Infinite games on graphs

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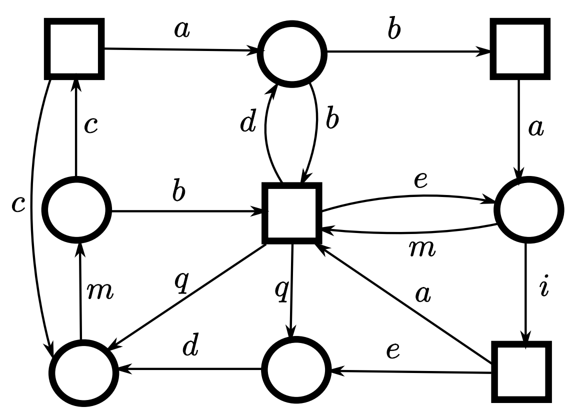
1 Transition systems

are couples

$$\mathcal{A} = (S, \Delta)$$

where
S— a finite set of states,
C— a set of colors,
$\Delta \subset S \times C \times S$ — a finite set of *actions*.



For each action

$$e = (s, c, t) \in \Delta$$

$s = \text{source}(e)$ — the *source*,
$t = \text{target}(e)$ — the *target*,
$c = \gamma_C(e)$ — the colour of e.

$$\Delta(s) = \{e \in \Delta \mid \text{source}(e) = s\}$$

the set of actions *available at* s.

A *play* is an infinite path in $\mathcal{A}$:

$$p = e_1 e_2 e_3 \ldots \quad \forall i \geq 0, e_i \in \Delta \quad \text{and} \quad \text{target}(e_i) = \text{source}(e_{i+1}).$$

2 Arenas

are tuples

$$\mathcal{A} = (S, \Delta, \pi)$$

where

- (S, Δ) – a transition system without sink states , i.e. $\Delta(s) \neq \emptyset$ for all $s \in S$,
- $\pi : S \to \{\mathrm{Min}, \mathrm{Max}\}$ is a mapping designating for each state $s \in S$ the player $\pi(s)$ controlling s.

3 Outcomes

An *outcome* of an infinite play p is

$$\gamma_C(p) = \gamma_C(e_1 e_2 \ldots) = \gamma_C(e_1)\gamma_C(e_2)\ldots.$$

The set of outcomes

$$\mathcal{O}(C) = \bigcup_{\substack{B \subset C \\ B \text{ finite nonempty}}} B^\omega.$$

4 Preference relation

A preference relation is a binary relation $\sqsupseteq$ over the set $\mathcal{O}(C)$ of outcomes which is

- reflexive, i.e. $u \sqsupseteq u$, for all $u \in \mathcal{O}(C)$,
- transitive, i.e. $u \sqsupseteq v$ and $v \sqsupseteq w$ imply $u \sqsupseteq w$, for $u, v, w \in \mathcal{O}(C)$ and
- total, either $u \sqsupseteq v$ or $v \sqsupseteq u$, for all $u, v \in \mathcal{O}(C)$.

A preference relation = a total preorder relation over the set $\mathcal{O}(C)$ of outcomes.
If

$$u \sqsupseteq v, \quad u, v \in \mathcal{O}(C).$$

then u is no worse than v.
The player *strictly prefers* u to v, $u \sqsupset v$, if $u \sqsupseteq v$ but not $v \sqsupseteq u$.
If $u \sqsupseteq v$ and $v \sqsupseteq u$ then the player is indifferent between u and v.
$\sqsubseteq$ — the inverse of $\sqsupseteq$.

5 Two-person strictly antagonistic game

is a couple

$$(\mathcal{A}, \sqsupseteq),$$

where $\mathcal{A}$ is an arena and $\sqsupseteq$ is a preference relation for Max. The preference relation for player Min is $\sqsubseteq$.

The obvious aim of each player is to obtain the most advantageous outcome with respect to his preference relation.

