

# Game theory and abstract convexity methods in static analysis of programs

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Synthesis of works with: [Éric Goubault](#), [Sylvie Putot](#) (CEA/MeASI),  
[Ankur Taly](#) (IIT Mumbai), [Sarah Zenouh](#) (EADS), [Assale Adje](#) (LIX/MeASI)

How to prove (**automatically**) that ?

```
void main() {  
  i = 1; j = 10;  
  while (i <= j){ //1  
    i = i + 2;  
    j = j - 1; }  
}
```

$$i \leq +\infty$$

$$i \geq 1$$

$$j \leq 10$$

$$j \geq -\infty$$

$$i \leq j$$

$$i + 2j \leq 21$$

$$i + 2j \geq 21$$

$(i, j) \in [(1, 10), (7, 7)]$  (exact result).

Answer:

**convex analysis** (including **generalized convexity**)

and **zero-sum games**

## Static analysis of programs by abstract interpretation

*Cousot 77*: finding invariants of a program reduces to computing the smallest fixed point of a monotone self-map of a complete lattice  $L$

To each breakpoint  $i$  of the program, is associated a set  $x^i \in L$  which is an overapproximation of the set of reachable values of the variables, at this breakpoint.

$x^i$  may be a Cartesian product of intervals (one interval for each variable of the program) or belong to a more sophisticated domain

The best  $x$  is the smallest solution of a fixed point problem  $x = f(x)$  with  $f$  order preserving  $L^n \rightarrow L^n$  ( $n \leq \#$  breakpoints).

```

void main() {
    int x=0;           // 1
    while (x<100) {   // 2
        x=x+1;       // 3
    }                 // 4
}

```

$$\begin{aligned}
 x_1 &= [0, 0] \\
 x_2 &= ] - \infty, 99] \cap (x_1 \cup x_3) \\
 x_3 &= x_2 + [1, 1] \\
 x_4 &= [100, +\infty[ \cap (x_1 \cup x_3)
 \end{aligned}$$

Let  $x_2^+ := \max x_2$ . After some elimination, we arrive at

$$x_2^+ = \min(99, \max(0, x_2^+ + 1)) .$$

The smallest  $x_2^+$  is 99. This fixed point problem involves the Shapley operator of a zero-sum game with a stopping option.

*How to solve efficiently the previous fixed point problem ?*

Naive value iteration yields 99 iterations, because MIN prefers to pay 1 and continue to play rather than pay the final cost 99, if the horizon is  $< 99$  (“après moi le déluge”):

$$x_2^+ = \min(99, \max(0, x_2^+ + 1)) .$$

*When does the fixed point problem of abstract interpretation reduce to a game problem ?*

*Does it work for More general programs ? More general domains ?*

**Answer** (SG, Goubault, Taly, Zennou, Adje; ESOP'07, MTNS'08; see also an early version in CAV'05).

1. There is a correspondence between zero-sum game and static analysis problems.

2. One basic technique in control/game theory, policy iteration, yields an alternative family of methods in static analysis.

## *Game to Static Analysis Dictionary*

Games	Abstract interpretation
dynamical system	program
Shapley operator	functional
state space	(# breakpts.) $\times$ (# deg. freedom)
horizon $n$ problem	execution of $n$ logical steps
limit of the value in horizon $n$	optimal invariant (bound)
value iteration	Kleene iteration
mean payoff value $\leq 0$	$f(v) \leq v, v \in \mathbb{R}^n$ (finite invariant!)



## Some useful domains

1. **Zones** (Miné). Sets of the form

$$Z = \{x \in \mathbb{R}^n \mid x_i - x_j \leq M_{ij}\}$$



a zone is coded by the matrix  $M \in (\mathbb{R} \cup \{+\infty\})^{n \times n}$ .

by setting  $x_0 := 0$  and projecting, we see that Zones  $\supset$  Intervals.

2. **Polyhedra** (Cousot, Halbwachs 78. . . )

but the number of extreme points or faces may grow exponentially during the run of Kleene iteration

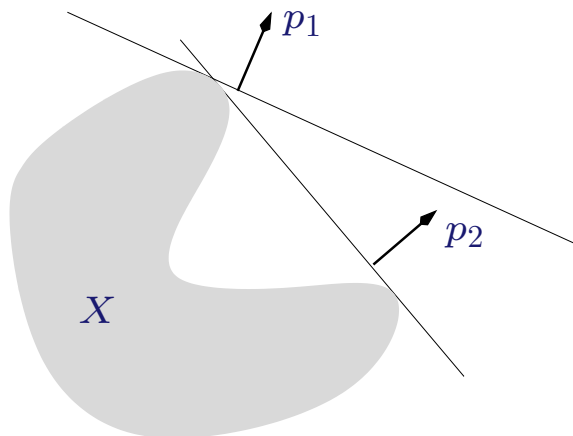
**3. Templates** S. Sankaranarayanan and H. Sipma and Z. Manna  
(VMCAI'05)

almost as expressive as polyhedra but scalable.

I'll give a **convex analytic view of templates**.

