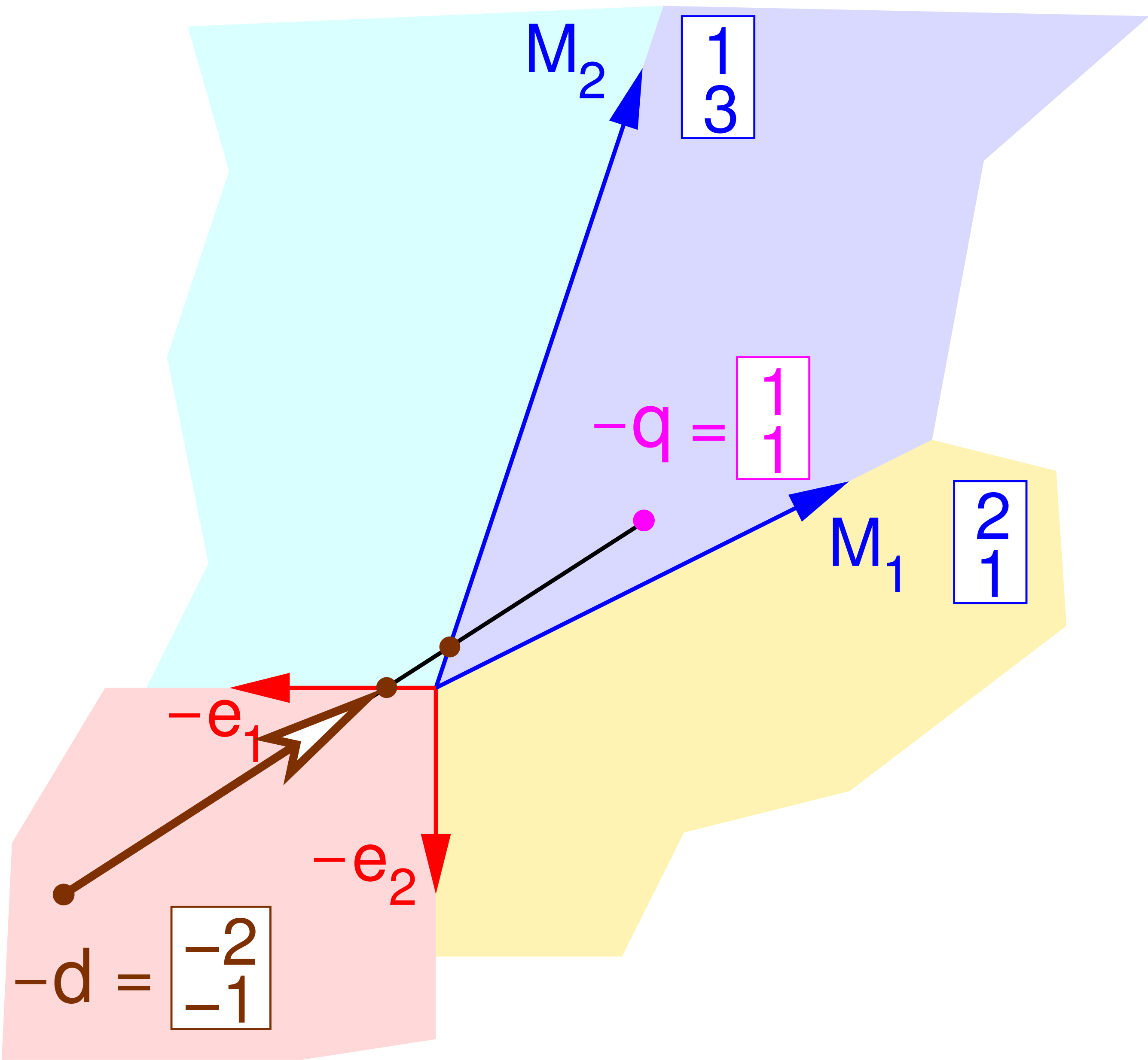
$$\Leftrightarrow \qquad Mx^+(1/\mathbf{t}) + x^-(1/\mathbf{t}) = (-d)(1-\mathbf{t})/\mathbf{t} + (-q)$$

$$\Leftrightarrow \qquad \boxed{Mz - w = (-d)z_0 + (-q)}\,, \qquad z \ge 0 \perp w \ge 0\,.$$

