HIGHER-DIMENSIONAL REWRITING

Yves GUIRAUD

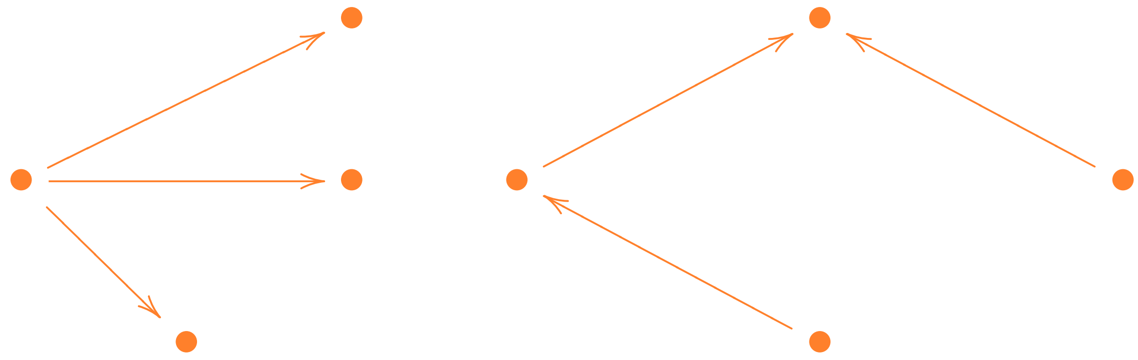
Institut de Mathématiques de Luminy
http://iml.univ-mrs.fr/~guiraud

GÉOCAL 2006 – Algebra and computation week
Marseille – 2nd February 2006

Examples

Rewriting in dimension 1

Consider a set (points, "0-cells"):



Directed equations (arrows, "1-cells") yield an equivalence relation (connex components)

Paths ("1-arrows") represent computations seeking representatives

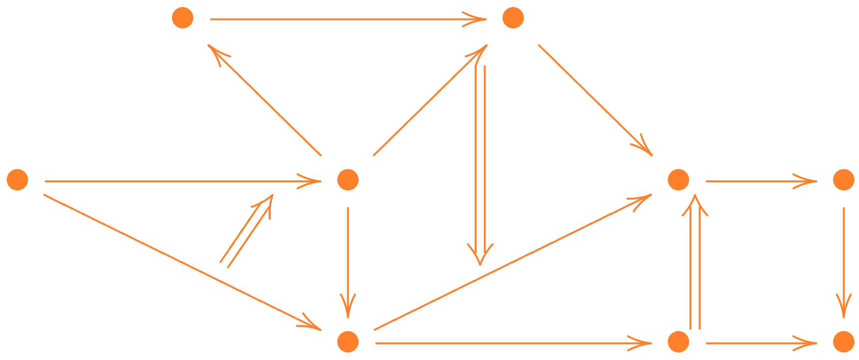
One-dimensional rewriting: computational properties of 1-arrows in "1-polygraphs"

- Termination: no infinite path $\rightsquigarrow$ Existence of computable representatives
- Confluence: every diagram $\cdot \leftarrow \cdot \rightarrow \cdot$ can be extended into a square $\rightsquigarrow$ Unicity of computable representatives
- Convergence: termination + confluence $\rightsquigarrow$ Decision procedure for equality

Example: *abstract rewriting systems* (sets equipped with binary relations)

Rewriting in dimension 2

Consider a graph (0-cells and 1-cells):



Directed equations between parallel 1-arrows ("2-cells") yield an equivalence relation (2-connex components)

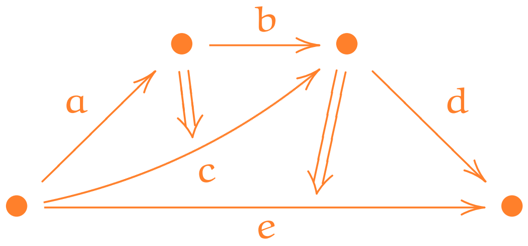
Two-dimensional paths ("2-arrows") represent computations seeking representatives

Two-dimensional rewriting: computational properties (termination and confluence) of 2-arrows in "2-polygraphs"

Rewriting in dimension 2

Main example: word/string rewriting systems or monoid presentations

Generators are 1-cells and equations are 2-cells between 1-arrows:



This is a directed presentation of the following monoid:

$$\langle a, b, c, d, e \rangle / (ab = c, cd = e)$$

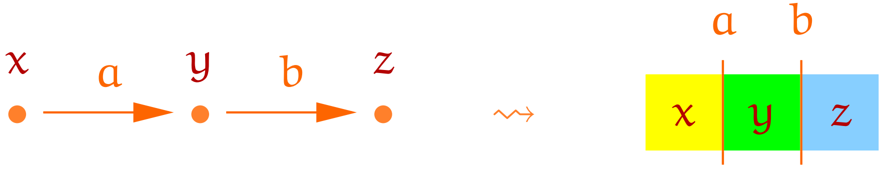
Any convergent 2-polygraph presenting this monoid yields a decision procedure for the word problem (equality in the quotient)