The **support function**  $\sigma_X$  of  $X \subset \mathbb{R}^n$  is defined by

$$\sigma_X(p) = \sup_{x \in X} p \cdot x$$



Legendre-Fenchel duality tells that  $\sigma_X = \sigma_Y$  iff  $X$  and  $Y$  have the same closed convex hull.

$\sigma_X(\alpha p) = \alpha \sigma_X(p)$  for  $\alpha > 0$ , so it is enough to know  $\sigma_X(p)$  for all  $p$  in the unit sphere.

Idea: discretize the unit sphere and represent  $X$  by  $\sigma_X$  restricted to the discretization points.

So fix  $\mathcal{P} \subset \mathbb{R}^n$  a finite set of directions.

$L(\mathcal{P})$  lattice of sets of the form

$$Z = \{x \mid p \cdot x \leq \gamma(p), \forall p \in \mathcal{P}\}, \quad \gamma : \mathcal{P} \rightarrow \mathbb{R} \cup \{+\infty\}.$$

$Z$  is coded by  $\gamma := \sigma_Z \upharpoonright_{\mathcal{P}}$ .

$Z$  is a polyhedron every facet of which is orthogonal to some  $p \in \mathcal{P}$ .

We may think of  $L(\mathcal{P})$  as a lattice of abstract convex sets (Rubinov, Singer, . . . )

Specialization:  $\mathcal{P} = \{\pm e_i, i = 1, \dots, n\}$  gives intervals,  $\mathcal{P} = \{\pm(e_i - e_j), 1 \leq i < j \leq n\}$  gives Miné's templates.

```
void main() {  
    i = 1; j = 10;  
    while (i <= j){ //1  
        i = i + 2;  
        j = j - 1; }  
}
```

$$i \leq +\infty$$

$$i \geq 1$$

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$$j \geq -\infty$$

$$i \leq j$$

$$i + 2j \leq 21$$

$$i + 2j \geq 21$$

```

void main() {
  i = 1; j = 10;
  while (i <= j){ //1
    i = i + 2;
    j = j - 1; }
}

```

$$\gamma(e_1) = +\infty$$

$$\gamma(-e_1) = -1$$

$$\gamma(e_2) = 10$$

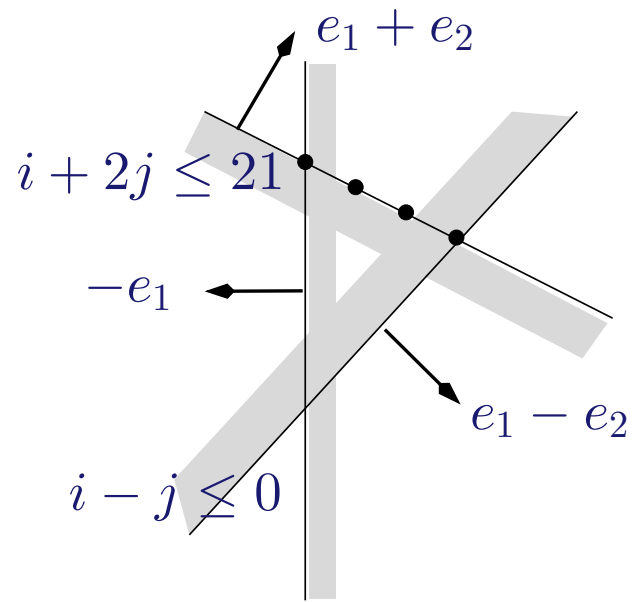
$$\gamma(-e_2) = -\infty$$

$$\gamma(e_1 - e_2) = 0$$

$$\gamma(e_1 + 2e_2) = 21$$

$$\gamma(-e_1 - 2e_2) = -21 .$$

$\mathcal{P} = \{\pm e_1, \pm e_2, e_1 - e_2, \pm(e_1 + 2e_2)\}$ ,  $\gamma$ : breakpoint 1





To show this, we must solve the fixed point problem:

$$\gamma(p) = ((1, 10) \cdot p) \vee (\bar{\gamma}(p) + (2, -1) \cdot p), \quad \forall p \in \mathcal{P} \setminus \{e_1 - e_2\}$$
$$\gamma(e_1 - e_2) = 0 \wedge (-9 \vee (\bar{\gamma}(e_1 - e_2) - 3)), \quad \bar{\gamma} = \text{convex hull}(\gamma)$$

```
void main() {  
    i = 1; j = 10;  
    while (i <= j){ //1  
        i = i + 2;  
        j = j - 1; }  
}
```

$$\bar{\gamma}(p) = \text{convex hull}(\gamma)(p)$$

$$= \sup_{x \in \mathbb{R}^n} p \cdot x; \quad q \cdot x \leq \gamma(q), \quad \forall q \in \mathcal{P},$$

$$\gamma(p) = ((1, 10) \cdot p) \vee (\bar{\gamma}(p) + (2, -1) \cdot p), \quad \forall p \in \mathcal{P} \setminus \{e_1 - e_2\}$$

$$\gamma(e_1 - e_2) = 0 \wedge (-9 \vee (\bar{\gamma}(e_1 - e_2) - 3)),$$

```
void main() {  
    i = 1; j = 10;  
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}
```

