Termination orders for operad presentations

Yves Guiraud

Institut de Mathématiques de Luminy
Université Aix-Marseille 2
`guiraud@iml.univ-mrs.fr`

Rencontres GÉOCAL : Complexité implicite et logique
Université Paris-Nord
6 et 7 septembre 2004

Overview :

Overview :

- Equational theories and term rewrite systems

Overview :

- Equational theories and term rewrite systems
- Resource management

Overview :

- Equational theories and term rewrite systems
- Resource management
- The presentation $L(\mathbb{Z}_2)$

Equational theories and term rewrite systems

The equational theory of monoids :

The equational theory of monoids :

- Generators (operators) :
  - Product $\mu : 2 \to 1$
  - Unit $\eta : 0 \to 1$

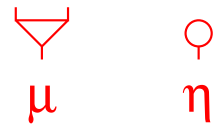
The equational theory of monoids :

- Generators (operators) :
  - Product $\mu : 2 \to 1$
  - Unit $\eta : 0 \to 1$
- Relations (equations, equalities) :
  - Associativity $\mu(\mu(x, y), z) = \mu(x, \mu(y, z))$
  - Left neutral $\mu(\eta, x) = x$
  - Right neutral $\mu(x, \eta) = x$

Examples of terms in the theory of monoids :

Examples of terms in the theory of monoids :

- Operator representations :

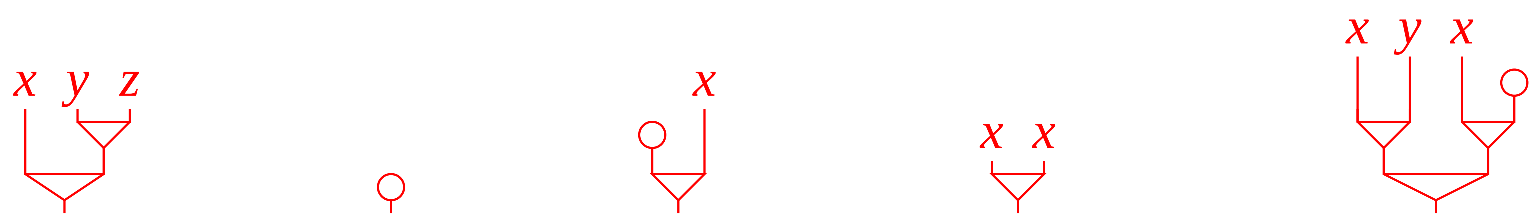


Examples of terms in the theory of monoids :

- Operator representations :

μ η

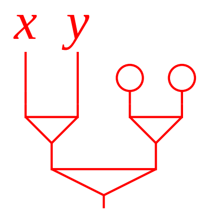
- Representations for five terms :



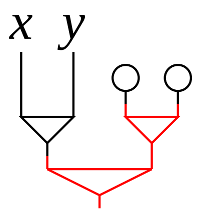
$\mu(x,\mu(y,z))$ η $\mu(\eta,x)$ $\mu(x,x)$ $\mu(\mu(x,y),\mu(x,\eta))$

Equality between two terms :

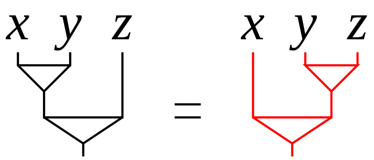
Equality between two terms :



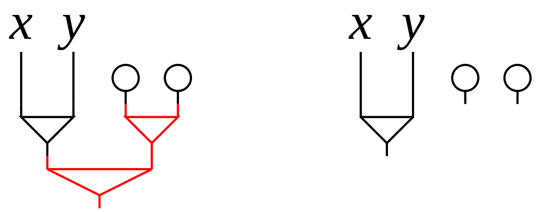
Equality between two terms :



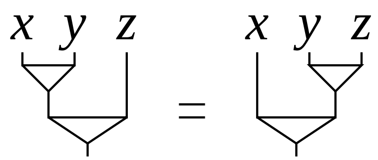
Associativity equation :



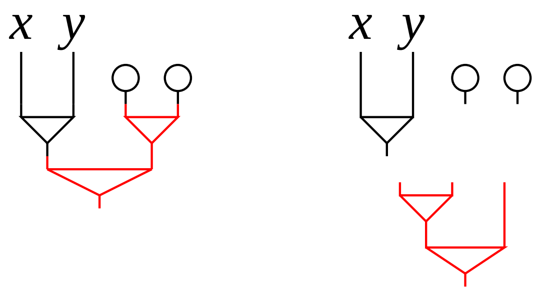
Equality between two terms :



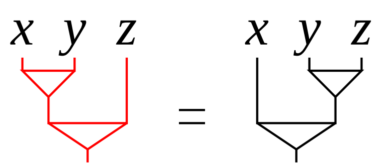
Associativity equation :



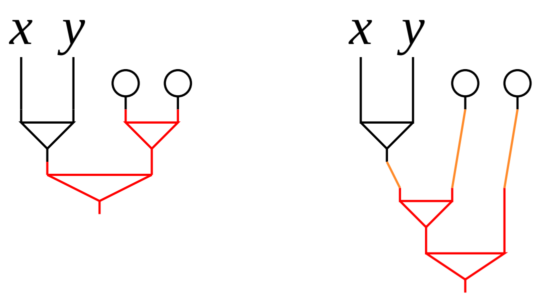
Equality between two terms :



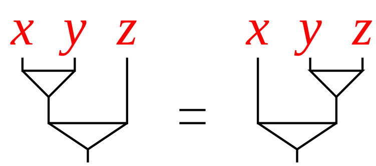
Associativity equation :



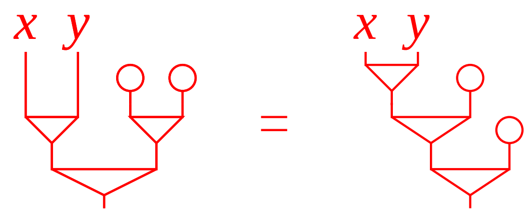
Equality between two terms :



Associativity equation :

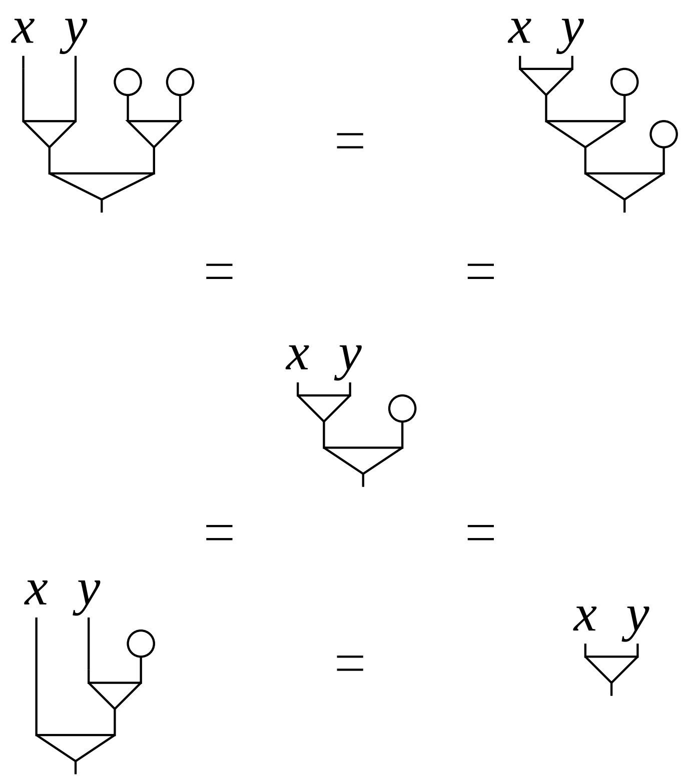


Equality between two terms :

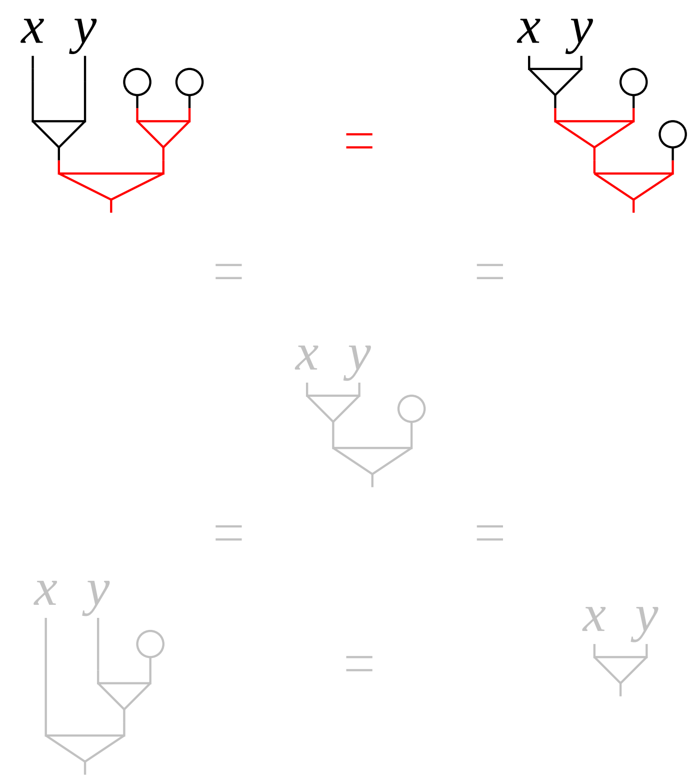


Five terms, equal in the theory of monoids :