Representations of 2-arrows

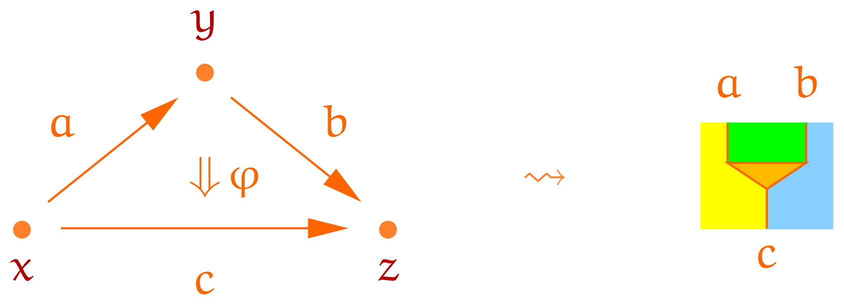
Cell-like representations for 2-arrows are not practical ⇝ *string diagrams*

- 0-cells are labelled parts of the plane:

- 1-cells are labelled vertical lines, separating two parts of the plane:

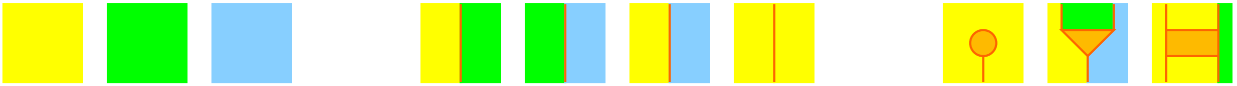


- 2-cells are "circuit components":

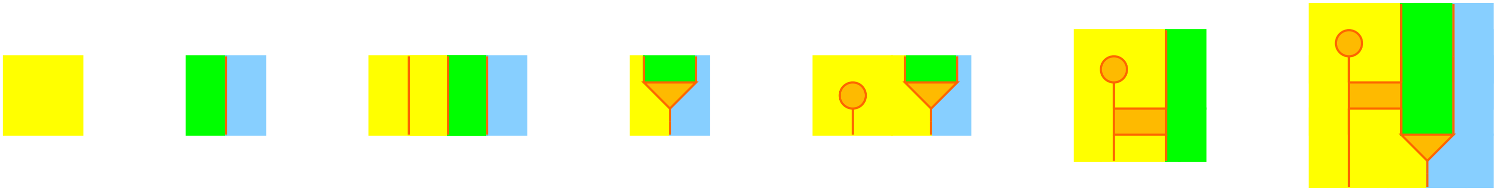


Rewriting in dimension 3

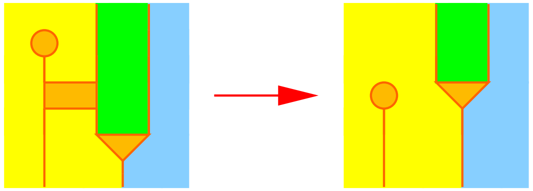
Consider a 2-polygraph (0-cells, 1-cells and 2-cells):



Its 2-arrows are all the circuits that can be built with these cells:



An equivalence relation on circuits can be specified by directed equations ("3-cells"):

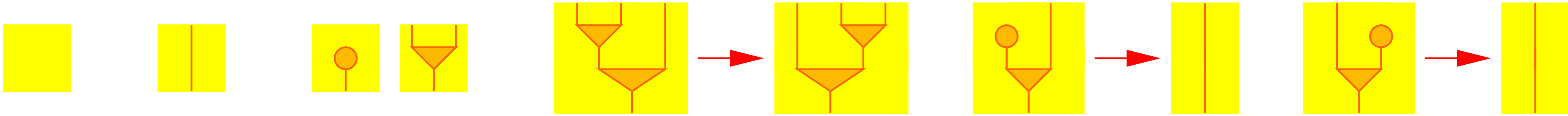


Three-dimensional rewriting: termination/confluence of paths generated by 3-cells

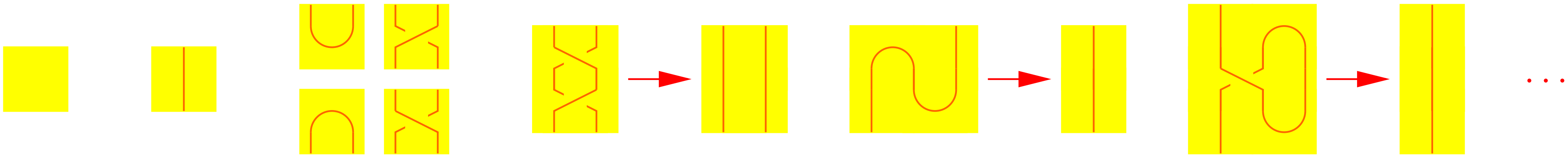
Rewriting in dimension 3

Many examples:

