Taylor expansion for Call-By-Push-Value

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2 Quantitative semantics

3 Quantitative syntax

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6 Conclusions

Programs as transformations

$P : A \to B$ transforms elements of A into elements of B.

A and B can be ordinary types, or more complex objects as power series, probability distributions,. . .

But P is still a transformation, P *does* something.

Resource consumption

A program $P : A \to B$, as every natural process, needs energy to run. It *consumes* the elements of A.

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Quantitative semantics

Interpret precisely duplication and erasing in the semantics.

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Denotational semantics of Λ

Calculus	$M, N ::= x \mid \lambda x M \mid MN$
Operations	$(\lambda x M)N \to_\beta M[N/x]$
Interpretation	$[\![M]\!]_{\mathcal{M}}$ (something in $\mathcal{M}$ invariant under $\to_\beta$)

For $M : A \to B$, $[\![M]\!]$ can be *e.g* a function/relation from $[\![A]\!]$ to $[\![B]\!]$.

- Status of $[\![M]\!]$ in $(\lambda x(xx))M$? *wrt* $(\lambda xx)M$?
- Interpret probabilistic reduction ?
- ...

Girard (Normal functors, 1988)

Uses of arguments $\rightsquigarrow$ degree of a monomial in a power series.

Types: $[\![A]\!] \subseteq \mathbb{S}^{|A|}$ where $\mathbb{S}$ is a semiring
Programs : power series

Quantitative Semantics

Example : multirelations

$\mathbb{S}$	$\rightsquigarrow$	Boolean semiring
Types	$\rightsquigarrow$	$[\![A \to B]\!] = \mathcal{M}_{\text{fin}}(\mid A \mid) \times \mid B \mid$
Programs	$\rightsquigarrow$	$P : A \to B \Rightarrow [\![P]\!] \subseteq \mathcal{M}_{\text{fin}}(\mid A \mid) \times \mid B \mid$
Invariance	$\rightsquigarrow$	Composition of multirelations.

Key idea

let $M : A \to B,\ N : A$.
$([a_1, \ldots, a_k], b) \in [\![M]\!]$ will match with k uses of the argument N in the application (MN).

- Probabilistic coherence spaces (Danos, Ehrhard). (Fully abstract for probabilistic PCF (Ehrhard, Pagani, Tasson 2015))
- Weighted relational model (Laird, McCusker, Manzonetto, Pagani)
- Finiteness spaces (Ehrhard)
- Convenient vector spaces (Blute, Ehrhard, Tasson) (Kerjean).

From Semantics to Syntax

Quantitative syntax

Extract a calculus from the multiset theoretic-semantics in order to provide a step-by-step analysis of duplication.

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Resource calculus

A quantitative syntax

$$m, n ::= x \mid \lambda x m \mid \langle m \rangle [n_1, \ldots, n_k] \text{ (k-linear application)}$$

$$\langle \lambda x m \rangle [n_1, \ldots, n_k] \to_\partial \sum_{\sigma \in \mathfrak{S}_n} m \left[n_{\sigma(1)}/x_1, \ldots, n_{\sigma(k)}/x_k \right]$$

$$\langle \lambda x \langle x \rangle [x, x] \rangle [z] \to_\partial 0 \quad {}_\partial\!\leftarrow \langle \lambda x x \rangle [z, z, z]$$

λ-calculus		resource calculus
MN	$\rightsquigarrow$	$\langle m \rangle [n_1, \ldots, n_k]$
$(\lambda x (x) x) z$	$\rightsquigarrow$	$\langle \lambda x \langle x \rangle [x] \rangle [z, z]$
$\downarrow_\beta$		$\downarrow_\partial$
zz	$\rightsquigarrow$	$\langle z \rangle [z] + \langle z \rangle [z]$

Multilinear Approximations

We define the approximation relation $m \lhd M$:

- $x \lhd x$
- $\lambda x n \lhd \lambda x N$ if $n \lhd N$.
- $\langle m \rangle [n_1, \ldots, n_k] \lhd MN$ for all $k \in \mathbb{N}$ if $m \lhd M$ and $n_i \lhd N$.

(Resource terms can be seen as polynomials that approximate power series.)