6 Preferences versus payoff mappings

Payoff mapping

$$f : \mathcal{O}(C) \to \mathbb{R} \cup \{-\infty, +\infty\}$$

induces preference $\sqsupseteq_f$,

$$u \sqsupseteq_f v \quad \text{if} \quad f(u) \geq f(v).$$

7 Strategies and equilibria

$\mathcal{A} = (S, \Delta, \pi)$ – an arena.

$$S_{\mathrm{Max}} = \{s \in S \mid \pi(s) = \mathrm{Max}\}$$

states controlled by player Max

$$S_{\mathrm{Min}} = S \setminus S_{\mathrm{Max}}$$

states controlled by player Min.

$\mathscr{P}(\mathcal{A})$ – the set of finite paths in $\mathcal{A}$ (including for each state s the empty path λ_s with the source and target s).

A *strategy for player* $\mu \in \{\mathrm{Max}, \mathrm{Min}\}$ is a mapping

$$\sigma_\mu : \{p \in \mathscr{P}(\mathcal{A}) \mid \mathrm{target}(p) \in S_\mu\} \to \Delta,$$

such that $\sigma_\mu(p) \in \Delta(s)$, where $s = \mathrm{target}(p)$.

8 Plays consistent with a strategy

A (finite or infinite) play $p = e_0 e_1 e_2 \ldots$ is *consistent* with player μ's strategy σ_μ if, for each factorization $p = p'p''$, such that

- p'' is nonempty
- and $\mathrm{target}(p') = \mathrm{source}(p'')$ is controlled by player μ,

$\sigma_\mu(p')$ is the first action in p''.

Positional strategies. A *positional* (or memoryless) strategy for player μ

$$\sigma_\mu : S_\mu \to \Delta$$

such that, for all $s \in S_\mu$,

$$\sigma_\mu(s) \in \Delta(s)$$

A *strategy profile* is a pair (σ, τ) of strategies, where σ is a strategy for player Max and τ is a strategy for player Min.

$$p_{\mathcal{A},s}(\sigma, \tau)$$

is the unique play with source s consistent with σ and τ.

A strategy profile $(\sigma^\#, \tau^\#)$ is an *equilibrium* if for all states $s \in S$ and all strategies σ and τ,

$$\gamma_C(p_{\mathcal{A},s}(\sigma^\#, \tau)) \sqsupseteq \gamma_C(p_{\mathcal{A},s}(\sigma^\#, \tau^\#)) \sqsupseteq \gamma_C(p_{\mathcal{A},s}(\sigma, \tau^\#)) \ .$$

An equilibrium $(\sigma^\#, \tau^\#)$ is said to be *positional* if the strategies $\sigma^\#$ and $\tau^\#$ are positional.

9 Examples

Mean-payoff games.

$$C = \mathbb{R} \times \mathbb{R}_+$$

$$(r_1, t_1)(r_2, t_2)(r_3, t_3) \ldots \sqsupseteq (r'_1, t'_1)(r'_2, t'_2)(r'_3, t'_3) \ldots$$

if

$$\limsup_n \frac{r_1 t_1 + r_2 t_2 + \cdots + r_n t_n}{t_1 + t_2 + \cdots + t_n} \geq \limsup_n \frac{r'_1 t'_1 + r'_2 t'_2 + \cdots + r'_n t'_n}{t'_1 + t'_2 + \cdots + t'_n}$$

But then

$$1000, 1000, ..., 1000, 0^\omega \approx 0^\omega$$

Overtaking.

$$(r_1, t_1)(r_2, t_2)(r_3, t_3) \ldots \sqsupseteq (r'_1, t'_1)(r'_2, t'_2)(r'_3, t'_3) \ldots$$

if

$$\exists k, \forall n > k, \quad \frac{r_1 t_1 + r_2 t_2 + \cdots + r_n t_n}{t_1 + t_2 + \cdots + t_n} \geq \frac{r'_1 t'_1 + r'_2 t'_2 + \cdots + r'_n t'_n}{t'_1 + t'_2 + \cdots + t'_n}$$

Weighted limits.

$$C = \mathbb{R}, \quad \alpha \in [0, 1]$$

$$f_\alpha(r_1 r_2 r_3 \ldots) = \alpha \cdot \limsup_i r_i + (1 - \alpha) \cdot \liminf_i r_i$$

10 Extended preference relation and $\succeq$- equilibria

The *extended preference relation* $\succeq$ is defined as follows:

$$\text{for } x, y \in \mathcal{O}(C), \quad x \succeq y \quad \text{if} \quad \forall u \in C^*, ux \sqsupseteq uy.$$

Obviously, if $x \succeq y$ then $x \sqsupseteq y$.