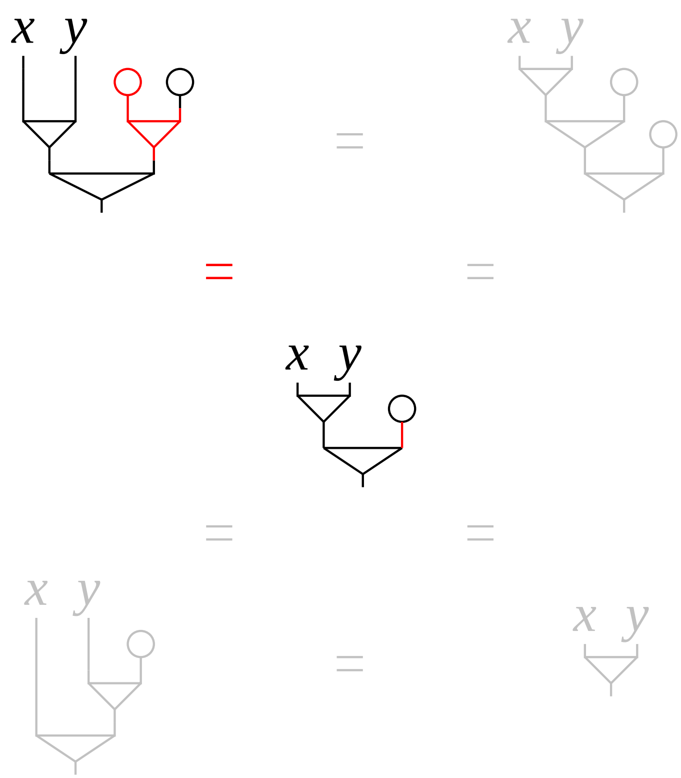
Five terms, equal in the theory of monoids :



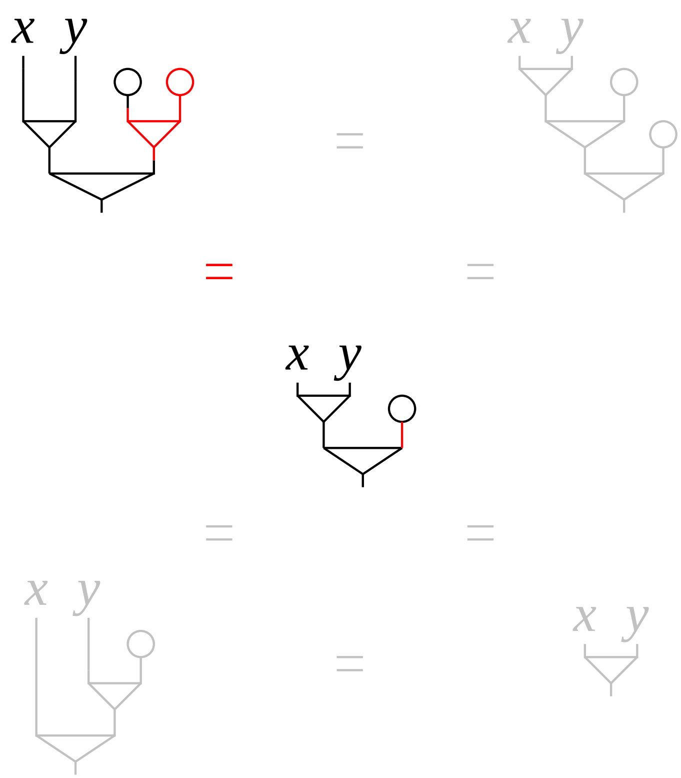
Five terms, equal in the theory of monoids :



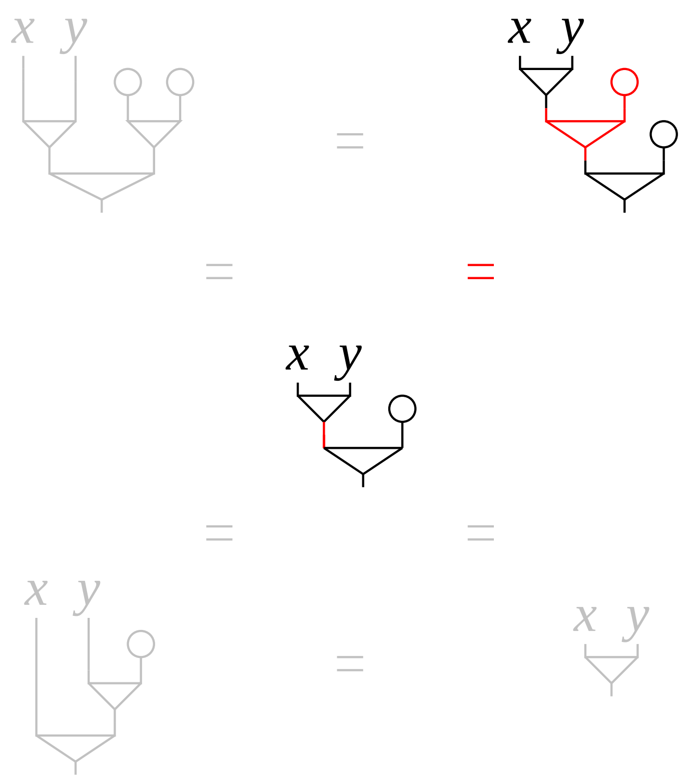
Five terms, equal in the theory of monoids :



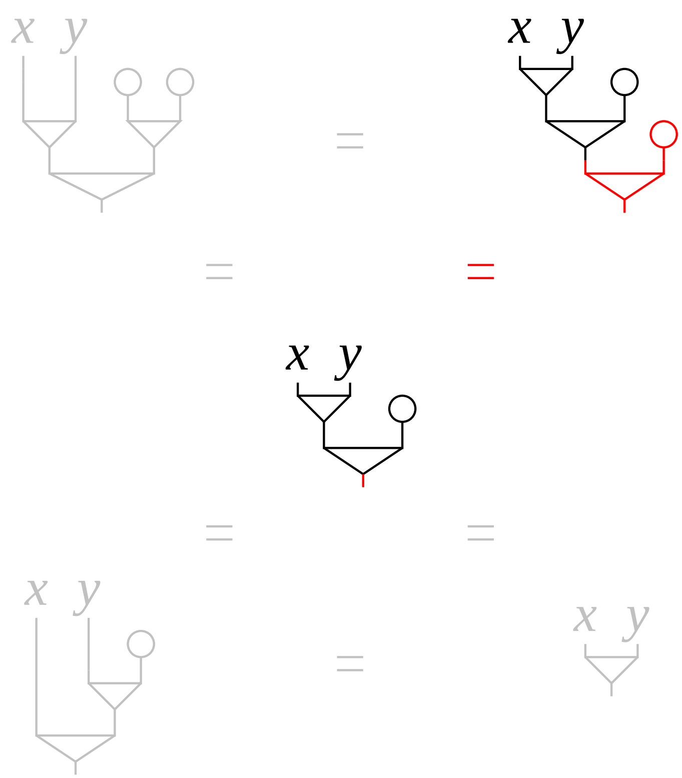
Five terms, equal in the theory of monoids :



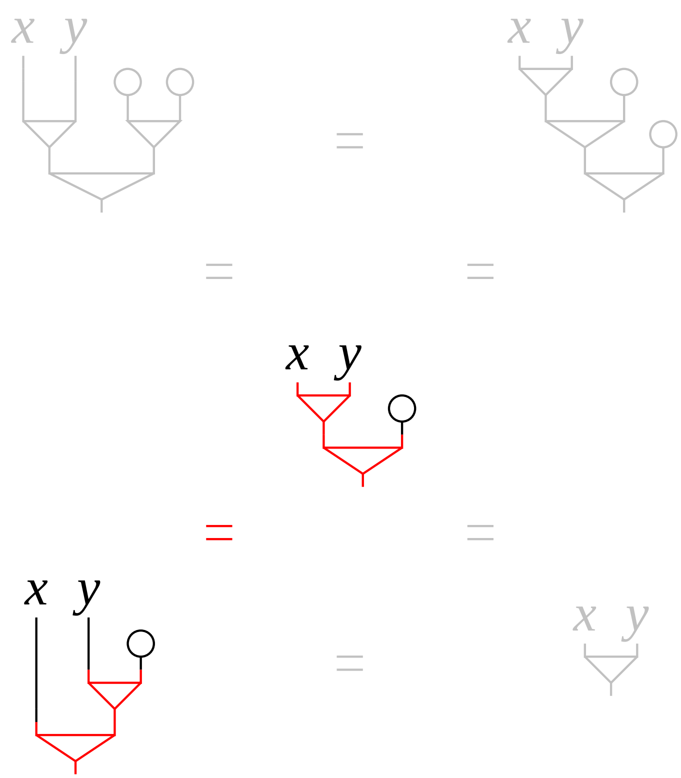
Five terms, equal in the theory of monoids :