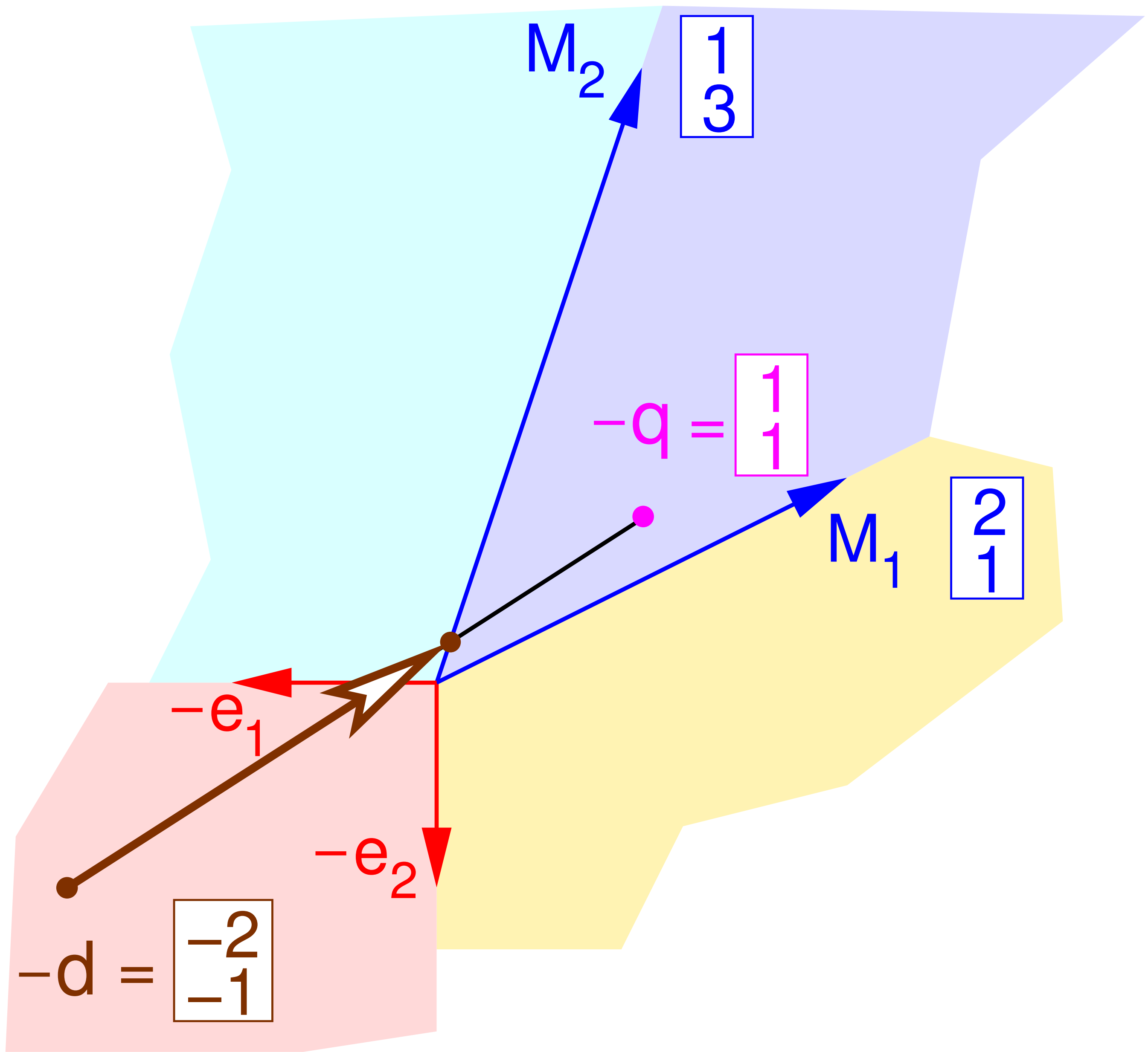
# Inverting the LCP map F



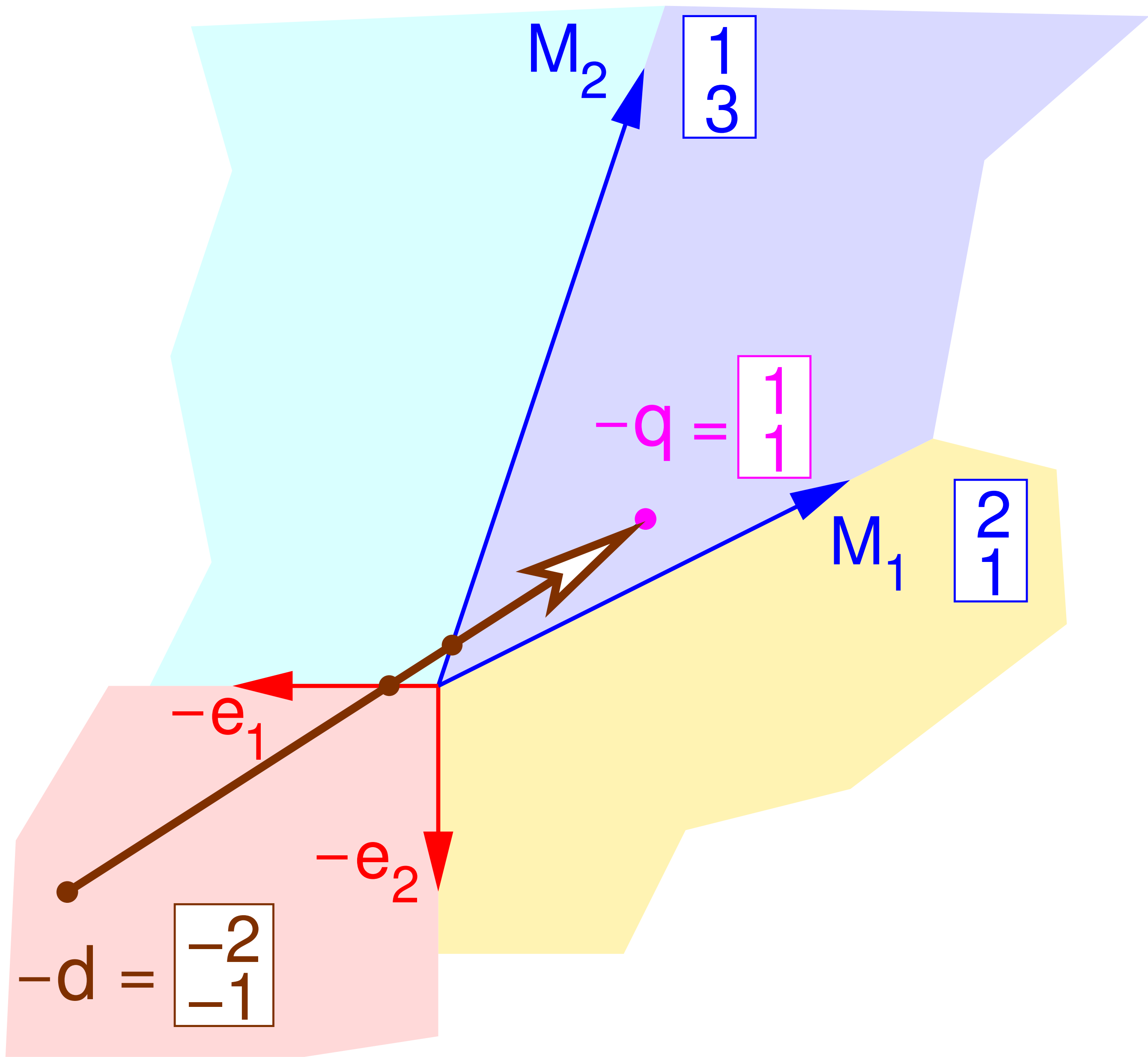
# Inverting the LCP map F



# Inverting the LCP map F



# Inverting the LCP map F



# Lemke-Howson: $-d$ = unit vector

**Theorem:**

|  |  |
|---|---|
|  | Symmetric Lemke-Howson with missing label $k$ |
| = | Lemke started at $-d = e_{\mathbf{k}}$ in cone $\mathbf{C}(\{k\})$ |

