A Bialgebra on Hypertree and Partition Posets

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SLC 72
A Bialgebra on Hypertree and Partition \textcolor{red}{Bounded} Posets

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1 Incidence Hopf Algebra of a Family of Bounded Posets
   - Incidence Hopf Algebra
   - Moebius number

2 Hypertree Posets
   - From Hypergraphs to Hypertrees
   - Hypertree Posets

3 Construction of a Bialgebra on Hypertree and Partition Bounded Posets
   - From the Incidence Hopf Algebra to a simpler Bialgebra
   - Computation of the Coproduct in this Bialgebra
   - Application: Computation of Moebius numbers of Hypertree Posets
Bounded poset = a poset with a least and a greatest element.

We consider posets up to isomorphisms of posets. Considered a family $\mathcal{P}$ of bounded posets which is
- Interval closed,
- Stable under direct product.

We endow the $\mathbb{Q}$-vector space $V_{\mathcal{P}}$ generated by $\mathcal{P}$ with
- a coproduct defined for all $P \in V_{\mathcal{P}}$ by:

$$\Delta[P] = \sum_{x \in P} [0_P, x] \otimes [x, 1_P],$$

- the direct product of posets.
We endow the \( \mathbb{Q} \)-vectorial space \( V_{\mathcal{P}} \) generated by \( \mathcal{P} \) with

- a **coproduct** defined for all \( P \in V_{\mathcal{P}} \) by:

\[
\Delta[P] = \sum_{x \in P} [0_P, x] \otimes [x, 1_P],
\]

- the direct product of posets.

**Theorem (W.R. Schmitt, 1994)**

\( (V_{\mathcal{P}}, \Delta, \times) \) is a Hopf Algebra, called *Incidence Hopf algebra*. 
Moebius number

**Definition**

For any poset $P$ the Moebius function is defined by:

\[
\mu(x, x) = 1, \quad \forall x \in P
\]

\[
\mu(x, y) = - \sum_{x \leq z < y} \mu(x, z), \quad \forall x < y \in P.
\]

If $P$ is bounded, the Moebius number of $P$ is $\mu(P) := \mu(\hat{0}, \hat{1})$
Idea:
The **coproduct** on the Incidence Hopf algebra enables us to **compute** **Moebius numbers** of posets in this algebra!
1 Incidence Hopf Algebra of a Family of Bounded Posets

2 Hypertree Posets
   - From Hypergraphs to Hypertrees
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3 Construction of a Bialgebra on Hypertree and Partition Bounded Posets
Hypergraphs and hypertrees

Definition (Berge, 1989)

A hypergraph (on a set $V$) is an ordered pair $(V, E)$ where:

- $V$ is a finite set (vertices)
- $E$ is a collection of subsets of cardinality at least two of elements of $V$ (edges).

The valency of a vertex $v$ in $H$ is the number of edges containing $v$.

Example of a hypergraph on $[1; 7]$
Walk on a hypergraph

Definition

Let $H = (V, E)$ be a hypergraph. A walk from $d$ to $f$ in $H$ is an alternating sequence of vertices and edges beginning by $d$ and ending by $f$:

$$(d, \ldots, e_i, v_i, e_{i+1}, \ldots, f)$$

where for all $i$, $v_i \in V$, $e_i \in E$ and $\{v_i, v_{i+1}\} \subseteq e_i$.

Examples of walks

![Diagram showing examples of walks in a hypergraph](image)
Hypertrees

Definition

A hypertree is a non-empty hypergraph $H$ such that, given any distinct vertices $v$ and $w$ in $H$,

- there exists a walk from $v$ to $w$ in $H$ with distinct edges $e_i$, ($H$ is connected),
- and this walk is unique, ($H$ has no cycles).

Example of a hypertree
The hypertree poset

Definition

Let $I$ be a finite set of cardinality $n$, $S$ and $T$ be two hypertrees on $I$.

$S \preceq T \iff$ Each edge of $S$ is the union of edges of $T$

We write $S \prec T$ if $S \preceq T$ but $S \neq T$.

Example with hypertrees on four vertices

$$
\begin{array}{cccc}
\spadesuit & \heartsuit & \spadesuit & \heartsuit \\
\clubsuit & \heartsuit & \spadesuit & \heartsuit \\
\spadesuit & \heartsuit & \spadesuit & \heartsuit \\
\end{array}
\quad \preceq \\
\begin{array}{cccc}
\clubsuit & \heartsuit & \spadesuit & \heartsuit \\
\spadesuit & \heartsuit & \spadesuit & \heartsuit \\
\spadesuit & \heartsuit & \spadesuit & \heartsuit \\
\end{array}
\quad \prec \\
\begin{array}{cccc}
\spadesuit & \heartsuit & \spadesuit & \heartsuit \\
\spadesuit & \heartsuit & \spadesuit & \heartsuit \\
\spadesuit & \heartsuit & \spadesuit & \heartsuit \\
\end{array}
\quad \text{but not}
\begin{array}{cccc}
\clubsuit & \heartsuit & \spadesuit & \heartsuit \\
\spadesuit & \heartsuit & \spadesuit & \heartsuit \\
\spadesuit & \heartsuit & \spadesuit & \heartsuit \\
\end{array}
\end{array}
$$
• Triangle-like poset
• \( HT_n = \) hypertree poset on \( n \) vertices.
• Möbius number : \( (n - 1)^{n-2} \) [McCammond and Meier 2004]
- Triangle-like poset
- $HT_n =$ hypertree poset on $n$ vertices.
- Möbius number : $(n - 1)^{n-2}$ [McCammond and Meier 2004]