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Property : simulation of $\to_\beta$ with approximants

If $m \lhd M$ and $M \to M'$, $m \to 0$ or $\exists m'$ s.t. $m \rightrightarrows_\partial m'$ and $m' \lhd M'$.

Parallel reduction

Definition

We extend $\to_\partial$ to a parallel reduction $\rightrightarrows_\partial$.

Example

- $MN \to MN'$
- Let $\langle m\rangle[n_1, \ldots, n_k] \lhd MN$.
- $\langle m\rangle[n_1, \ldots, n_k] \rightrightarrows_\partial \langle m\rangle[n'_1, \ldots, n'_k] \lhd MN'$ if $n_i \rightrightarrows_\partial n'_i$ for **all** i.

In the resource setting, Taylor expansion consists in taking infinite sums of resource terms.

The idea is that taken together, the combination of all $m \lhd M$, behave exactly as M.

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Taylor expansion - Combining approximants

A bridge between syntax and semantics

Semantic approach : Interpret a term/function as an infinite series of approximants.

Syntactic Taylor expansion :

$$\mathcal{T}(MN) = \sum_{k \in \mathbb{N}} \frac{1}{k!} \langle \mathcal{T}(M) \rangle [\mathcal{T}(N), \dots, \mathcal{T}(N)]_k$$

$\mathcal{T}(\lambda x M) = \lambda x \mathcal{T}(M) \qquad \mathcal{T}(x) = x.$

Remark

$\mathcal{T}(M)$ is a weighted sum of <u>all</u> resource nets m s.t. $m \lhd M$

Wanted result: correction

Extend $\rightrightarrows_\partial$ to infinite sums of terms ($\Rightarrow_\partial$), in order to have
$M \to_\beta N \Rightarrow \mathcal{T}(M) \Rightarrow_\partial \mathcal{T}(N)$.

Simulation

A convergence problem

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Problem

Can $\Rightarrow_\partial$ be always well-defined ?

Simulation

A convergence problem

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Problem

Can $\Rightarrow_\partial$ be always well-defined ?

No

$$\sum_{k\in\mathbb{N}} \langle \lambda xx \rangle [\langle \lambda xx \rangle \dots [y]]] \dots] \Rightarrow_\partial \infty \cdot y$$

If $\mathbb{S}$ is not a complete semiring, the reduction is not defined for any series.

Some correction results

$\Rightarrow_\partial$ is well defined and simulates $\to_\beta$ in Taylor expansion:

- Classical Λ: Ehrhard Regnier – 2007.
- Non deterministic Λ (finite sums): Pagani, Tasson, Vaux-Auclair – CSL 2016.
- Algebraic calculus: Vaux-Auclair – CSL 2017.
- Linear Logic proof nets: Chouquet, Vaux-Auclair – CSL 2018
- Call-By-Value, PCF, Call-By-Need: Chouquet – ~~CSL~~ MFPS 2019

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Call-By-Push-Value

A Linear Logic inspired-presentation (Ehrhard)

$$M, N ::= x \mid \lambda x M \mid \langle M \rangle N \mid (M, N) \mid \pi_i(M) \mid \iota_i(M) \mid$$
$$\mathbf{case}(M, y \cdot N_1, z \cdot N_2) \mid M^! \mid \mathbf{der}(M) \mid \mathbf{fix}_x(M)$$

$$V, U ::=: x \mid \lambda x M \mid M^! \mid (V, U) \mid \iota_i(V)$$

$$A, B ::= !I \mid A \otimes B \mid A \oplus B \qquad (positives)$$
$$I, J ::= A \mid A \multimap B \mid \top \qquad (general)$$