## Correspondence theorem (SG, Goubault, Taly, Zennou, ESOP'07)

When the arithmetics of the program is *affine* (no product or division of variables), abstract interpretation over a lattice of templates reduces to finding the smallest fixed point of a map  $f : (\mathbb{R} \cup \{+\infty\})^n \rightarrow (\mathbb{R} \cup \{+\infty\})^n$  of the form

$$f_i(x) = \inf_{a \in A(i)} \sup_{b \in B(i,a)} (r_i^{ab} + M_i^{ab} x)$$

with  $M_i^{ab} := (M_{ij}^{ab})$ ,  $M_{ij}^{ab} \geq 0$ , but possibly  $\sum_j M_{ij}^{ab} > 1$ , i.e. **negative discount**: payment  $r_{i_1 i_2} + \alpha r_{i_2 i_3} + \alpha^2 r_{i_3 i_4} + \dots$ ,  $\alpha := \exp(-\text{rate}) > 1$



The cornerstone of the operator approach to game theory is the **nonexpansiveness**:

$$\|f(x) - f(y)\|_\infty \leq \|x - y\|_\infty \iff \text{nonnegative discount}$$

## Sketch of proof.

Let  $y = Ax + b$ ; Knowing that  $x \in Z^1 := \{z \mid p \cdot z \leq \gamma^1(p), \forall p \in \mathcal{P}\}$ , find the best  $Z^2 := \{z \mid p \cdot z \leq \gamma^2(z), \forall p \in \mathcal{P}\}$  such that  $y \in Z^2$  ?

$$\begin{aligned}\gamma^2(p) &= \sup_{x \in Z^1} p \cdot (Ax + b) = \sup_{x \in Z^1} p \cdot (Ax + b); q \cdot x \leq \gamma^1(q), \forall q \in \mathcal{P} \\ &= \sup_{x \in \mathbb{R}^n} \inf_{\lambda \in \mathbb{R}_+^{\mathcal{P}}} p \cdot (Ax + b) + \sum_{q \in \mathcal{P}} \lambda(q) (\gamma^1(q) - q \cdot x) \\ &= \inf_{\lambda \in \mathbb{R}_+^{\mathcal{P}}} p \cdot b + \sum_{q \in \mathcal{P}} \lambda(q) \gamma^1(q); \lambda(q) \geq 0, A^T p = \sum_{q \in \mathcal{P}} \lambda(q) q\end{aligned}$$

(strong duality).

The inf is attained at an extreme point of the feasible set, so this is in fact a min of finitely many affine forms of  $\gamma$ .

Other operations:

$$\sigma_{X \cap Y} = \text{convex hull}(\inf(\sigma_X, \sigma_Y)).$$

Convex hull reduces to a finite min by a similar argument.

Modelling the dataflow yields maxima, because  $\sigma_{X \cup Y} = \sup(\sigma_X, \sigma_Y)$

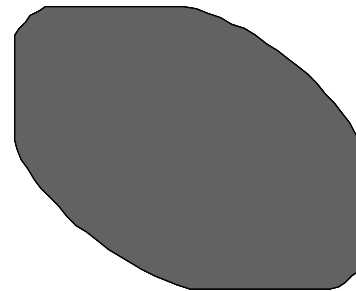
Recent work on nonlinear templates (Adje, SG, Goubault).

$$Z = \{x \mid p(x) \leq \gamma(p), \forall p \in \mathcal{P}\}$$

$p$  is now a **non-linear** map (e.g. quadratic, e.g. Lyapunov function).

$$\ddot{x} + c\dot{x} + x = 0$$

```
x:=[0,1];  
v:=[0,1];  
while(true){ //2  
w:=v;  
v:=v(1-h)-hx;  
x=x+hw;  
}
```



$$\{-1.8708 \leq x \leq 1.8708, -1.5275 \leq v \leq 1.5275, 2x^2 + 3v^2 + 2xv \leq 7\}$$

**Theorem.** *The correspondence games  $\leftrightarrow$  static analysis carries over to quadratic templates  $p$ . We solve  $\gamma = \bar{f}(\gamma)$  for some relaxed functional  $\bar{f} \geq f$ .*

The functional  $\bar{f}$  is constructed using Shor's relaxation scheme, the action space of MIN is now infinite (but SDP yields a minimization oracle).

Recall that nonconvex quadratic programming is NP-hard, so we really need to replace  $f$  by some  $\bar{f} \geq f$ .

Could also use SOS relaxations, but Shor is scalable: evaluating the functional remains polynomial time. . .

The max-plus bases methods (Fleming, McEneaney; Akian, SG, Lakhoua, eg SICON 08) approximate the value function  $v$  of a control problem by

$$v = \sup_{1 \leq i \leq m} \lambda_i + p_i, \quad \lambda_i \in \mathbb{R}, \quad v : \mathbb{R}^n \rightarrow \mathbb{R} .$$

The sublevel set

$$Z = \{x \in \mathbb{R}^n \mid v(x) \leq 0\} = \{x \in \mathbb{R}^n \mid p_i(x) \leq -\lambda_i\}$$

is defined by nonlinear templates!