**Proof:**

- initialize by pivoting $z_0$ in, $w_{\mathbf{k}}$ out (still infeasible!), $w_{\mathbf{k}}$ stays in negative unit column
- pivot $z_{\mathbf{k}}$ in (note $M_{\mathbf{k}} > \mathbf{0}$), gives start in cone $\mathbf{C}(\{k\})$

# Example with missing label 1

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} z_1 + \begin{pmatrix} 1 \\ 3 \end{pmatrix} z_2 + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \boxed{z_0}$$

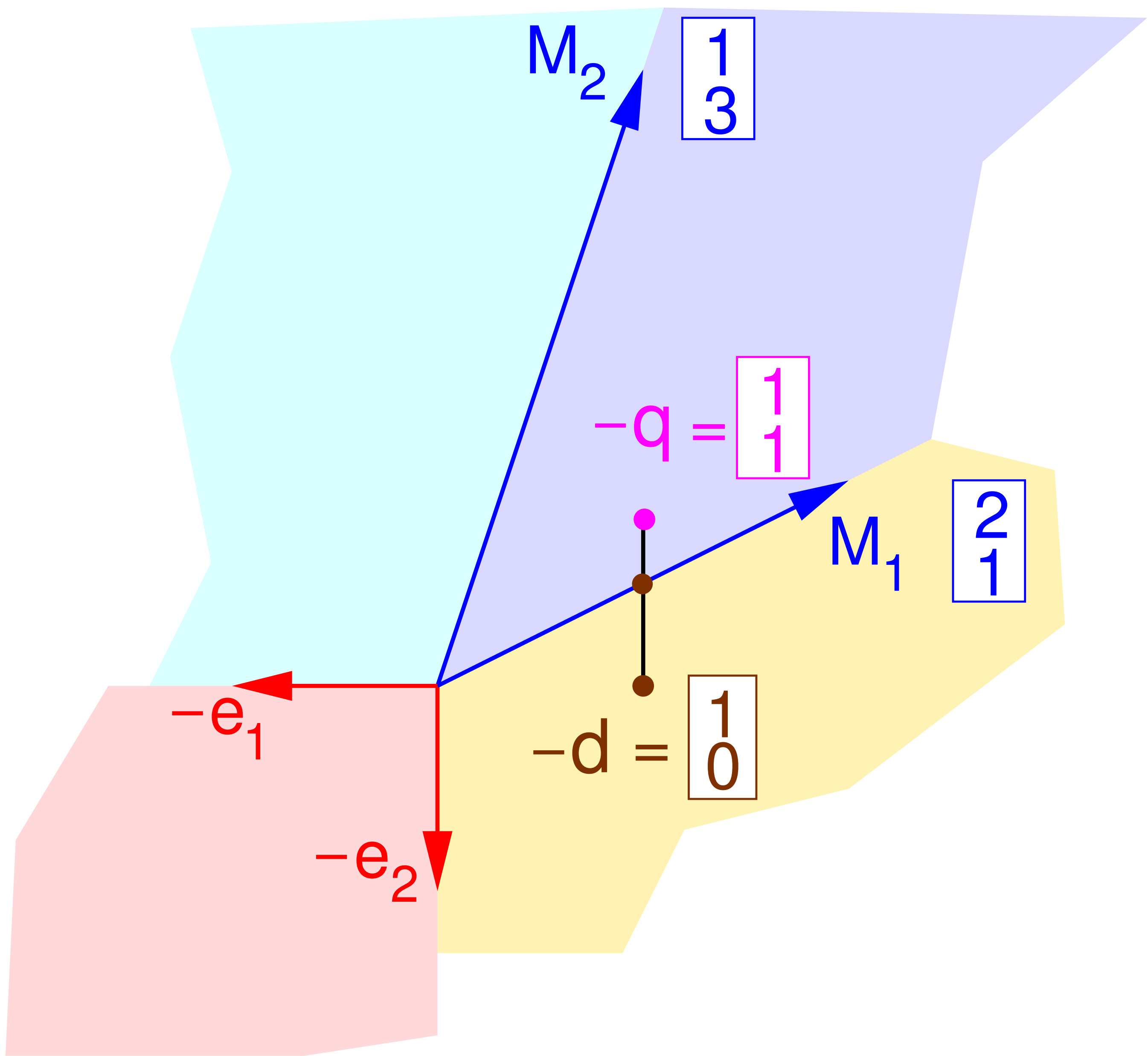
$$\begin{pmatrix} z_0 \\ w_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \boxed{z_1} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} z_2 + \begin{pmatrix} -1 \\ 0 \end{pmatrix} w_1$$