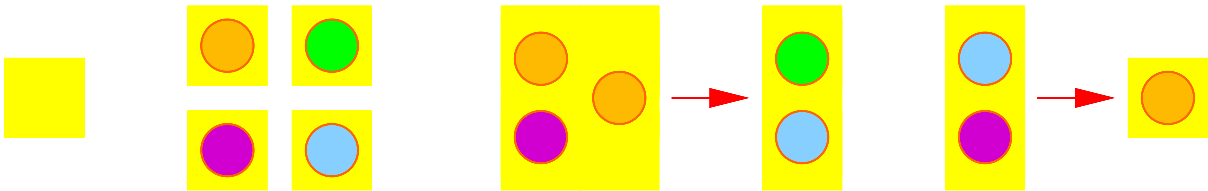
- From algebra: monoids



- From topology: tangles



- From computer science: Petri nets



Rewriting in dimension 3

Here we only study monochromatic 3-dimensional rewriting:

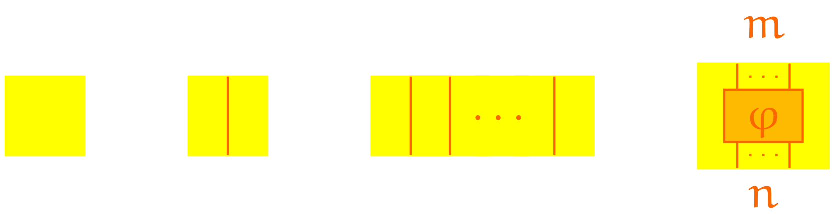
- 3-dimensional because it is the most frequent case
- Monochromatic (one 0-cell and one 1-cell) because:
  - The most frequent in 3-dimensional rewriting
  - Simplifies the definitions but conveys the same ideas
  - General case can often be reduced to monochromatic one

Polygraphs

Monochromatic 2-polygraphs

A *monochromatic 2-polygraph* is a graph $\mathcal{C} \rightrightarrows \mathbb{N}$ over the set $\mathbb{N}$ of natural numbers

To recover the intuitions of part 1:

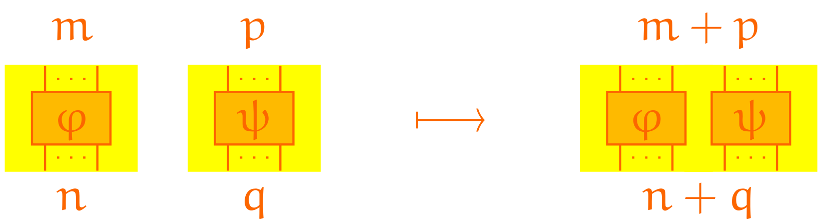


- The number 0 is the only 0-cell
- The number 1 is the only 1-cell
- The number n is the 1-arrow made of n times the 1-cell: $n = 1 \otimes \cdots \otimes 1$
- An arrow $\varphi : m \to n$ is a 2-cell with m inputs and n outputs

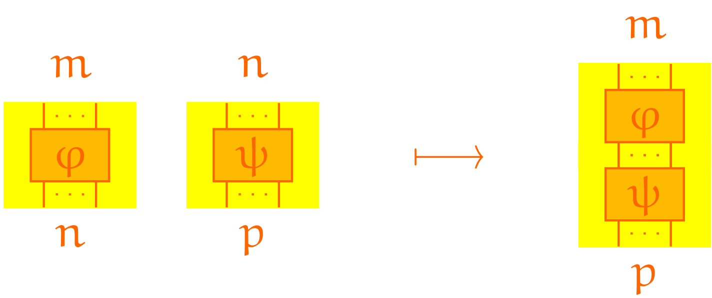
Monochromatic 2-polygraphs

The 2-arrows of a monochromatic 2-polygraph are built this way:

- Every natural number n is a 2-arrow $n \to n$
- Every 2-cell $\varphi : m \to n$ is a 2-arrow $m \to n$
- Two-arrows $\varphi : m \to n$ and $\psi : p \to q$ yield a new 2-arrow $\varphi \otimes \psi : m + p \to n + q$



- Two-arrows $\varphi : m \to n$ and $\psi : n \to p$ yield a new 2-arrow $\psi \circ \varphi : m \to p$



Monochromatic 2-polygraphs

The two-arrows are identified by the following relations:

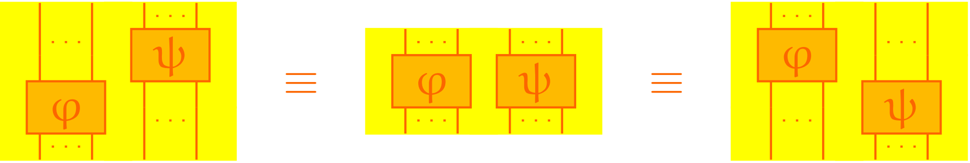
- The operation $\otimes$ is $+$ on 1-arrows ($m \otimes n \equiv m + n$)
- The operation $\otimes$ is associative and 0 is its neutral element
- The operation $\circ$ is associative (when defined)

For the moment: these equations are invisible on string diagrams

- Local units for $\circ$: for every $\varphi : m \to n$, $n \circ \varphi \equiv \varphi \equiv \varphi \circ m$

- Exchange relation: for every $\varphi : m \to n$ and $\psi : p \to q$,

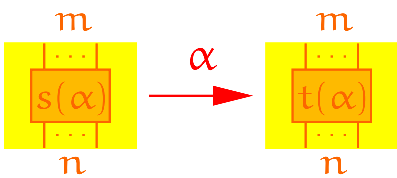
$$(\varphi \otimes q) \circ (m \otimes \psi) \equiv \varphi \otimes \psi \equiv (n \otimes \psi) \circ (\varphi \otimes p)$$



Monochromatic 3-polygraphs

A *monochromatic 3-polygraph* is a data $\mathcal{C} = (\mathcal{C}_2, \mathcal{C}_3, s, t)$ such that:

- $\mathcal{C}_2$ is a 2-polygraph
- $\mathcal{C}_3$ is a set (*3-cells*)
- s and t are maps from $\mathcal{C}_3$ to the set of 2-arrows of $\mathcal{C}_2$
- for every α in $\mathcal{C}_3$, $s(\alpha)$ and $t(\alpha)$ are parallel 2-arrows



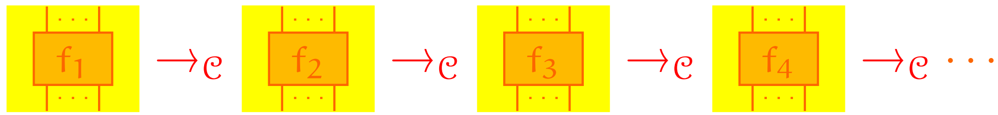
Such a 3-polygraph generates a binary relation $\to_{\mathcal{C}}$ on 2-arrows (*reduction relation*)

Definition: $f \to_{\mathcal{C}} g$ whenever there exist 2-arrows h and k such that:

Termination

Definitions

A monochromatic 3-polygraph $\mathcal{C}$ *terminates* when there is no infinite path:



A *termination order* for a 3-polygraph $\mathcal{C}$ is an order relation $\leq$ on 2-arrows such that:

- There is no infinite sequence $f_1 > f_2 > \cdots > f_n > \cdots$
- The compositions $\otimes$ and $\circ$ are strictly monotone in both arguments
- Every 3-cell α satisfies $s(\alpha) > t(\alpha)$

Lemma: if a 3-polygraph can be equipped with a termination order then it terminates

Thus: building a termination order is a good way to prove termination

Question: how?

Termination order – first try

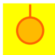
Let us proceed as in term rewriting:

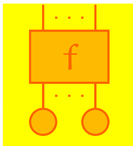
- Fix a set X with an order $\leq$ such that there is no infinite sequence $x_1 > x_2 > \cdots$
- Send each 2-cell $\varphi : m \to n$ onto a monotone map $\varphi_* : X^m \to X^n$
- Extend $(\cdot)_*$ by functoriality
- Get back $\leq$ on 2-arrows:

  if $\varphi, \psi : m \to n$, then $\varphi \leq \psi$ whenever $\varphi(\vec{x}) \leq \psi(\vec{x})$ for every $\vec{x} \in X^m$
- Prove that $s(\alpha) > t(\alpha)$ for every 3-cell α

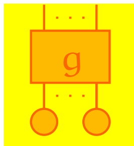
This works with terms but not with polygraphs

Indeed: $f > g \;\not\Rightarrow\; h \circ f > h \circ g$

Example: if there is a 2-cell : $1 \to 0$ then, for every parallel f and g,



$_* \;=\;$



$_* \;=\; (X^m \twoheadrightarrow \{*\})$

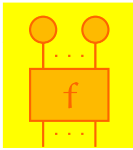
Termination order – second try

Let use a dual interpretation:

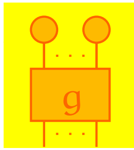
- Send each 2-cell $\varphi : m \to n$ onto a monotone map $\varphi^* : Y^n \to Y^m$
- $\ldots$

We get the dual problem: $f > g \;\not\Rightarrow\; f \circ h > g \circ h$

If there is some : $0 \to 1$, then:



$^* =$



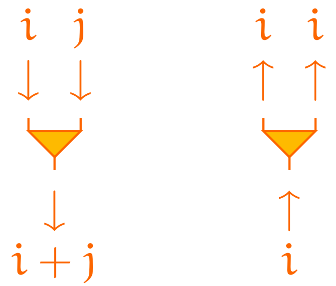
$^* = \left(Y^n \twoheadrightarrow \{*\}\right)$

Termination order – third try

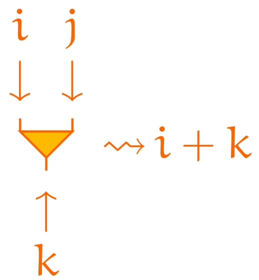
Let us keep both previous interpretations $(\cdot)_*$ and $(\cdot)^*$

Each 2-cell φ is seen as an electronic component

The maps φ_* and φ^* tell how φ transmits currents, such as:



A third interpretation $[\varphi]$ is built, giving the "heat" produced by φ:

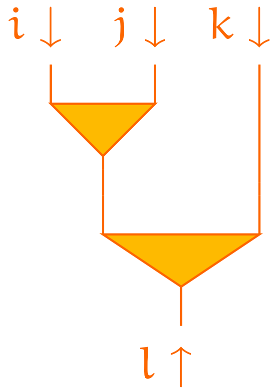


The heat produced by a 2-arrow is the sum of the heats produced by each 2-cell

Two-arrows are compared according to the heat they produce

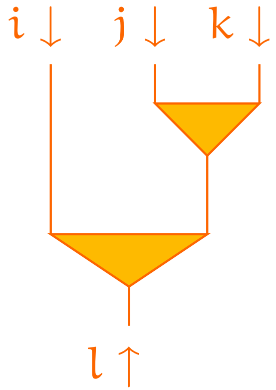
Termination order – third try

Example 1:



$$\rightsquigarrow \ 2i + j + 2l$$

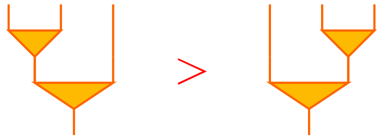
Example 2:



$$\rightsquigarrow \ i + j + 2l$$

Termination order – third try

Conclusion:



This can be used for any polygraph with a 3-cell corresponding to *associativity*

Such interpretations yield termination orders for 3-polygraphs

They have been used to prove:

- The termination of $L(\mathbb{Z}_2)$: a 3-polygraph for the structure of $\mathbb{Z}/2\mathbb{Z}$-vector spaces

  Important example: contains a 3-cell for commutativity

  This kind of rule cannot be contained in a terminating term rewriting system

- The existence of the 3-polygraph $\Delta(\Sigma, R)$ associated to a term rewriting system (Σ, R)

- The equivalence between the termination of (Σ, R) and $\Delta(\Sigma, R)$

Confluence

Definitions

We denote by $\twoheadrightarrow_{\mathcal{C}}$ the reflexive and transitive closure of $\rightarrow_{\mathcal{C}}$

The 3-polygraph $\mathcal{C}$ is *confluent* if:

For every 2-arrows $\mathfrak{f}$, $\mathfrak{g}$ and $\mathfrak{h}$ such that
$$\begin{array}{ccc} \mathfrak{f} & \overset{\mathcal{C}}{\twoheadrightarrow} & \mathfrak{g} \\ {\scriptstyle \mathcal{C}}\big\downarrow & & \\ \mathfrak{h} & & \end{array}$$

There exists a 2-arrow $\mathfrak{k}$ such that
$$\begin{array}{ccc} \mathfrak{f} & \overset{\mathcal{C}}{\twoheadrightarrow} & \mathfrak{g} \\ {\scriptstyle \mathcal{C}}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{C}} \\ \mathfrak{h} & \underset{\mathcal{C}}{\twoheadrightarrow} & \mathfrak{k} \end{array}$$