In control: curse of dimensionality - record of maxplus methods dimension  $n = 6$  (Desphande, McEneaney, SG ACC'08), but we require accurate numerical results - in static analysis aim typically at  $n = 10^5$ , but the problem is so difficult that any bound can be useful.



## How to solve the fixed point problem ?

Classically: Kleene (fixed point iteration) is slow or may even not converge, so widening and narrowing have been used, leading to an overapproximation of the solution.

An alternative: **Policy iteration.**

method developed by Howard (60) in stochastic control, extended by Hofman and Karp (66) to some special (nondegenerate) stochastic games. Extension of Newton method  $\implies$  **experimentally fast**, complexity still open.

extended by Costan, SG, Goubault, Martel, Putot, CAV'05) to fixed point problems in static analysis (novelty: complete lattice setting).

experiments: PI often yields more accurate fixed points (because it avoids widening), small number of iterations.

A **strategy** is a map  $\pi$  which to a state  $i$  associates an action  $\pi(i) \in A(i)$ .

Consider the **one player** dynamic programming operator:

$$f_i^\pi(x) := \sup_{b \in B(i, \pi(i))} (r_i^{\pi(i)b} + M_i^{\pi(i)b} x)$$

$$f = \inf_{\pi} f^\pi$$

and the set  $\{f^\pi \mid \pi \text{ strategy}\}$  has a *selection*:

$$\forall v \in \mathbb{R}^n, \exists \pi \quad f(v) = f^\pi(v) .$$

Since  $f^\pi$  is convex and piecewise affine, finding the smallest finite fixed point of  $f^\pi$  (if any) can be done by linear programming:

$$\min \sum_i v_i; \quad f^\pi(v) \leq v .$$

Can we compute the smallest fixed point of  $f$  from the smallest fixed points of the  $f^\pi$ ?

We denote by  $f^-$  the smallest fixed point of a monotone self-map  $f$  of a complete lattice  $\mathcal{L}$ , whose existence is guaranteed by Tarski's fixed point theorem.

**Theorem** (Costan, SG, Goubault, Martel, Putot CAV'05). *Let  $\mathcal{G}$  denote a family of monotone self-maps of a complete lattice  $\mathcal{L}$  with a lower selection, and let  $f = \inf \mathcal{G}$ . Then  $f^- = \inf_{g \in \mathcal{G}} g^-$ .*

For templates, a strategy is a selection of extreme points of the dual linear programs constructed in the proof of the game representation of the functional.

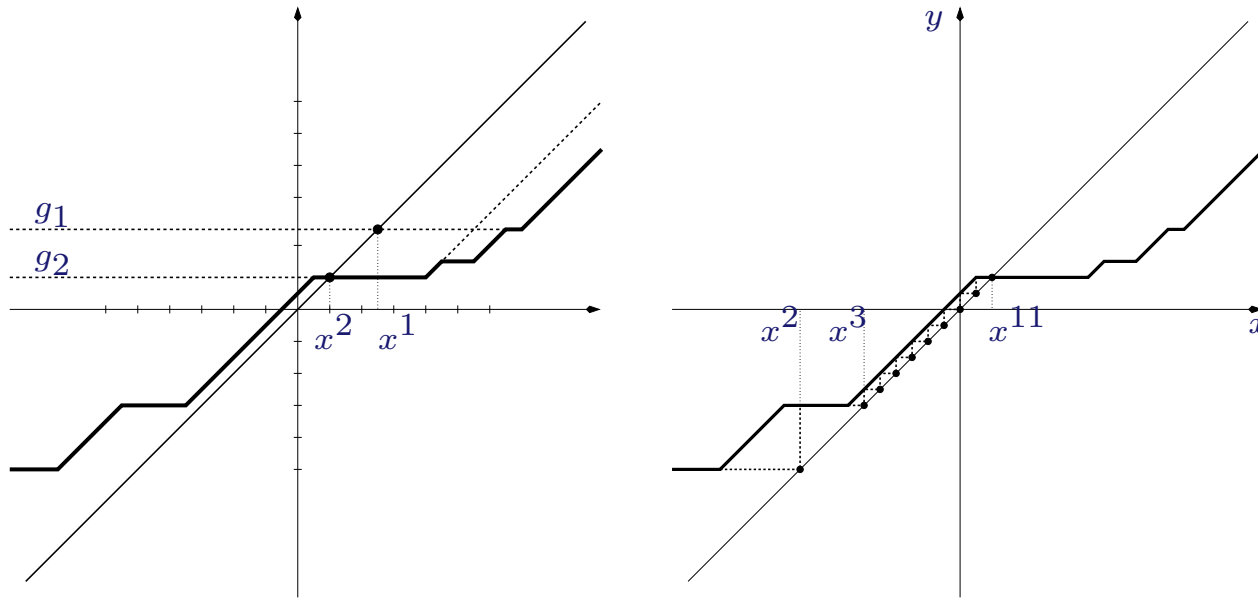
Eg, for Mine's zone (difference bound matrices), some constraints involve the shortest path weights  $M_{ij}^*$ , a strategy turns out to be an **extreme point** in the shortest path polytope, namely, an **elementary path**.