- Subsumes Call-By-Name and Call-By-Value at the operational and denotational level
- Results in quantitative semantics (Ehrhard-Tasson)
- Coinductive datatypes

$$\frac{}{\Gamma, x : A \vdash x : A} \qquad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x M : A \multimap B}$$

$$\frac{\Gamma \vdash M : A_1 \otimes A_2}{\Gamma \vdash \pi_i(M) : A_i}\ i \in \{1,2\} \qquad \frac{\Gamma \vdash M : A \multimap I \qquad \Delta \vdash N : A}{\Gamma, \Delta \vdash \langle M \rangle N : I}$$

$$\frac{\Gamma \vdash M : A \qquad \Delta \vdash N : B}{\Gamma, \Delta \vdash (M, N) : A \otimes B} \qquad \frac{\Gamma \vdash M : A_i}{\Gamma \vdash \iota_i(M) : A_1 \oplus A_2}\ i \in \{1,2\}$$

$$\frac{\Gamma \vdash m : !A}{\Gamma \vdash \mathbf{der}(m) : A} \qquad \frac{\Gamma, x : !I \vdash M : I}{\Gamma \vdash \mathbf{fix}_x(M) : I} \qquad \frac{\Gamma \vdash M : I}{\Gamma \vdash M^! : !I}$$

$$\frac{\Gamma \vdash M_1 : A \oplus B \qquad \Delta \vdash M_2 : I \qquad \Theta \vdash M_3 : I}{\Gamma, \Delta, \Theta \vdash \mathbf{case}(M_1, y \cdot M_2, z \cdot M_3) : I}$$

$\langle \lambda x M \rangle V \to_{\mathsf{pv}} M[V/x]$

$\pi_i(V_1, V_2) \to_{\mathsf{pv}} V_i$

$\mathbf{case}(\iota_i(V), x_1 \cdot M_1, x_2 \cdot M_2) \to_{\mathsf{pv}} M_i[V/x_i]$

$\mathbf{der}(M^!) \to_{\mathsf{pv}} M$

$\mathbf{fix}_x(M) \to_{\mathsf{pv}} M[(\mathbf{fix}_x(M))^!/x]$

Duplication : Exponentials VS coalgebras morphisms

	Λ	Λ_{pv}
Application	$(\lambda x M)N$	$\lambda x M(V_1, V_2)$
Approximation	$\langle \lambda x m\rangle[n_1, \ldots, n_k]$	$\langle \lambda x m\rangle(v_1, v_k)$
Interpretation	$N : !A$	$(V_1, V_1) : P_1 \otimes P_2$

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$$P_1 \otimes P_2 \xrightarrow{h} (P_1 \otimes P_2) \otimes \ldots\ldots \otimes (P_1 \otimes P_2)$$

h is a morphism coming from the coalgebra structure in the interpretation of positive types (the duplicable ones).

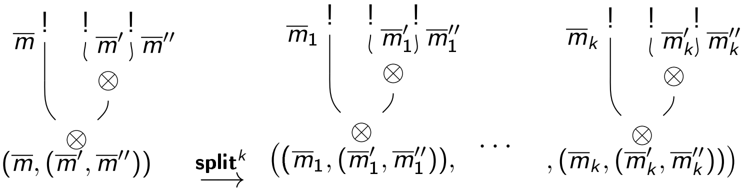
If duplication exists in the semantics, what about syntactic Taylor expansion ?

Resource reduction with splitting

Splitting operator « **split** »

$\mathbf{split}(([m,m],[m,m])) = ([m],[m]),([m],[m])$

In general : $\mathbf{split}^k(v) = (v_1,\ldots,v_k)$ where the v_i have the same syntactic tree than v.



Call-By-Push-Value resource reduction

$$\langle \lambda x m\rangle v \to \sum_{(v_1,\ldots,v_k)\in \mathbf{split}^k(v)} m[v_1/x_1,\ldots,v_k/x_k]$$

Then we have a resource calculus for Call-By-Push-Value and we can define Taylor expansion.

Theorem

For any Call-By-Push-Value term M, if $M \to N$, then $\mathcal{T}(M) \Rightarrow \mathcal{T}(N)$

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- Taylor expansion is a bridge between syntax and semantics
- Its definition and consistence *w.r.t* the models may be tricky and depends on the calculus
- Following the semantics, we can build a convenient resource calculus for Call-By-Push-Value giving a syntactic account to coalgebras morphisms, and prove the correction of Taylor expansion

Some perspectives :

- Link these results to Linear Logic proof nets
- Extend Taylor expansion's correction to non uniform settings of Call-By-Push-Value
- Go to lunch