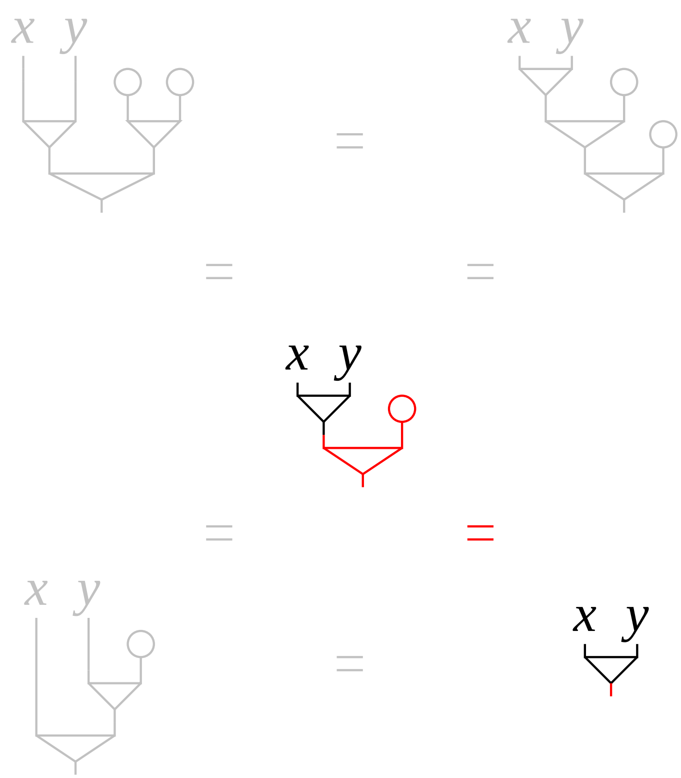
Five terms, equal in the theory of monoids :



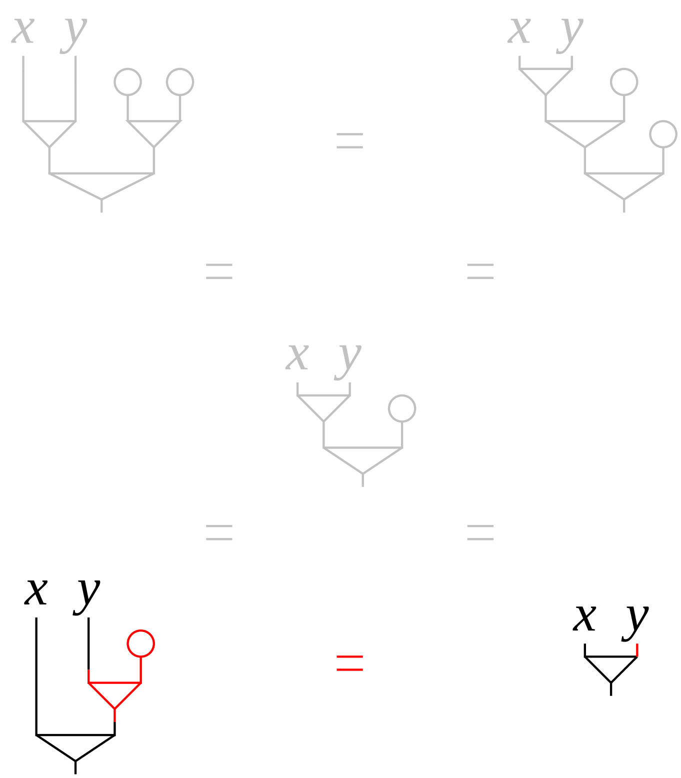
Five terms, equal in the theory of monoids :



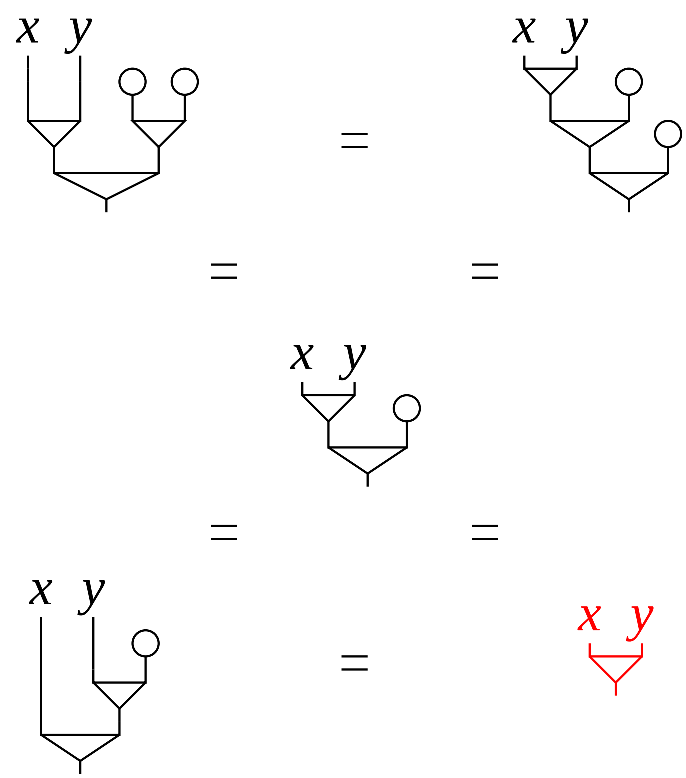
Five terms, equal in the theory of monoids :



Five terms, equal in the theory of monoids :



Five terms, equal in the theory of monoids :



Each term represents the operation $(x, y) \mapsto \mu(x, y)$.

A term rewrite system built from the theory of monoids :

A term rewrite system built from the theory of monoids :

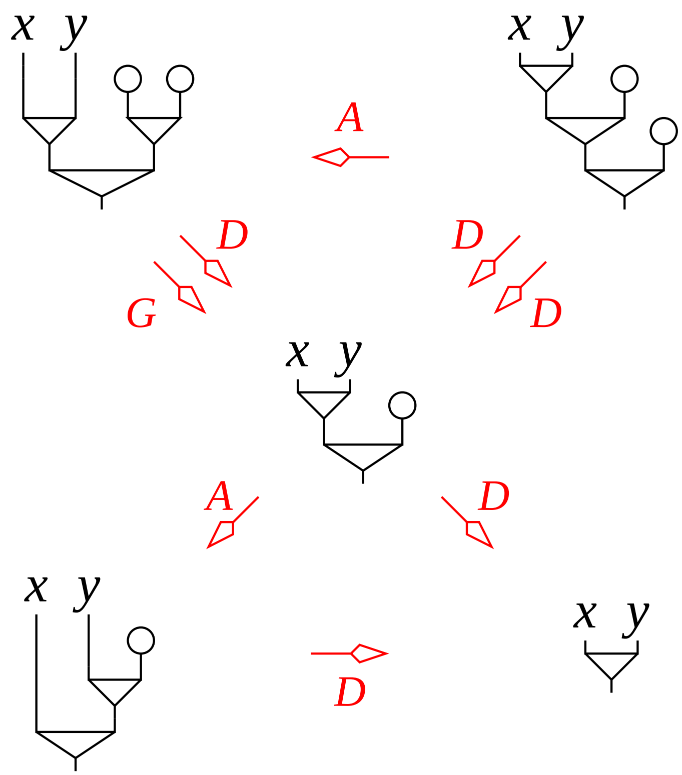
- Operators (Σ) :
  - Product $\mu : 2 \rightarrow 1$
  - Unit $\eta : 0 \rightarrow 1$

A term rewrite system built from the theory of monoids :

- Operators (Σ) :
  - Product $\mu : 2 \to 1$
  - Unit $\eta : 0 \to 1$
- Rules (R_0) :
  - Associativity $\mu(\mu(x,y),z) \to_A \mu(x,\mu(y,z))$
  - Left neutral $\mu(\eta,x) \to_G x$
  - Right neutral $\mu(x,\eta) \to_D x$

A reduction graph in the rewrite system (Σ, R_0) :

A reduction graph in the rewrite system (Σ, R_0) :



Some interesting properties for a rewrite system :

Some interesting properties for a rewrite system :

- Termination :

  A rewrite system *terminates* if it does not contain any <u>infinite</u> reduction path :

$$u_0 \longrightarrow u_1 \longrightarrow u_2 \longrightarrow \cdots \longrightarrow u_n \longrightarrow u_{n+1} \longrightarrow \cdots$$

Some interesting properties for a rewrite system :

- Termination :

  A rewrite system *terminates* if it does not contain any <u>infinite</u> reduction path :

$$u_0 \longrightarrow u_1 \longrightarrow u_2 \longrightarrow \cdots \longrightarrow u_n \longrightarrow u_{n+1} \longrightarrow \cdots$$

  Consequence :

  Every term u has one *normal form* $\hat{u}$ <u>at least</u> :

$$u \twoheadrightarrow \hat{u} \quad \text{and} \quad \hat{u} \text{ is irreducible.}$$

Some interesting properties for a rewrite system :

- Confluence :

  A rewrite system is *confluent* if every diagram $\begin{array}{ccc} u & \twoheadrightarrow & v \\ \downarrow & & \\ w & & \end{array}$ can be closed : $\begin{array}{ccc} u & \twoheadrightarrow & v \\ \downarrow & & \downarrow \\ w & \twoheadrightarrow & t \end{array}$

Some interesting properties for a rewrite system :

- Confluence :

  A rewrite system is *confluent* if every diagram

  $$\begin{array}{ccc} u & \twoheadrightarrow & v \\ \downarrow & & \\ w & & \end{array}$$

  can be closed :

  $$\begin{array}{ccc} u & \twoheadrightarrow & v \\ \downarrow & & \downarrow \\ w & \twoheadrightarrow & t \end{array}$$

  Consequence :

  Every term has one normal form <u>at most</u>.