$$\begin{bmatrix} z_0 \\ w_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} z_1 + \begin{bmatrix} 1 \\ 3 \end{bmatrix} z_2 + \begin{bmatrix} -1 \\ 0 \end{bmatrix} w_1$$

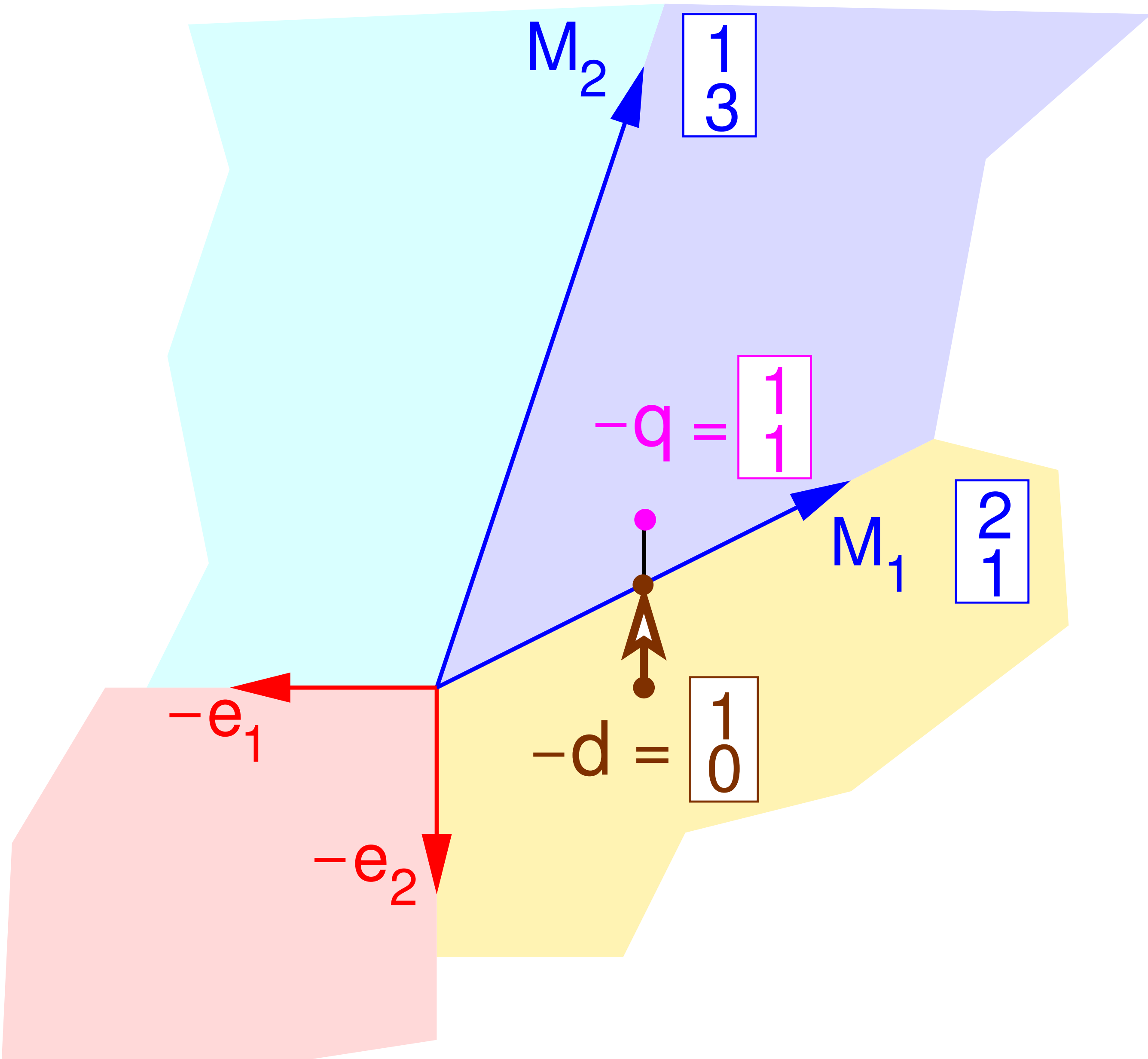
$$\begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} w_2 + \begin{bmatrix} -5 \\ -3 \end{bmatrix} z_2 + \begin{bmatrix} -1 \\ 0 \end{bmatrix} w_1$$

$$\begin{bmatrix} z_2 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix} + \begin{bmatrix} 0.4 \\ -0.2 \end{bmatrix} w_2 + \begin{bmatrix} -0.2 \\ 0.6 \end{bmatrix} z_0 + \begin{bmatrix} -0.2 \\ 0 \end{bmatrix} w_1$$