$\succeq$ is transitive and reflexive, but can be not total.
A strategy profile $(\sigma^\#, \tau^\#)$ is a $\succeq$-*equilibrium* if for all strategies σ, τ

$$\gamma_C(p_{\mathcal{A},s}(\sigma^\#, \tau)) \succeq \gamma_C(p_{\mathcal{A},s}(\sigma^\#, \tau^\#)) \succeq \gamma_C(p_{\mathcal{A},s}(\sigma, \tau^\#)) \ .$$

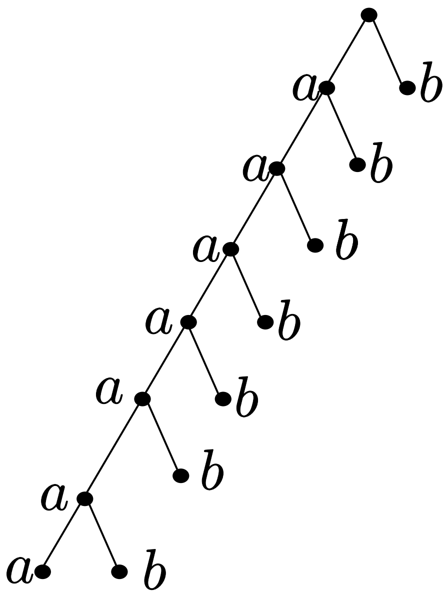
11 Adherence operator

$$[\![\ ]\!] : 2^{C^*} \to 2^{C^\omega}$$

For $L \subseteq C^*$,

$$[\![L]\!] = \{u \in C^\omega \mid \mathrm{Pref}(u) \subset \mathrm{Pref}(L)\} \ .$$

Example $[\![a^* b]\!] = \{a^\omega\}$



12 Why adherence?

$\mathcal{A} = (S, \Delta)$ an arena. Then $L_s^\omega(\mathcal{A}) = [\![L_s(\mathcal{A})]\!]$ where $L_s^\omega(\mathcal{A})$ the set of infinite words labelling infinite paths starting at s, $L_s(\mathcal{A})$ the set of finite words labelling finite paths starting at s.

If $L \in \mathrm{Rec}(C^*)$ then $L_s^\omega(\mathcal{A}) = [\![L]\!]$ for some arena $\mathcal{A}$.

13 Conditions for positional equilibria

Let $u \in \mathcal{O}(C)$ and $X \subset \mathcal{O}(C)$.

Notation.

$$u \succeq X$$

if, for all $x \in X$, $u \succeq x$.

Ultimately periodic infinite words Let $u, w \in C^*$ and $v \in C^+$.

An infinite word of the form

$$uv^\omega,$$

is called *ultimately periodic.*

Simple periodic languages Languages of the form uv^*, here $u \in C^*$ and $v \in C^+$. Note

$$[\![uv^*]\!] = \{uv^\omega\}$$

Union selection. $\succeq$ satisfies *union selection* condition if, for all ultimately periodic words $u_1u_2^\omega$ and $v_1v_2^\omega$, either

$$u_1u_2^\omega \succeq v_1v_2^\omega$$

or

$$v_1v_2^\omega \succeq u_1u_2^\omega$$

We can rewrite this condition as

$$\exists x \in \{u_1 u_2^\omega, v_1 v_2^\omega\}, \quad x \succeq [\![u_1 u_2^* \cup v_1 v_2^*]\!] \,.$$