Some interesting properties for a rewrite system :

- Convergence :
  A rewrite system is *convergent* if it is both terminating ang confluent.

Some interesting properties for a rewrite system :

- Convergence :

  A rewrite system is *convergent* if it is both terminating ang confluent.

  Consequence :

  Every term has one normal form exactly.

Some interesting properties for a rewrite system :

- Convergence :

  A rewrite system is *convergent* if it is both terminating ang confluent.

  Consequence :

  Every term has one normal form exactly.

  Consequence :

  A procedure to decide if two terms are equal.

Some interesting properties for a rewrite system :

- Convergence :

  A rewrite system is *convergent* if it is both terminating ang confluent.

  Consequence :
  Every term has one normal form exactly.

  Consequence :
  A procedure to decide if two terms are equal.

$u \qquad v$

Some interesting properties for a rewrite system :

- Convergence :

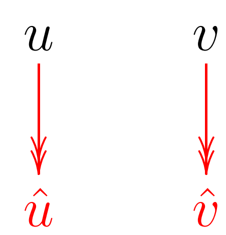
  A rewrite system is *convergent* if it is both terminating ang confluent.

  Consequence :

  Every term has one normal form exactly.

  Consequence :

  A procedure to decide if two terms are equal.



Some interesting properties for a rewrite system :

- Convergence :

  A rewrite system is *convergent* if it is both terminating ang confluent.

  Consequence :

  Every term has one normal form exactly.

  Consequence :

  A procedure to decide if two terms are equal.

$$\begin{array}{ccc} u & & v \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ \hat{u} & =? & \hat{v} \end{array}$$

The theory of commutative monoids :

The theory of commutative monoids :

- Operators :
  - Product $\mu : 2 \to 1$
  - Unit $\eta : 0 \to 1$

The theory of commutative monoids :

- Operators :
  - Product $\mu : 2 \to 1$
  - Unit $\eta : 0 \to 1$
- Equations :
  - Associativity $\mu(\mu(x,y),z) = \mu(x,\mu(y,z))$
  - Left neutral $\mu(\eta,x) = x$
  - Right neutral $\mu(x,\eta) = x$
  - Commutativity $\mu(x,y) = \mu(y,x)$

The theory of commutative monoids :

- Operators :
  - Product $\mu : 2 \to 1$
  - Unit $\eta : 0 \to 1$
- Equations :
  - Associativity $\mu(\mu(x,y),z) = \mu(x,\mu(y,z))$
  - Left neutral $\mu(\eta,x) = x$
  - Right neutral $\mu(x,\eta) = x$
  - Commutativity $\mu(x,y) = \mu(y,x)$

A term rewrite system built from the theory of commutative monoids.

A term rewrite system built from the theory of commutative monoids.

- Operators (Σ) :
  - Product $\mu : 2 \to 1$
  - Unit $\eta : 0 \to 1$

A term rewrite system built from the theory of commutative monoids.

- Operators (Σ) :
  - Product $\mu : 2 \to 1$
  - Unit $\eta : 0 \to 1$
- Rules (R_1) :
  - Associativity $\mu(\mu(x,y),z) \to_A \mu(x,\mu(y,z))$
  - Left neutral $\mu(\eta,x) \to_G x$
  - Right neutral $\mu(x,\eta) \to_D x$
  - Commutativity $\mu(x,y) \to_C \mu(y,x)$

A term rewrite system built from the theory of commutative monoids.

- Operators (Σ) :
  - Product $\mu : 2 \to 1$
  - Unit $\eta : 0 \to 1$
- Rules (R_1) :
  - Associativity $\mu(\mu(x, y), z) \to_A \mu(x, \mu(y, z))$
  - Left neutral $\mu(\eta, x) \to_G x$
  - Right neutral $\mu(x, \eta) \to_D x$
  - Commutativity $\mu(x, y) \to_C \mu(y, x)$

Infinite path in (Σ, R_1) :

Infinite path in (Σ, R_1) :

$$\mu(u, v)$$

Infinite path in (Σ, R_1) :

$$\mu(u, v) \rightarrow_C \mu(v, u)$$

Commutativity rule :

$$\mu(x, y) \rightarrow_C \mu(y, x)$$

Infinite path in (Σ, R_1) :

$$\mu(u,v) \rightarrow_C \ \mu(v,u) \rightarrow_C \ \mu(u,v)$$

Commutativity rule :

$$\mu(x,y) \rightarrow_C \ \mu(y,x)$$

Infinite path in (Σ, R_1) :

$$\mu(u,v) \rightarrow_C \mu(v,u) \rightarrow_C \mu(u,v) \rightarrow_C \mu(v,u) \rightarrow_C \cdots$$

Commutativity rule :

$$\mu(x,y) \rightarrow_C \mu(y,x)$$

The theory of $\mathbb{Z}/2\mathbb{Z}$-vector spaces :

The theory of $\mathbb{Z}/2\mathbb{Z}$-vector spaces :

- Operators :
  - Product $\mu : 2 \to 1$
  - Unit $\eta : 0 \to 1$

The theory of $\mathbb{Z}/2\mathbb{Z}$-vector spaces :

- Operators :
  - Product $\mu : 2 \to 1$
  - Unit $\eta : 0 \to 1$
- Equations :
  - Associativity $\mu(\mu(x,y),z) = \mu(x,\mu(y,z))$
  - Left neutral $\mu(\eta,x) = x$
  - Right neutral $\mu(x,\eta) = x$
  - Commutativity $\mu(x,y) = \mu(y,x)$
  - Self-inverse $\mu(x,x) = \eta$

The theory of $\mathbb{Z}/2\mathbb{Z}$-vector spaces :

- Operators :
  - Product $\mu : 2 \to 1$
  - Unit $\eta : 0 \to 1$
- Equations :
  - Associativity $\mu(\mu(x, y), z) = \mu(x, \mu(y, z))$
  - Left neutral $\mu(\eta, x) = x$
  - Right neutral $\mu(x, \eta) = x$
  - Commutativity $\mu(x, y) = \mu(y, x)$
  - Self-inverse $\mu(x, x) = \eta$

A term rewrite system built from the theory of $\mathbb{Z}/2\mathbb{Z}$-vector spaces :

A term rewrite system built from the theory of $\mathbb{Z}/2\mathbb{Z}$-vector spaces :

- Operators (Σ) :
  - Product $\mu : 2 \to 1$
  - Unit $\eta : 0 \to 1$

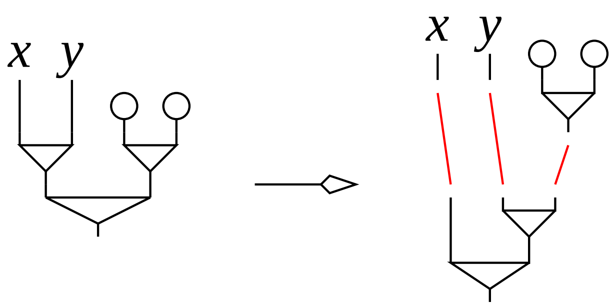
A term rewrite system built from the theory of $\mathbb{Z}/2\mathbb{Z}$-vector spaces :

- Operators (Σ) :
  - Product $\mu : 2 \to 1$
  - Unit $\eta : 0 \to 1$
- Rules (R_2) :
  - Associativity $\mu(\mu(x,y),z) \to_A \mu(x,\mu(y,z))$
  - Left neutral $\mu(\eta,x) \to_G x$
  - Right neutral $\mu(x,\eta) \to_D x$
  - Commutativity $\mu(x,y) \to_C \mu(y,x)$
  - Self-inverse $\mu(x,x) \to_I \eta$

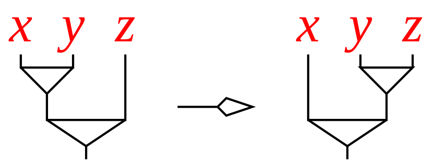
Resource management

Reminder :

Reminder :



Associativity rule :



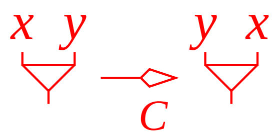
Resource management operations :

Resource management operations :

- Permutation :

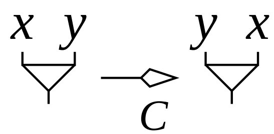
Resource management operations :

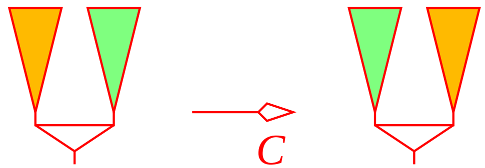
- Permutation :



Resource management operations :

- Permutation :



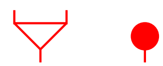


Resource management operations :

- Erasure :

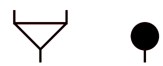
Resource management operations :

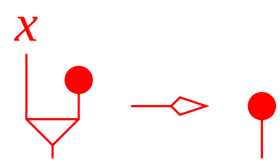
- Erasure :



Resource management operations :