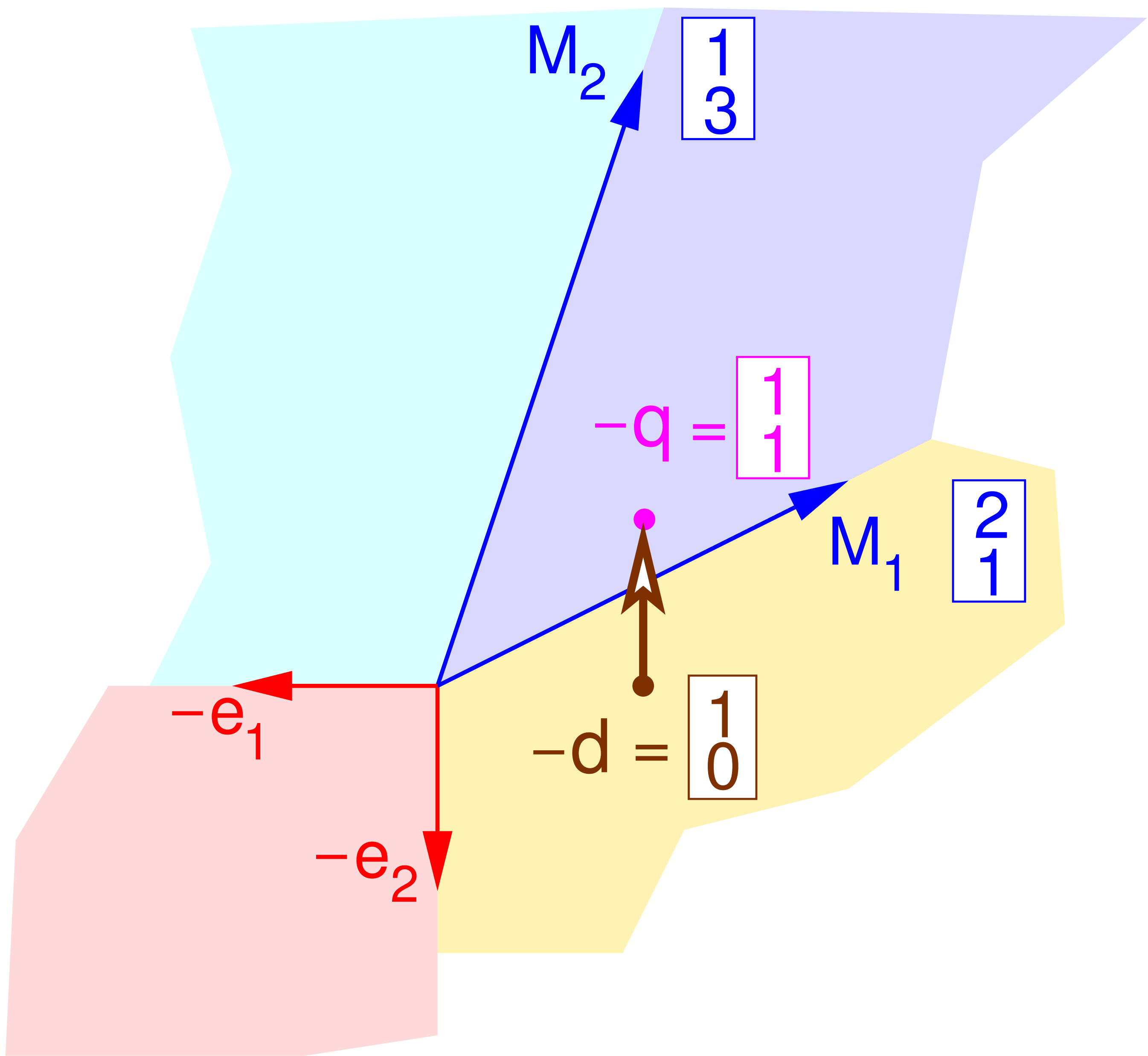
# Start at unit vector



# Start at unit vector



# Start at unit vector



# Complexity

A result of **Morris** implies that the symmetric LH can be best-case **exponential** (i.e., for **any** missing label).

**Savani & von Stengel** showed that for bimatrix games LH can be best-case **exponential** (i.e., for **any** missing label).

**Murty** and **Goldfarb** (independently):
Lemke's algorithm derived from an **LP** can be **exponential** for the specific covering vectors $(0,\dots,0,1,\dots,1)^T$ resp. $(1,\dots,1,0,\dots,0)^T$.

**Megiddo**: Lemke for **random** M (not > **0**) has **expected**

- **exponential** running time when $\mathbf{d} = (1,1,\dots,1)^T$
- **quadratic** running time when $\mathbf{d} = (\varepsilon, \varepsilon^2,\dots, \varepsilon^n)^T$.

# Starting Lemke anywhere

F is surjective for $M > 0$ .

So we can use any **d** provided we know
x with F(x) = –**d** .

**Example**: Choose x in an arbitrary cone and
**let** **d** = –F(x).

**Open question:**

Running time for **random** starting point?

# What is new?

**So far:**

- Lemke only for **d** > **0**
- No complementary cones view of Lemke-Howson

**Now:**

- **Unified view** of Lemke-Howson and Lemke
- **Surjectivity** of LCP map F for M > 0
- **Extension:** start in arbitrary cones
- **Open question:**
  challenge instances for the new algorithm?