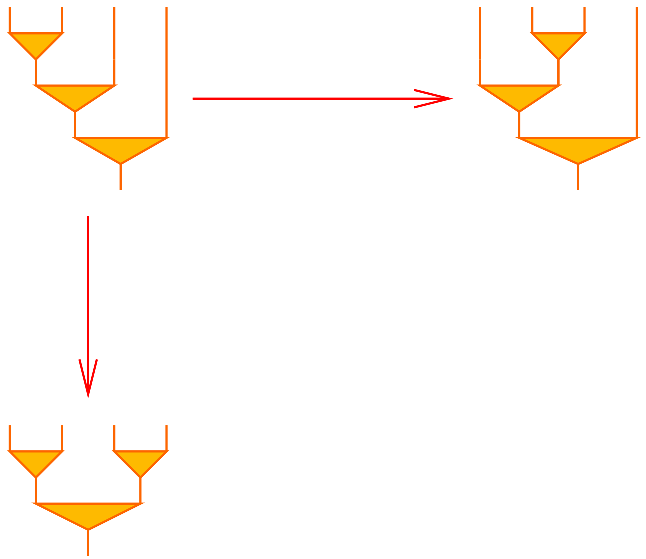
The *critical pairs* are central in confluence issues

Critical pairs

A *critical pair* of $\mathfrak{C}$ is a diagram
$$\begin{array}{ccc} \mathfrak{f} & \xrightarrow{\mathfrak{C}} & \mathfrak{g} \\ {\scriptstyle \mathfrak{C}}\downarrow & & \\ \mathfrak{h} & & \end{array}$$
such that:

- $\mathfrak{f}$ is an "overlap" of minimal size of the sources of two 3-cells α and β
- $\mathfrak{g}$ is produced from $\mathfrak{f}$ by the application of α
- $\mathfrak{h}$ is produced from $\mathfrak{f}$ by the application of β

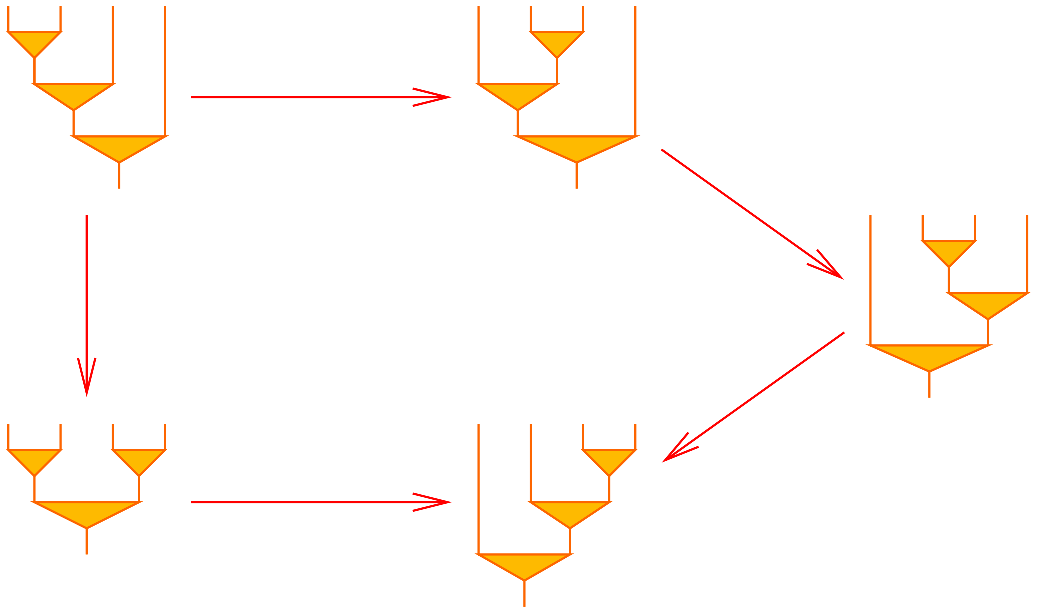
Example: the associativity rule generate the following critical pair with itself



Critical pairs

A critical pair $\begin{array}{ccc} \mathfrak{f} & \xrightarrow{\mathfrak{e}} & \mathfrak{g} \\ {\scriptstyle\mathfrak{e}}\downarrow & & \\ \mathfrak{h} & & \end{array}$ is *confluent* if there exists $\begin{array}{ccc} \mathfrak{f} & \xrightarrow{\mathfrak{e}} & \mathfrak{g} \\ {\scriptstyle\mathfrak{e}}\downarrow & & \downarrow\!\!\downarrow{\scriptstyle\mathfrak{e}} \\ \mathfrak{h} & \underset{\mathfrak{e}}{\twoheadrightarrow} & \mathfrak{k} \end{array}$

Example: the critical pair of associativity is confluent



Critical pairs

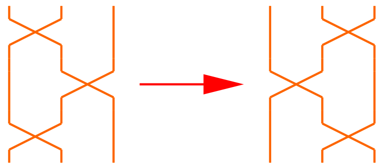
If all the critical pairs of $\mathcal{C}$ are confluent, $\mathcal{C}$ is *locally confluent*