The input of the following algorithm consists of a finite set  $\mathcal{G}$  of monotone self-maps of a lattice  $\mathcal{L}$  with a lower selection. When the algorithm terminates, its output is a fixed point of  $f = \inf \mathcal{G}$ .

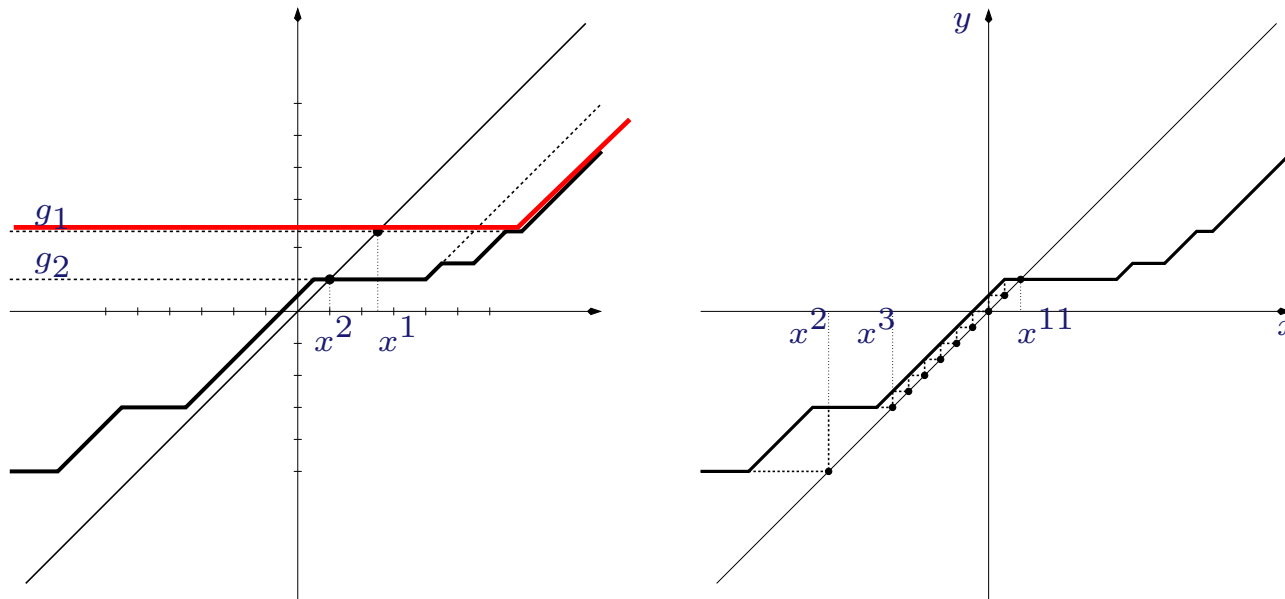
1. *Initialization.* Set  $k = 1$  and select any map  $g_1 \in \mathcal{G}$ .
2. *Value determination.* Compute a fixed point  $x^k$  of  $g_k$ .
3. Compute  $f(x^k)$ .
4. If  $f(x^k) = x^k$ , return  $x^k$ .
5. *Policy improvement.* Take  $g_{k+1}$  such that  $f(x^k) = g_{k+1}(x^k)$ . Increment  $k$  and goto Step 2.

The algorithm does terminate when at each step, the smallest fixed-point of  $g_k$ ,  $x^k = g_k^-$  is selected.

*Example.* Take  $\mathcal{L} = \overline{\mathbb{R}}$ , and consider the self-map of  $\mathcal{L}$ ,  $f(x) = \inf_{1 \leq i \leq m} \max(a_i + x, b_i)$ , where  $a_i, b_i \in \mathbb{R}$ . The set  $\mathcal{G}$  consisting of the  $m$  maps  $x \mapsto \max(a_i + x, b_i)$  admits a lower selection.