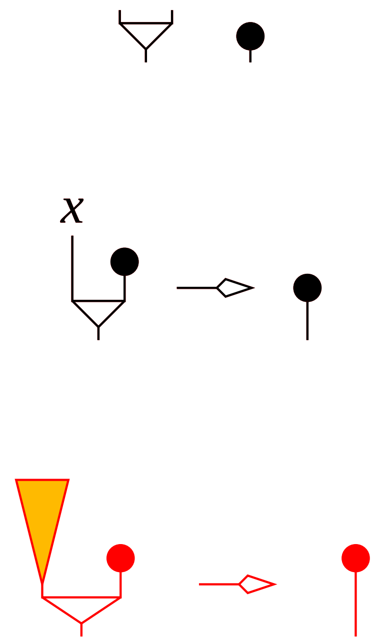
- Erasure :





Resource management operations :

- Erasure :



Resource management operations :

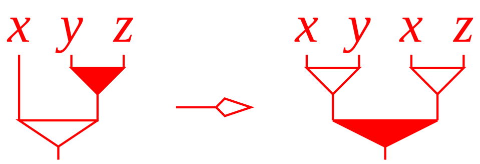
- Duplication :

Resource management operations :

- Duplication :

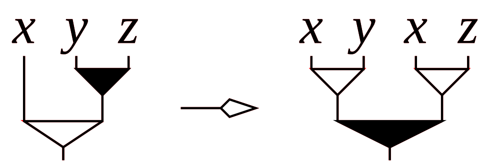
Resource management operations :

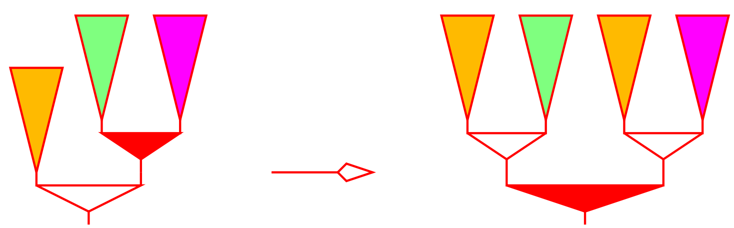
- Duplication :



Resource management operations :

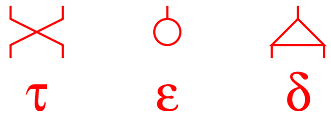
- Duplication :



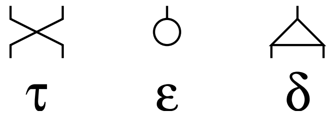


Resource management operators (Δ) :

Resource management operators (Δ) :



Resource management operators (Δ) :

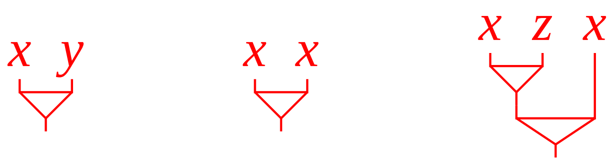


Desired interpretations :

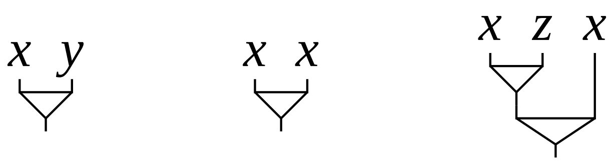
- $\tau(x, y) = (y, x)$
- $\varepsilon(x) = *$
- $\delta(x) = (x, x)$

From variables to resource management operators :

From variables to resource management operators :



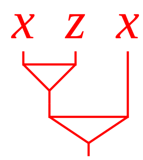
From variables to resource management operators :



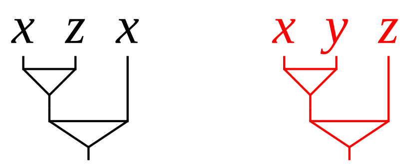
Formal operations represented by these terms :

$$(x, y) \mapsto \mu(x, y), \quad x \mapsto \mu(x, x), \quad (x, y, z) \mapsto \mu(\mu(x, z), x).$$

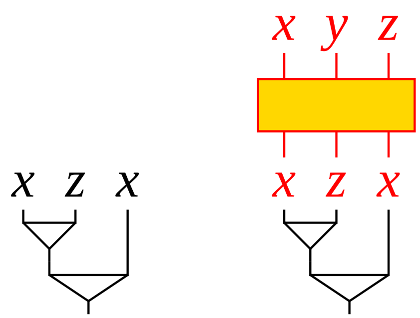
From variables to resource management operators :



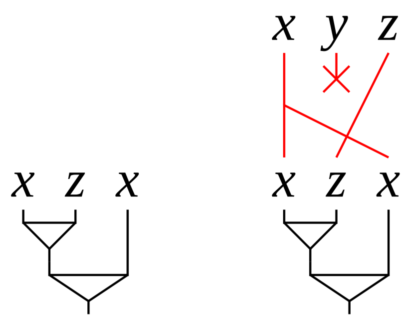
From variables to resource management operators :



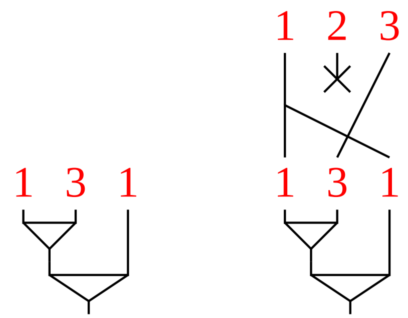
From variables to resource management operators :



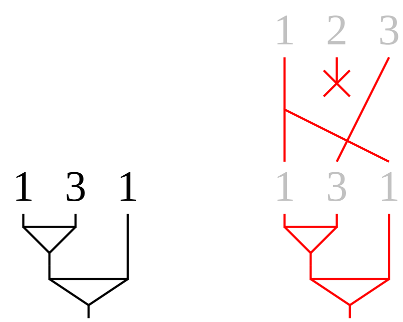
From variables to resource management operators :



From variables to resource management operators :

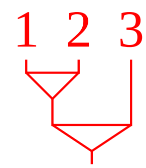


From variables to resource management operators :

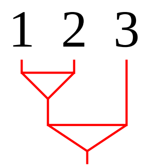


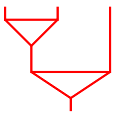
Translation of terms from the theory of $\mathbb{Z}/2\mathbb{Z}$-vector spaces :

Translation of terms from the theory of $\mathbb{Z}/2\mathbb{Z}$-vector spaces :

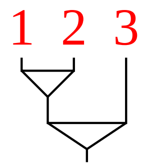


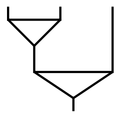
Translation of terms from the theory of $\mathbb{Z}/2\mathbb{Z}$-vector spaces :



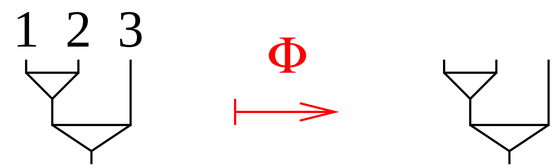


Translation of terms from the theory of $\mathbb{Z}/2\mathbb{Z}$-vector spaces :

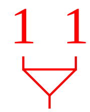




Translation of terms from the theory of $\mathbb{Z}/2\mathbb{Z}$-vector spaces :

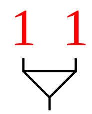


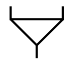
Translation of terms from the theory of $\mathbb{Z}/2\mathbb{Z}$-vector spaces :



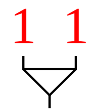
Translation of terms from the theory of $\mathbb{Z}/2\mathbb{Z}$-vector spaces :

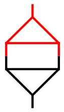
Translation of terms from the theory of $\mathbb{Z}/2\mathbb{Z}$-vector spaces :



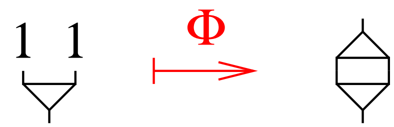


Translation of terms from the theory of $\mathbb{Z}/2\mathbb{Z}$-vector spaces :

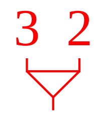




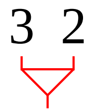
Translation of terms from the theory of $\mathbb{Z}/2\mathbb{Z}$-vector spaces :

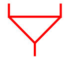


Translation of terms from the theory of $\mathbb{Z}/2\mathbb{Z}$-vector spaces :



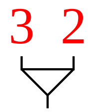
Translation of terms from the theory of $\mathbb{Z}/2\mathbb{Z}$-vector spaces :

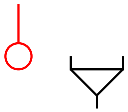




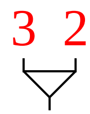
Translation of terms from the theory of $\mathbb{Z}/2\mathbb{Z}$-vector spaces :

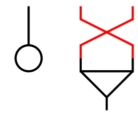
Translation of terms from the theory of $\mathbb{Z}/2\mathbb{Z}$-vector spaces :



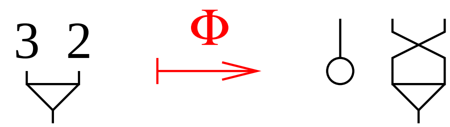


Translation of terms from the theory of $\mathbb{Z}/2\mathbb{Z}$-vector spaces :

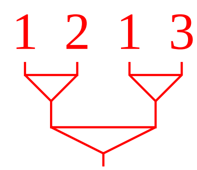




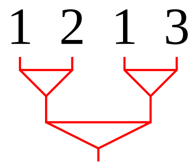
Translation of terms from the theory of $\mathbb{Z}/2\mathbb{Z}$-vector spaces :