**Goal:**

Construction of an analogue of Incidence Hopf algebra which enables us to compute again Möbius numbers of posets.
Incidence Hopf Algebra of a Family of Bounded Posets

Hypertree Posets

Construction of a Bialgebra on Hypertree and Partition Bounded Posets
- From the Incidence Hopf Algebra to a simpler Bialgebra
- Computation of the Coproduct in this Bialgebra
- Application: Computation of Moebius numbers of Hypertree Posets
Add a maximum element to triangle posets
Close by interval and product

⇒ Incidence Hopf algebra $\mathcal{H}$

Construction of a smaller bialgebra in which computation will be easier.
THE Bialgebra

Lemma (McCammond, Meier, 2004)

Let $\tau$ be a hypertree on $n$ vertices.

(a) The interval $[\hat{0}, \tau]$ is a direct product of partition posets (bounded posets),

(b) The half-open interval $[\tau, \hat{1})$ is a direct product of hypertree posets.

Family of direct products of hypertree posets and partition posets is interval closed and closed by direct product $\leadsto$ associated algebra $B$

We endow this algebra with the following coproduct:

$$\Delta(d) = \sum_{x \in d} [\hat{0}_d, x] \otimes [x, \hat{1}_d] \quad \text{and} \quad \Delta(t) = \sum_{x \in t} [\hat{0}_t, x] \otimes [x, \hat{1}_{\hat{t}}],$$

for a bounded poset $d \in B$ and a triangle poset $t \in B$, where $\hat{t}$ is the bounded poset obtained from $t$ by adding a greatest element.
THE Bialgebra

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for a bounded poset $d \in B$ and a triangle poset $t \in B$, where $\hat{\ell}$ is the bounded poset obtained from $t$ by adding a greatest element.

$B$ is a bialgebra.
Comparison between coproducts

- Same coproducts on bounded posets.

  In $\mathcal{H}$

  $$\Delta(\hat{t}) = \sum_{x \in \hat{t}} [\hat{0}, x] \otimes [x, \hat{1}]$$

  In $\mathcal{B}$

  $$\Delta(t) = \sum_{x \in t} [\hat{0}_t, x] \otimes [x, \hat{1}_t]$$

Why working in $\mathcal{B}$?

Because $[x, \hat{1}_t]$ can be written as a product of hypertree posets whereas $[x, \hat{1}]$ cannot!
Comparison between coproducts

- Same coproducts on bounded posets.
- In $\mathcal{H}$
  \[ \Delta(\hat{t}) = \sum_{x \in \hat{t}} [\hat{0}, x] \otimes [x, \hat{1}] \]
- In $B$
  \[ \Delta(t) = \sum_{x \in t} [\hat{0}_t, x] \otimes [x, \hat{1}_t] \]

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Comparison between coproducts

- Same coproducts on bounded posets.
- In $\mathcal{H}$
  \[ \Delta(\hat{t}) = \sum_{x \in \hat{t}} [\hat{0}, x] \otimes [x, \hat{1}] \]
- In $\mathcal{B}$
  \[ \Delta(t) = \sum_{x \in t} [\hat{0}_t, x] \otimes [x, \hat{1}_t] \]

Why working in $\mathcal{B}$?

Because $[x, \hat{1}_t]$ can be written as a product of hypertree posets whereas $[x, \hat{1}]$ cannot!
Computation of the Coproduct in this Bialgebra

Lemma (McCammond, Meier, 2004)

Let $\tau$ be a hypertree on $n$ vertices.

(a) The interval $[\hat{0}, \tau]$ is a direct product of partition posets, with one factor $p_j$ for each vertex in $\tau$ with valency $j$.

(b) The half-open interval $[\tau, \hat{1})$ is a direct product of hypertree posets, with one factor $HT_j = h_j$ for each edge in $\tau$ with size $j$.
Lemma (McCammond, Meier, 2004)

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\[
\Delta(h_n) = \sum_{(\alpha, \pi) \in P_n} c^n_{\alpha, \pi} p_\alpha \otimes h_\pi,
\]

where for all \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \) and \( \pi = (\pi_2, \pi_3, \ldots, \pi_l) \),
\[
p_\alpha = 1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k} \quad \text{and} \quad h_\pi = h_2^{\pi_2} h_3^{\pi_3} \ldots h_l^{\pi_l}.
\]
Lemma (McCammond, Meier, 2004)

Let $\tau$ be a hypertree on $n$ vertices.