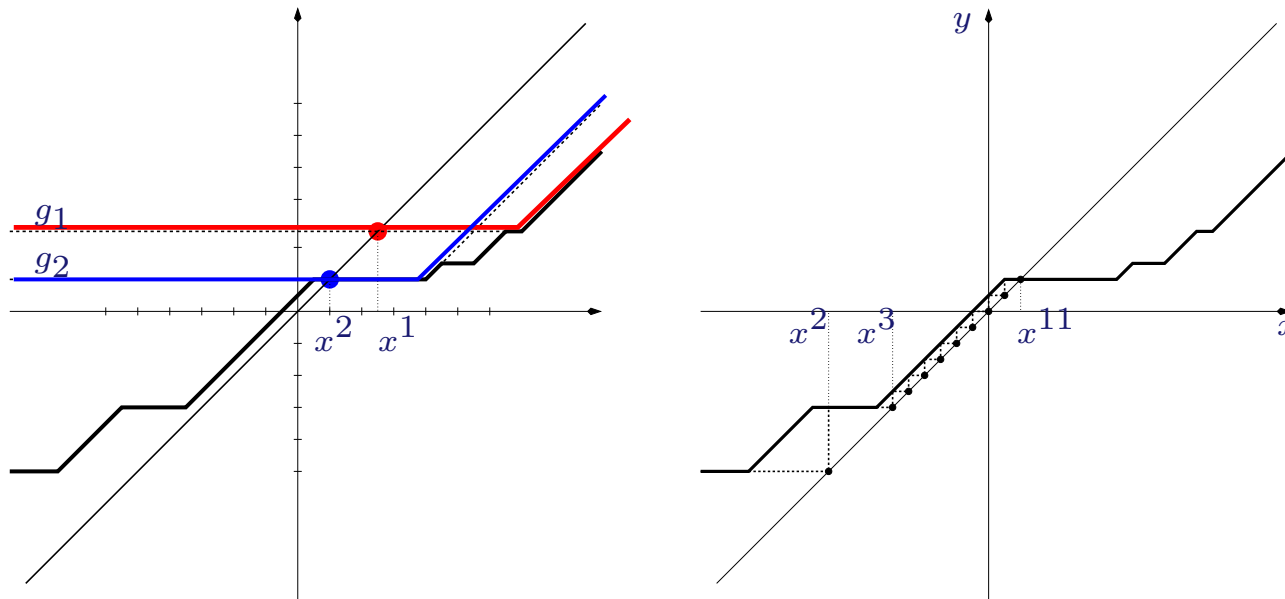
*Example.* Take  $\mathcal{L} = \overline{\mathbb{R}}$ , and consider the self-map of  $\mathcal{L}$ ,  $f(x) = \inf_{1 \leq i \leq m} \max(a_i + x, b_i)$ , where  $a_i, b_i \in \mathbb{R}$ . The set  $\mathcal{G}$  consisting of the  $m$  maps  $x \mapsto \max(a_i + x, b_i)$  admits a lower selection.



For this class of maps: deterministic games.



*Example.* Take  $\mathcal{L} = \overline{\mathbb{R}}$ , and consider the self-map of  $\mathcal{L}$ ,  $f(x) = \inf_{1 \leq i \leq m} \max(a_i + x, b_i)$ , where  $a_i, b_i \in \mathbb{R}$ . The set  $\mathcal{G}$  consisting of the  $m$  maps  $x \mapsto \max(a_i + x, b_i)$  admits a lower selection.



PI often more accurate than Kleene+widening/narrowing:

```

0   i = 150;
1   j = 175;
2   while (j >= 100){
3       i++;
4       if (j <= i){
5           i = i - 1;
6           j = j - 2;
7       }
8   }
9

```

$M_0$  = *context\_initialization*  
 $M_2$  = (*Assignment* ( $i \leftarrow 150, j \leftarrow 175$ )( $M_0$ ))\*  
 $M_3$  = ( $(M_2 \sqcup M_8) \sqcap (j \geq 100)$ )\*  
 $M_4$  = (*Assignment* ( $i \leftarrow i + 1$ )( $M_3$ ))\*  
 $M_5$  = ( $M_4 \sqcap (j \leq i)$ )\*  
 $M_7$  = (*Assignment* ( $i \leftarrow i - 1, j \leftarrow j - 2$ )( $M_5$ ))\*  
 $M_8$  = ( $(M_4 \sqcap (j > i))^* \sqcup M_7$ )  
 $M_9$  = ( $(M_2 \sqcup M_8) \sqcap (j < 100)$ )\*

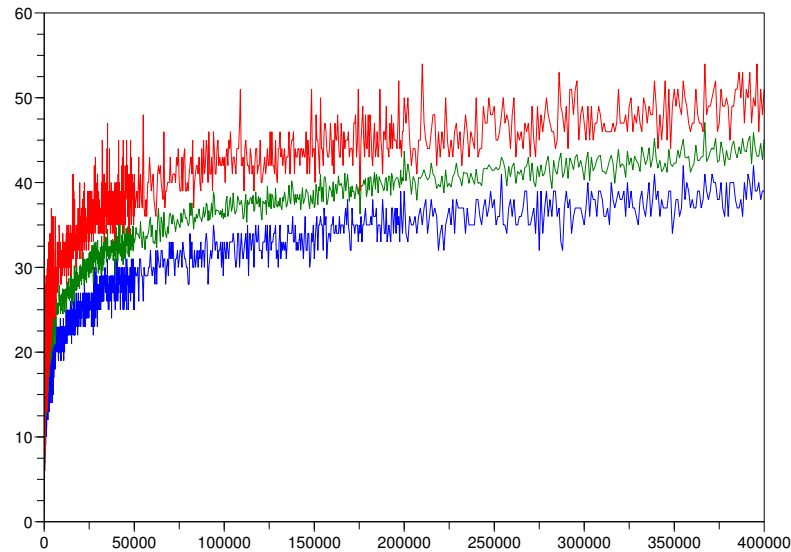
$$\text{IP} \left\{ \begin{array}{l} 150 \leq i \leq 174 \\ 98 \leq j \leq 99 \\ -76 \leq j - i \leq -51 \end{array} \right.$$

$$\text{Mine's Octagon} \left\{ \begin{array}{l} 150 \leq i \\ 98 \leq j \leq 99 \\ j - i \leq -51 \\ 248 \leq j + i \end{array} \right.$$

Prototype implementations of PI, to analyze fragments of C:

- Costan: domain of intervals;
- Taly and Zennou: domain of zones (in CAML+glpk);
- Adje: quadratic templates, tests in Matlab / YALMIP.