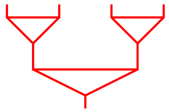


Translation of terms from the theory of $\mathbb{Z}/2\mathbb{Z}$-vector spaces :

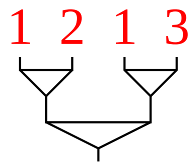


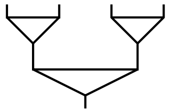
Translation of terms from the theory of $\mathbb{Z}/2\mathbb{Z}$-vector spaces :



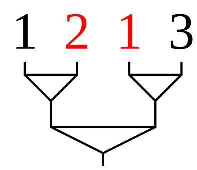


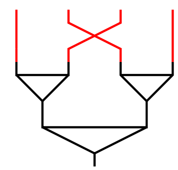
Translation of terms from the theory of $\mathbb{Z}/2\mathbb{Z}$-vector spaces :



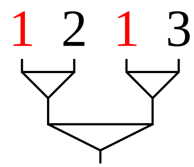


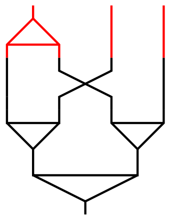
Translation of terms from the theory of $\mathbb{Z}/2\mathbb{Z}$-vector spaces :



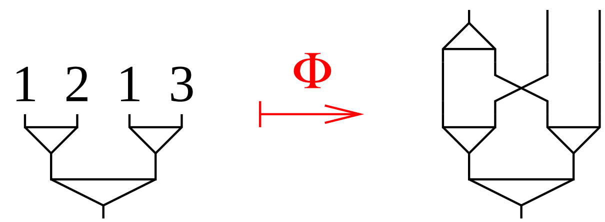


Translation of terms from the theory of $\mathbb{Z}/2\mathbb{Z}$-vector spaces :





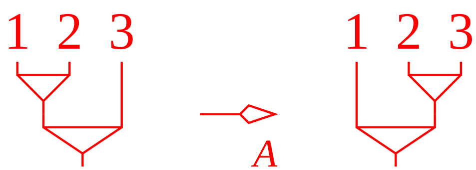
Translation of terms from the theory of $\mathbb{Z}/2\mathbb{Z}$-vector spaces :



Translation of the rules from R_2 :

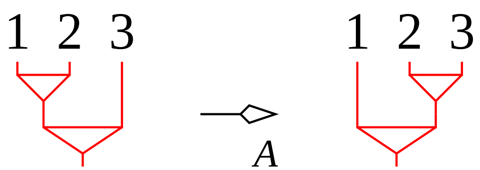
Translation of the rules from R_2 :

- Associativity :



Translation of the rules from R_2 :

- Associativity :

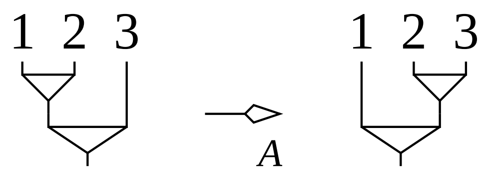


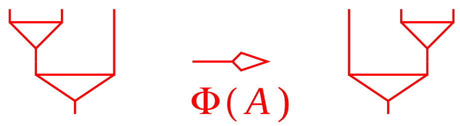
Translation of the rules from R_2 :

- Associativity :

Translation of the rules from R_2 :

- Associativity :





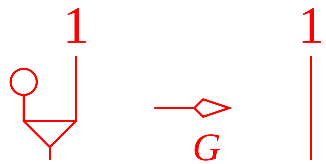
Translation of the rules from R_2 :

- Associativity :

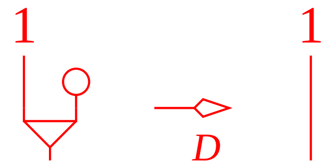
This rule is (left and right) linear.

Translation of the rules from R_2 :

- Left neutral :

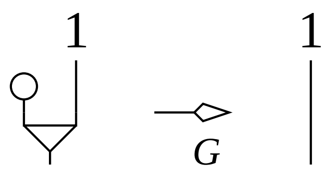


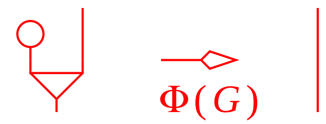
- Right neutral :



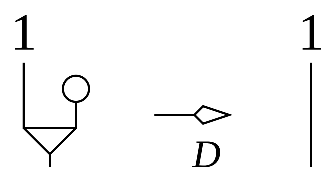
Translation of the rules from R_2 :

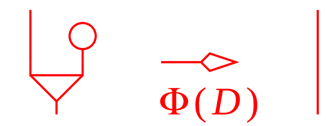
- Left neutral :





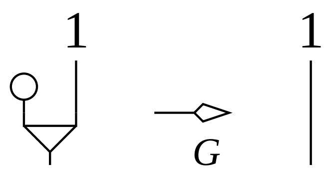
- Right neutral :

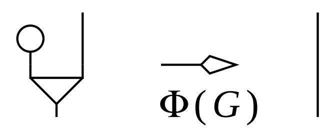




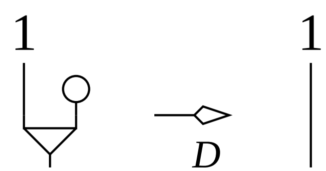
Translation of the rules from R_2 :

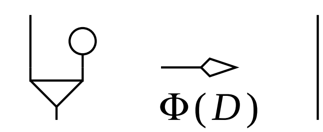
- Left neutral :





- Right neutral :

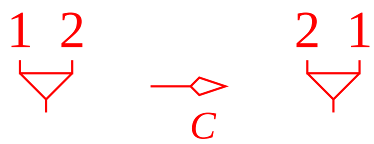




Both rules are linear.

Translation of the rules from R_2 :

- Commutativity :



Translation of the rules from R_2 :

- Commutativity :

Translation of the rules from R_2 :

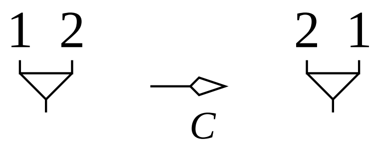
- Commutativity :

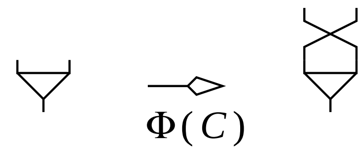
Translation of the rules from R_2 :

- Commutativity :

Translation of the rules from R_2 :

- Commutativity :

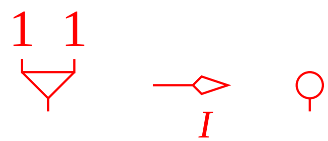




This rule is left linear only.

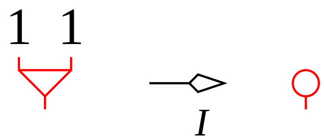
Translation of the rules from R_2 :

- Self-inverse :



Translation of the rules from R_2 :

- Self-inverse :



Translation of the rules from R_2 :

- Self-inverse :

Translation of the rules from R_2 :

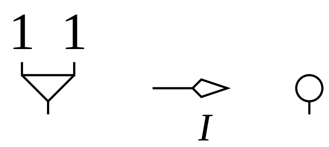
- Self-inverse :

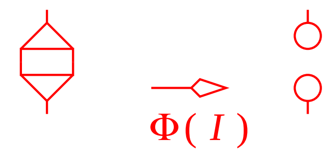
Translation of the rules from R_2 :

- Self-inverse :

Translation of the rules from R_2 :

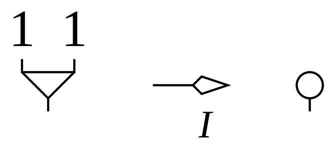
- Self-inverse :

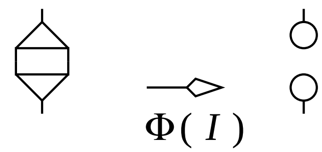




Translation of the rules from R_2 :

- Self-inverse :