Product selection. We say that $\succeq$ satisfies *product selection* condition for player Max if, for all $u, v, w, z \in C^*$ such that $|v| > 0$ and $|w| > 0$,

$$\exists x \in \{uv^\omega, uwz^\omega\}, \quad x \succeq [\![uv^* wz^*]\!] \,.$$

Note that

$$\{uv^\omega, uwz^\omega\} \subset uv^\omega \cup uv^* wz^\omega = [\![uv^* wz^*]\!] \,.$$

Star selection. $\succeq$ satisfies *star selection* condition for player Max if for each nonempty language $L \in \mathrm{Rec}(C^+)$

$$\exists x \in [\![L]\!] \cup \{u^\omega \mid u \in L\}, \quad x \succeq [\![L^*]\!] \,.$$

Note

$$[\![L]\!] \cup \{u^\omega \mid u \in L\} \subset L^* [\![L]\!] \cup L^\omega = [\![L^*]\!]$$

14 One player Max games

$\succeq$ satisfies all three selection conditions
if and only if
one-player Max games have optimal positional strategies for player Max.

Dual conditions.

$$\succeq \quad \leftrightarrow \quad \preceq \,.$$

15 Main result

The following result is a reformulation of [4] to appear in [5]

Theorem 1 (Gimbert, WZ). *Let $\sqsupseteq$ be a preference relation over $\mathcal{O}(C)$ and let $\succeq$ be the corresponding extended preference relation. The following conditions are equivalent:*

(1) There exist positional equilibria for all games $(\mathcal{A}, \sqsupseteq)$ over finite arenas.

(2) There exist positional $\succeq$-equilibria for all games $(\mathcal{A}, \sqsupseteq)$ over finite arenas.

(3) $\succeq$ satisfies union selection, product selection and star selection conditions for player Max *and player* Min.

(4) For all one-player games $(\mathcal{A}, \sqsupseteq)$ the player controlling the arena $\mathcal{A}$ has an optimal positional strategy.

(5) For all one-player games $(\mathcal{A}, \sqsupseteq)$ the player controlling the arena $\mathcal{A}$ has a $\succeq$-optimal positional strategy.

16 Games on infinite graphs

No restriction on the number of states and transitions.

$$W \subset C^\omega \quad \text{winning outcomes}$$

W prefix independent ($aW = W$ for each $a \in C$)

$$u \sqsupseteq_W v$$

if $u \in W$ or $v \notin W$.

Both players have optimal positional strategies in all games $(\mathcal{A}, \sqsupseteq_W)$ iff W described by a parity condition (Colcombet, Niwiński [1]).

If the set of states infinite but for each state the set of available transitions finite then there are positional games generalizing parity games (Grädel, Walukiewicz [6]).

17 Half-positional games

Finite graphs :

- the game has a value,
- player Max has an optimal positional strategy.

Some sufficient conditions (E. Kopczyński [7, 8]).

Remark: We lack concrete examples of such games.

18 Stochastic perfect information games

Finite number of states and actions.

Payoff mapping:

$$f : \mathcal{O}(C) \to \mathbb{R}$$

Theorem 2 (H. Gimbert, WZ [2, 9])**.** *The following conditions are equivalent:*

- *For each game $(\mathcal{A}, f)$ with $\mathcal{A}$ finite there exists an equilibrium profile $(\sigma^\#, \tau^\#)$ with $\sigma^\#$ and $\tau^\#$ pure positional,*
- *For each one player game $(\mathcal{A}, f)$ with $\mathcal{A}$ finite the unique player has an optimal pure positional strategy.*

19 One player perfect information games (MDP)

Theorem 3 (H. Gimbert [2, 3])**.** *Let*

$$f : \mathcal{O}(C) \to \mathbb{R}$$

prefix independent and such that for all $u, v \in \mathcal{O}(C)$

$$f(w) \leq \max\{f(u), f(v)\}$$

where w is the shuffle of u and v. Then player Max *has an optimal pure positional strategy for all one player stochastic perfect information games (Markov Decision Processes) $(\mathcal{A}, f)$ for $\mathcal{A}$ finite.*

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