Experiments. SG, Dhingra (Valuetools'06), combinatorial deterministic game



Sparse bipartite graphs.  $n$  nodes of each kind, every node has 2 successors drawn at random; , deterministic game, random weights. Number of iterations of minimizer  $N_{\min}$ .

Recall complexity of zero-sum games is an open problem (Condon, 92 : mean payoff games is in  $NP \cap co-NP$ ), see recent work by Jurdziński, Paterson, Zwick; Björklund, Sandberg, Vorobyov

## **Difficulty**

PI may return a nonminimal fixed point.

We know there is a policy yielding the minimal fixed point.

How to find it?

Define the semiderivative of  $f$  at point  $v$  by

$f(v + x) = f(v) + f'_v(x) + o(\|x\|)$  where  $f'_v$  is homogeneous of degree one, continuous, but possibly not linear.

E.g.,  $f(x) = \min(\max(1 + x_1, x_2), x_3)$ ,  $f'_{(0,0,0)}(x) = \min(x_1, x_3)$

*Theorem (Adje, SG, Goubault, MTNS'08). Assume that  $f$  is order preserving (sup norm) nonexpansive and semidifferentiable, and let  $v$  be such that  $f(v) = v$ .*

*Then,  $v$  is the smallest real fixed point of  $f$  if: ( $f'_v(x) = x$  and  $x \leq 0$  implies  $x = 0$ ). Moreover, this condition is necessary if  $f$  is piecewise affine.*

Without nonexpansiveness, we have only a local minimality.



Global minimality relies on: in finite dimension, the fixed point set of an order preserving (supnorm) nonexpansive map is a retract of the whole space, the retraction being also order preserving and nonexpansive

We have to solve an auxiliary (simpler) fixed point problem,

The condition of the theorem can be checked using results on spectral radius.

Let  $h : C \rightarrow C$  be M, continuous, and homogeneous of degree 1 (H).

Bonsall spectral radius

$$\tilde{r}(h) := \lim_k \|h^k\|^{1/k} \text{ where}$$

$$\|h\| = \sup_x \|h(x)\| / \|x\|$$

Cone eigenvalue spectral radius

$$r(h) := \sup\{\lambda \mid \exists v \in C \setminus 0, h(v) = \lambda v\}$$

Collatz-Wielandt value

$$\bar{r}(h) := \inf\{\lambda \mid \exists v \in \text{int } C, h(v) \leq \lambda v\}$$

$$r(h) \leq \tilde{r}(h) \leq \bar{r}(h)$$

Nussbaum and Mallet-Parret . . . showed that these coincide in reasonable circumstances ( $C = \mathbb{R}_+^n$  OK).

Let  $C := \mathbb{R}_+^n$ . To check that  $f'_v$  does not have a fixed point in  $C \setminus 0$ , it suffices that  $\tilde{r}(f'_v) < 1$ . Collatz-Wielandt value allows to check that (termination).

Eigenvectors give descent direction allowing one to improve the strategy.

Eigenvector can be computed by the power algorithm, or by policy iteration algorithms (convergence proof only in a nondegenerate case).

This leads to the

**Question.** Compute rapidly the spectral radius of a map (game with multiplicative reward):

$$f_i(x) = \inf_{a \in A(i)} \sup_{b \in B(i,a)} M_i^{ab} x$$

(the vectors  $M_i^{ab}$  being nonnegative).

Case where the transition matrix associated to every policy is irreducible is “easy”. In general, approximation reduces to stochastic games with ergodic payoff. Actually, we only need to check that  $r(f) < 1$ .

If negative discount is allowed, the previous result only allows us to find a locally minimal fixed point of the functional . . .

**Open question. How to find the smallest fixed point if  $f$  is not nonexpansive?**

Should make big jumps.

See also Gawlitza and Seidl (ESOP'07) in another context.

```

int x,int y,
x=[0,2];y=[10,15] //1
while (x<=y) { //2
    x=x+1; //3
    while (5<=y) { //4
        y=y-1; //5
    } //6
} //7

```

$$\begin{aligned}
 (x_1, y_1) &= ([0, 2], [10, 15]) \\
 x_2 &= (x_1 \cup x_6) \cap [-\infty, (y_1 \cup y_6)^+] \\
 y_2 &= (y_1 \cup y_6) \cap [(x_1 \cup x_6)^-, +\infty] \\
 (x_3, y_3) &= (x_2 + [1, 1], y_2) \\
 (x_4, y_4) &= (x_3, (y_3 \cup y_5) \cap [5, +\infty]) \\
 (x_5, y_5) &= (x_4, y_4 + [-1, -1]) \\
 (x_6, y_6) &= (x_5, (y_3 \cup y_5) \cap [-\infty, 4]) \\
 x_7 &= (x_1 \cup x_6) \cap [(y_1 \cup y_6)^- + 1, +\infty] \\
 y_7 &= (y_1 \cup y_6) \cap [-\infty, (x_1 \cup x_6)^+ - 1]
 \end{aligned}$$