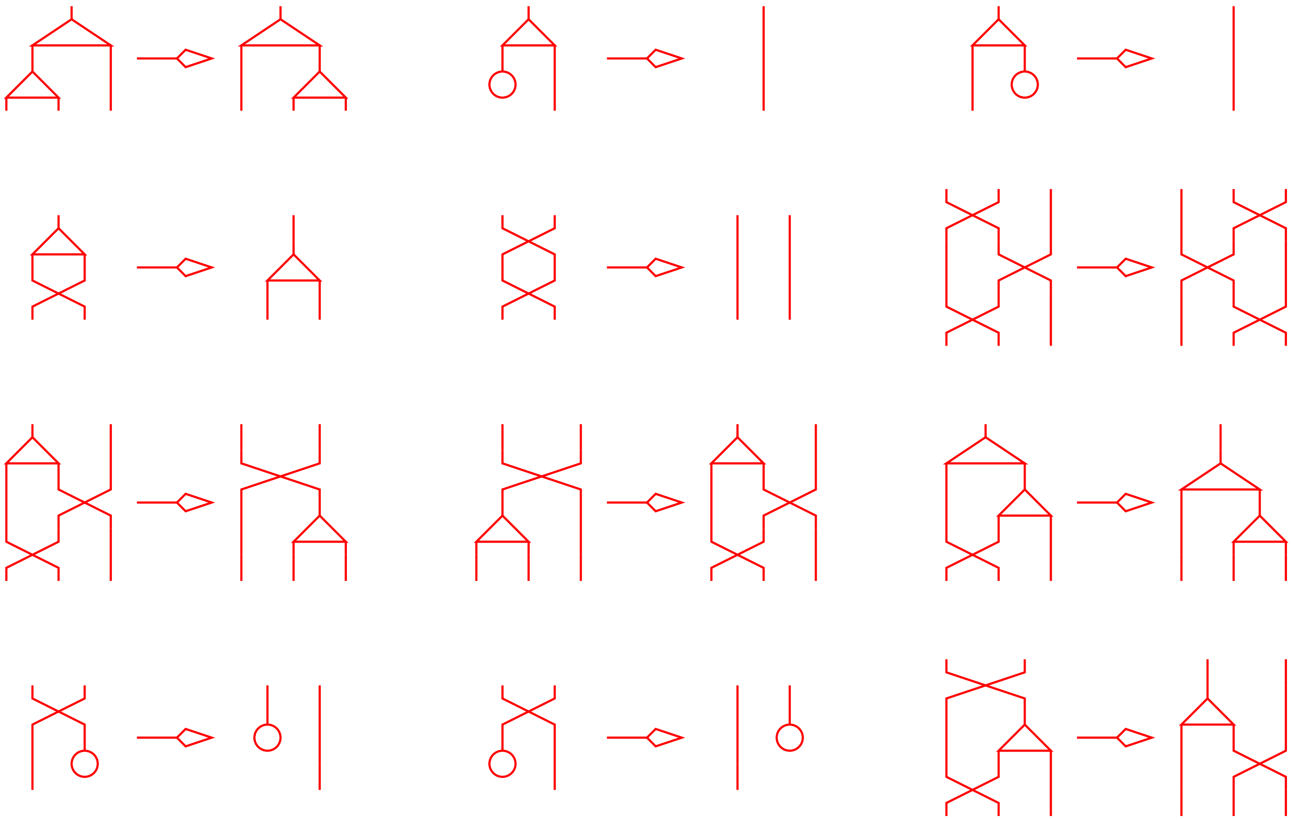


This rule is neither left nor right linear.

Additional rules $(R_{\Delta,\Sigma})$:

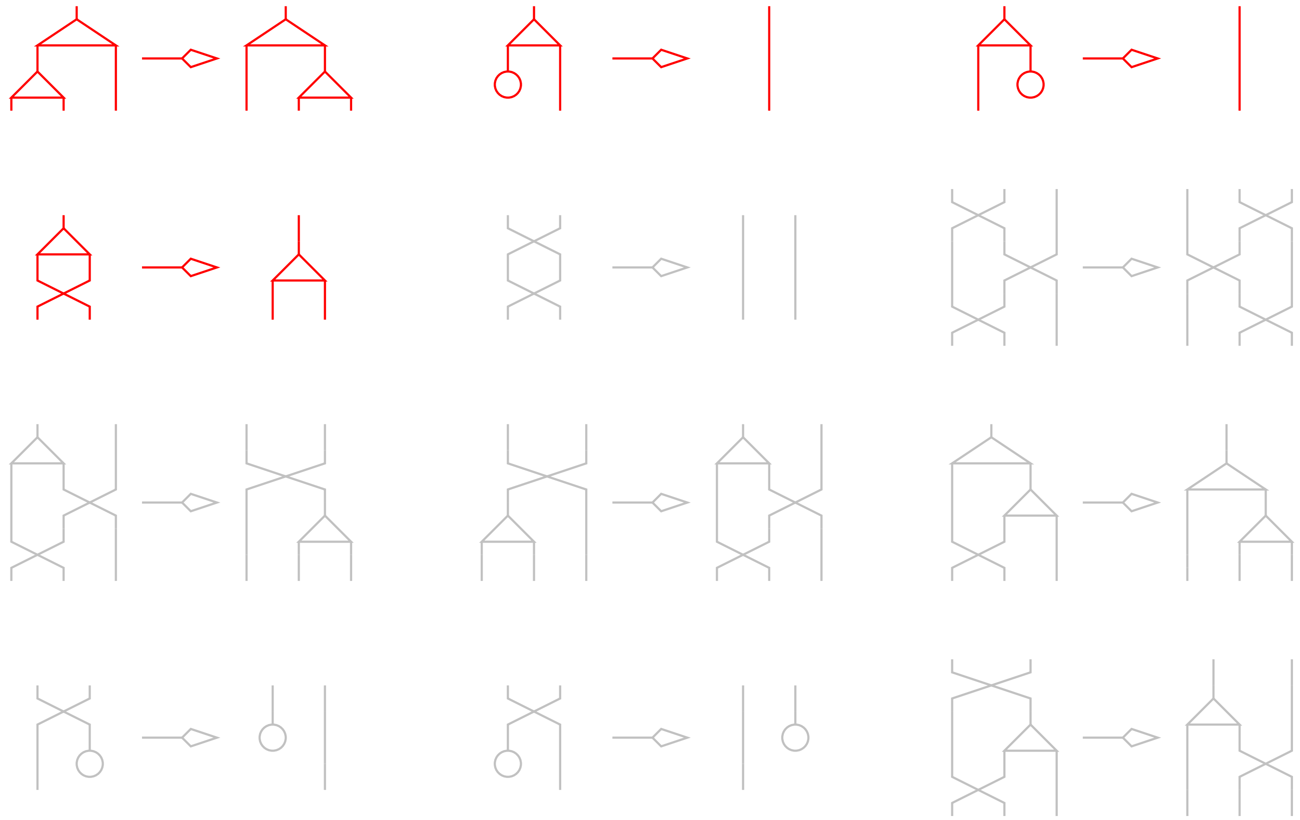
Additional rules ($R_{\Delta,\Sigma}$) :

- The first family (R_{Δ}) :



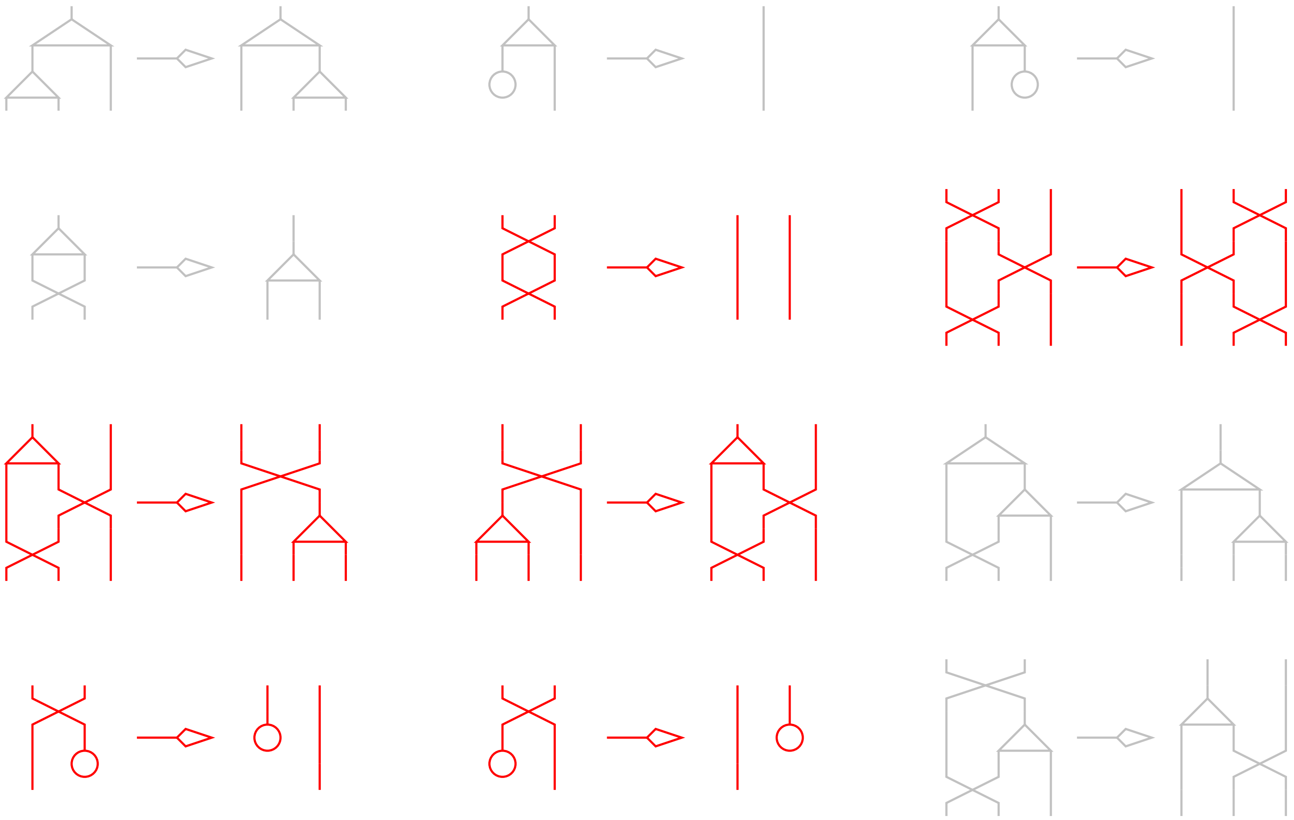
Additional rules $(R_{\Delta,\Sigma})$:

- The first family (R_{Δ}) :