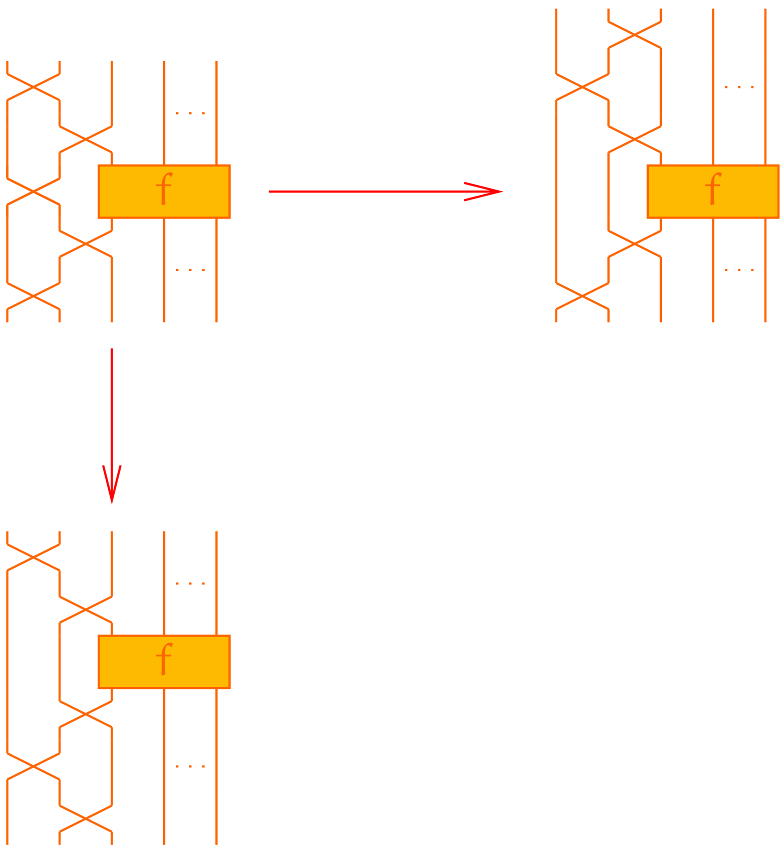
Moreover, if $\mathcal{C}$ terminates, then it is confluent (and thus convergent)

Hence: the determination and study of critical pairs is essential

But : much more complicated than in term rewriting

Critical pairs

Example: the Yang-Baxter 3-cell  generates



One for each 2-arrow $f : m + 1 \to n + 1$

Confluence depends on f

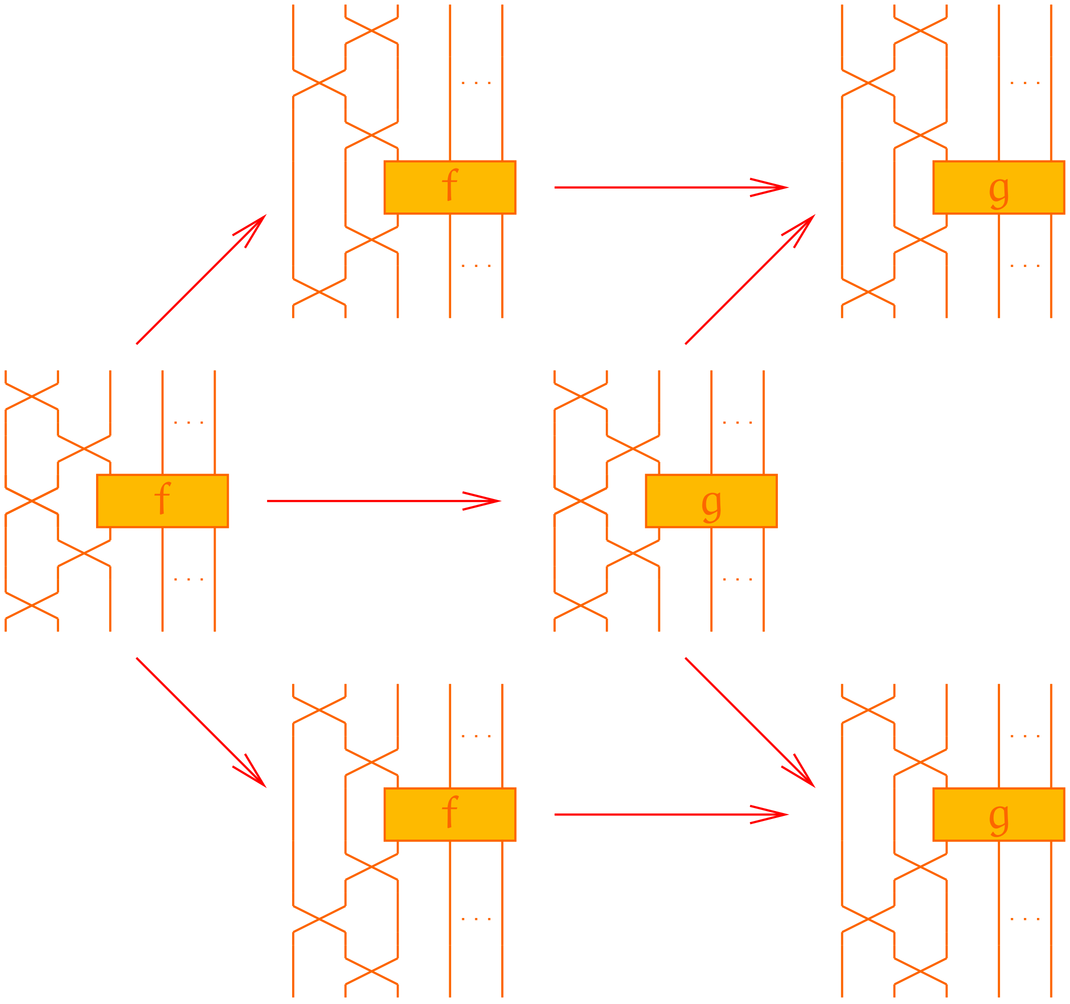
One 3-cell $\rightsquigarrow$ infinite number of checks