The monotone nonexpansive piecewise affine map  $f$  for the bounds of these intervals is:

$$f \begin{pmatrix} x \\ y \end{pmatrix} = f \begin{pmatrix} x_2^- \\ x_2^+ \\ x_7^- \\ x_7^+ \\ y_2^- \\ y_2^+ \\ y_4^- \\ y_4^+ \\ y_6^- \\ y_6^+ \\ y_7^- \\ y_7^+ \end{pmatrix} = \begin{pmatrix} 0 \quad \vee \quad (x_2^- - 1) \\ 2 \vee (x_2^+ + 1) \quad \wedge \quad \frac{15 \vee y_6^+}{15} \\ 0 \vee (x_2^- - 1) \quad \wedge \quad \frac{(-10 \vee y_6^-) - 1}{-10 \vee y_6^-} \\ 0 \quad \vee \quad (x_2^+ + 1) \\ \frac{0 \vee (x_2^- - 1)}{15} \quad \wedge \quad -10 \vee y_6^- \\ 15 \quad \vee \quad y_6^+ \\ y_2^- \vee (y_4^- + 1) \quad \wedge \quad \frac{-5}{y_4^+ - 1} \\ y_2^+ \quad \vee \quad y_4^+ - 1 \\ y_2^- \quad \vee \quad y_4^- + 1 \\ y_2^+ \vee (y_4^+ - 1) \quad \wedge \quad \underline{4} \\ -10 \quad \vee \quad y_6^- \\ 15 \vee y_6^+ \quad \wedge \quad \underline{(2 \vee (x_2^+ + 1)) - 1} \end{pmatrix}$$

The underlined terms represent the initial Policy. We find  $(\bar{x}, \bar{y}) = (0, 15, -1, 16, 0, 15, -5, 15, 0, 4, 0, 15)$ : it is a fixed point of  $f$ , and so policy iteration terminates in one step.

We calculate the semidifferential at  $(\bar{x}, \bar{y})$  in the direction  $(\delta x, \delta y)$ .

$$f'_{(\bar{x}, \bar{y})}(\delta\bar{x}, \delta\bar{y}) = \left( 0, 0, \delta\bar{x}_2^- \wedge \delta\bar{y}_6^-, \delta\bar{x}_2^+, 0 \wedge \delta\bar{y}_6^-, 0, 0, \delta\bar{y}_2^+, \delta\bar{y}_2^-, 0, \delta\bar{y}_6^-, 0 \wedge \delta\bar{x}_2^+, \right)$$

The power algorithm gives us  $h = (0, 0, -1, 0, -1, 0, 0, 0, -1, 0, -1, 0)$  (computed from the iterates of the vector with all coordinates equal to  $-1$ ). We know that there is an integer  $t < 0$  such that  $(\bar{x}, \bar{y}) - th$  is a fixed point of  $f$ .

$$f((\bar{x}, \bar{y}) - th) = f \begin{pmatrix} 0 \\ 15 \\ -1 + t \\ 16 \\ t \\ 15 \\ -5 \\ 15 \\ t \\ 4 \\ t \\ 15 \end{pmatrix} = \begin{pmatrix} 0 \\ 15 \\ 0 \wedge (-10 \vee t) - 1 \\ 16 \\ 0 \wedge -10 \vee t \\ 15 \\ t \vee -4 \wedge -5 \\ 15 \\ t \vee -4 \\ 4 \\ -10 \vee t \\ 15 \end{pmatrix}$$

The smallest such  $t$  is  $-4$ . We find a new fixed point  $(\tilde{u}, \tilde{v}) = (0, 15, -5, 16, -4, 15, -5, 15, -4, 4, -4, 15)$  for  $f$ . The semidifferential at  $(\tilde{u}, \tilde{v})$  is then:

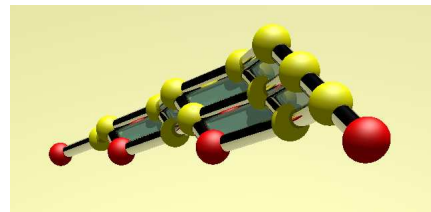


$$f'_{(\tilde{u}, \tilde{v})}(\delta\tilde{u}, \delta\tilde{v}) = \left(0, 0, \delta\tilde{v}_6^-, \delta\tilde{u}_2^+, \delta\tilde{v}_6^-, 0, 0, \delta\tilde{v}_2^+, \delta\tilde{v}_2^- \vee \delta\tilde{v}_4^-, 0, \delta\tilde{v}_6^-, 0 \wedge \delta\tilde{u}_2^+\right)$$

The power algorithm returns 0 (again with iterates of the vector identically equal to -1) , we conclude that  $(\bar{x}, \bar{y})$  is the smallest fixed point of  $f$ .

## Concluding remarks

- Open complexity/algorithmic issue: smallest fixed point of a Shapley operator, with negative discount.
- General use of nonconvex domains in static analysis, with SDP relaxations, current work with Adje and Goubault, to be developed.
- Use of tropical polyhedra to represent disjunctive constraints, with Allamigeon and Goubault (SAS'08).



Thank you!

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