(a) The interval $[\hat{0}, \tau]$ is a direct product of partition posets, with one factor $p_j$ for each vertex in $\tau$ with valency $j$.

(b) The half-open interval $[\tau, \hat{1})$ is a direct product of hypertree posets, with one factor $HT_j = h_j$ for each edge in $\tau$ with size $j$.

$$\Delta(h_n) = \sum_{(\alpha, \pi) \in \mathcal{P}_n} c_{\alpha, \pi}^n p_\alpha \otimes h_\pi,$$

where for all $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $\pi = (\pi_2, \pi_3, \ldots, \pi_l)$, $p_\alpha = 1^{\alpha_1}p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ and $h_\pi = h_2^{\pi_2}h_3^{\pi_3} \cdots h_l^{\pi_l}$.

$c_{\alpha, \pi}^n = \text{number of hypertrees in } h_n \text{ with:}$

- $\alpha_i$ vertices of valency $i$, $\forall i \geq 1$
- $\pi_j$ edges of size $j$, $\forall j \geq 2$
First criterion

Criterion for the vanishing of $c_{\alpha,\pi}^n$

$c_{\alpha,\pi}^n \neq 0 \iff \sum_{i=1}^{k} \alpha_i = n, \sum_{j=2}^{l} (j-1)\pi_j = n-1$ and $\sum_{i=1}^{k} i\alpha_i = n + \sum_{j=2}^{l} \pi_j - 1$. 
Counting hypertrees

A $\pi$-hooked partition $P$, for $\pi = (1, 2)$:

 Associated hypertree
Prüfer code

A $\pi$-hooked partition $P$, for $\pi = (1, 2)$:

Code:
Prüfer code

A $\pi$-hooked partition $P$, for $\pi = (1, 2)$:

Code:
Prüfer code

A $\pi$-hooked partition $P$, for $\pi = (1, 2)$:

Code : 1
Prüfer code

A $\pi$-hooked partition $P$, for $\pi = (1, 2)$:

Code: 1,
Prüfer code

A $\pi$-hooked partition $P$, for $\pi = (1, 2)$:

Code: 1, 6
Prufer code

A $\pi$-hooked partition $P$, for $\pi = (1, 2)$:

Code: 1, 6,
Prüfer code

A $\pi$-hooked partition $P$, for $\pi = (1, 2)$:

Code: 1, 6, 2
A \( \pi \)-hooked partition \( P \), for \( \pi = (1, 2) \):

Code: 1, 6, (2)
Prüfer code

A $\pi$-hooked partition $P$, for $\pi = (1, 2)$:

Code : 1, 6
Return of the Prüfer code

constructions of a rooted hypertree of valency set $\alpha$ from $P_\pi$

$\iff$

words on $[1, n]$, of length $k = \sum_{j\geq 2} \pi_j - 1$, with $\sum_{i\geq 2} \alpha_i$ different letters, where $\alpha_i$ letters appear $i - 1$ times, $\forall i \geq 2$

$\implies k! \times n! \over \prod_{i\geq 2} (i - 1)!^{\alpha_i \alpha_i}$.

Theorem (B.O.)

$\Delta(h_n) = \frac{1}{n} \times \sum_{(\alpha, \pi) \in \mathcal{P}(n)} \frac{n!}{\prod_{j\geq 2} (j - 1)!^{\pi_j \pi_j}} \times \frac{k! \times n!}{\prod_{i\geq 1} (i - 1)!^{\alpha_i \alpha_i}} \prod_{i=2}^{k} p_i^{\alpha_i} \otimes \prod_{j=2}^{l} h_j^{\pi_j}$. 
Application: Computation of Moebius numbers of Hypertree Posets

Theorem (McCammond and Meier 2004)

The Moebius number of the augmented hypertree poset on $n$ vertices is given by:

$$\mu(\widehat{HT}_n) = (-1)^{n-1}(n-1)^{n-2}.$$ 

The following equality holds:

$$(n - 1)^{n-2} = \sum_{(\alpha, \pi) \in \mathcal{P}(n)} \frac{(-1)^{i\alpha_{i-1}}}{n} \times \frac{n!}{\prod_{j \geq 2} (j-1)!^{\pi_j \pi_j!}} \times \frac{k! \times n!}{\prod_{i \geq 1} \alpha_i!},$$

where $\mathcal{P}(n) = (\alpha = (\alpha_1, \ldots, \alpha_k), \pi = (\pi_2, \ldots, \pi_l))$ satisfying:

$$\sum_{i=1}^{k} \alpha_i = n, \quad \sum_{j=2}^{l} (j-1)\pi_j = n - 1, \quad \text{and} \quad \sum_{i=1}^{k} i\alpha_i = n + \sum_{j=2}^{l} \pi_j - 1.$$
Thank you very much!