More complicated than term rewriting

Critical pairs

For the moment: no general theorem to study confluence

But: if $\mathsf{f} \rightarrow_{\mathcal{C}} \mathsf{g}$ then



Confluence of critical pair for f

iff

Confluence of critical pair for g

Critical pairs

If $\mathcal{C}$ is terminating
we only have to check confluence of the critical pairs for f in *normal form*

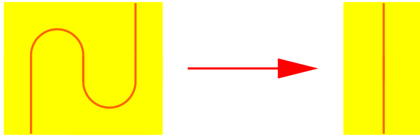
So far: terminating polygraphs with a finite number of possible shapes for normal forms

- The polygraph $L(\mathbb{Z}_2)$ of $\mathbb{Z}/2\mathbb{Z}$-vector spaces is *convergent*
- The polygraph $\Delta(\Sigma, R)$ associated to a *left-linear* term rewriting system (Σ, R) is *convergent* if and only if (Σ, R) is convergent

Future developments

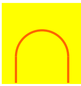
Termination

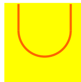
Termination technique can be enhanced

It cannot prove the termination of the 3-cell 

Indeed: for $(\cdot)_*$, the left part will be a constant map, while the right part is the identity

Set-theoretic maps φ_* and φ^* should be replaced by *linear* maps, such as:

 $_* : \; k \longrightarrow V \otimes V^*$, $\; 1 \longmapsto \sum_{i=1}^{n} e_i \otimes e_i^*$

 $_* : \; V^* \otimes V \longrightarrow k$, $\; e_i^* \otimes e_j \longmapsto \delta_{ij}$

Confluence

Create a general procedure for computation and classification of critical pairs

This will be a first step towards a *completion procedure* for 3-polygraphs:

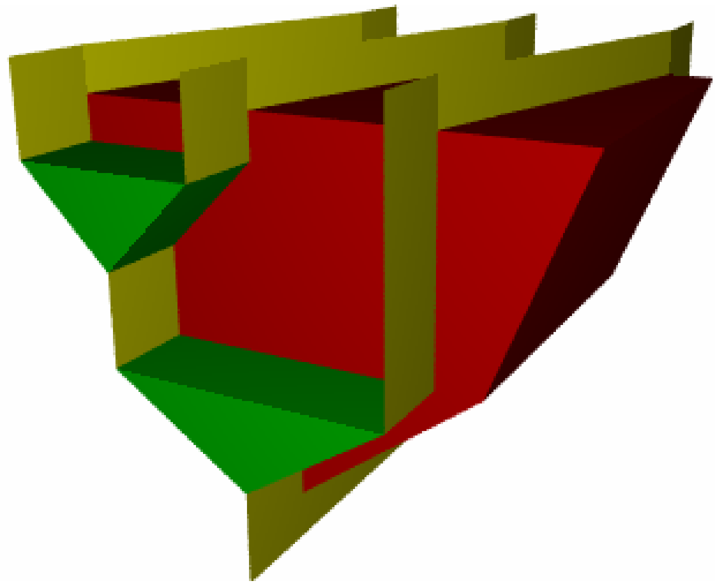
- Data: a 3-polygraph
- Goal: a convergent 3-polygraph with the same 3-connex components

The third dimension

Rewriting rules on 2-arrows are called *3-cells*

Because they *are* 3-dimensional objects

And can be represented as such:



Proofs of propositional calculus can be represented like this

Thus: computations on them can be described by 4-dimensional rewriting...

..we will see that in week 4