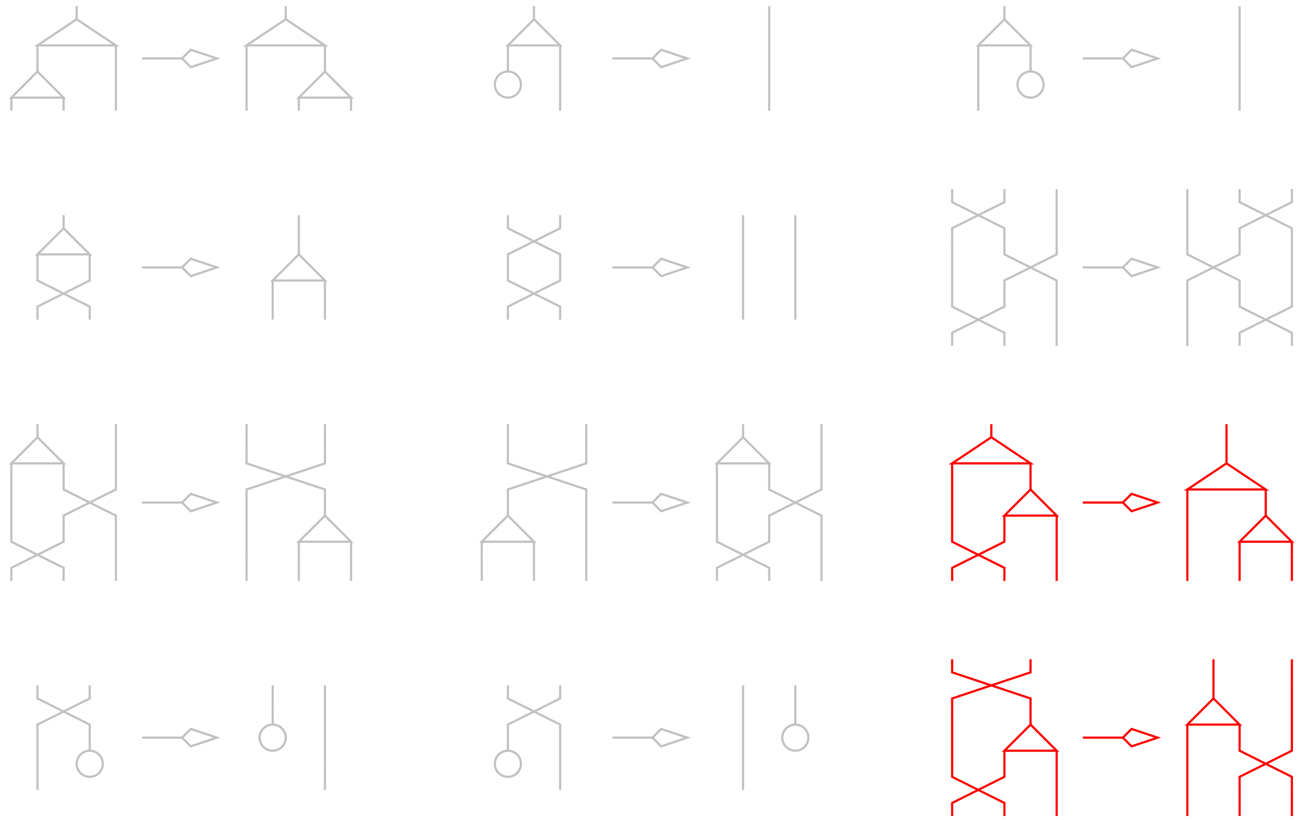
Additional rules ($R_{\Delta,\Sigma}$) :

- The first family (R_{Δ}) :



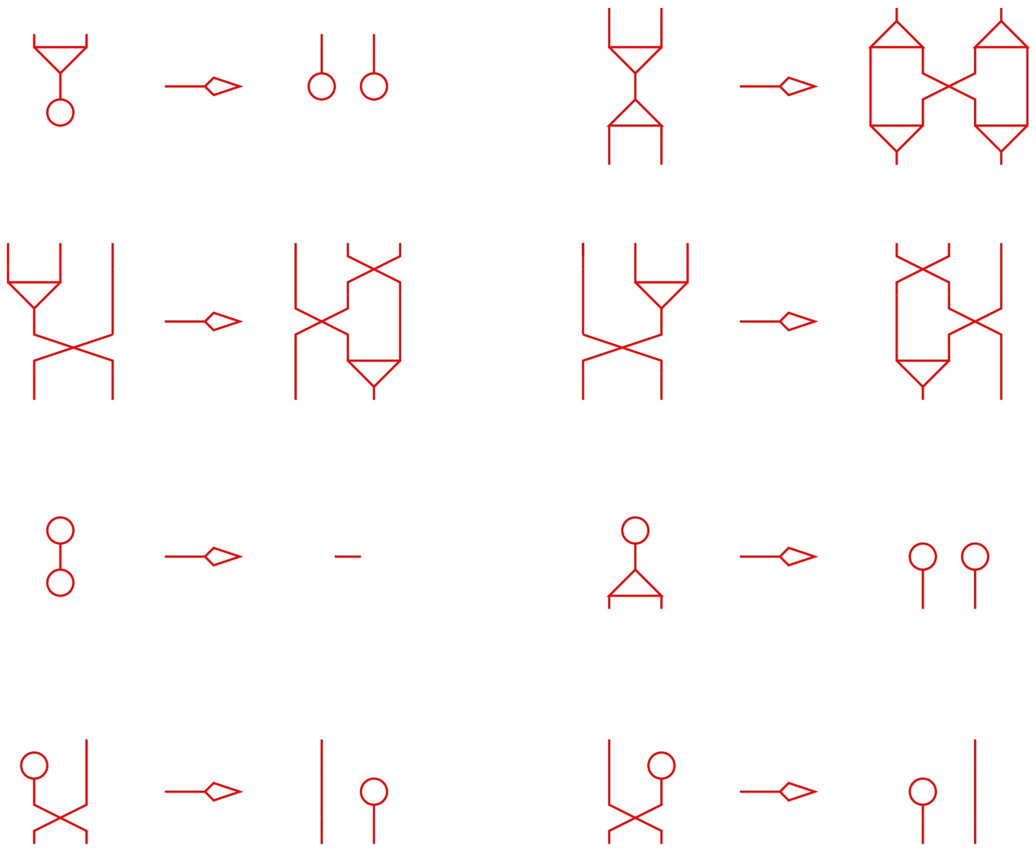
Additional rules ($R_{\Delta,\Sigma}$) :

- The first family (R_Δ) :



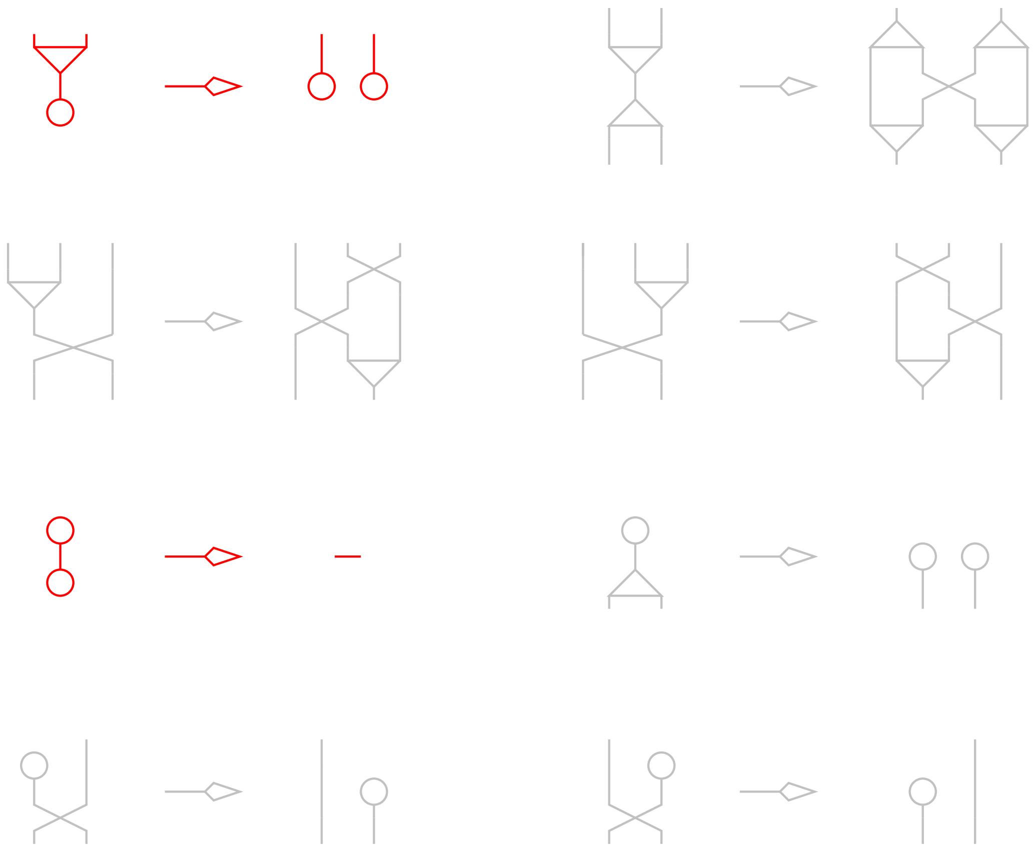
Additional rules $(R_{\Delta,\Sigma})$:

- The second family (R_{Σ}) :



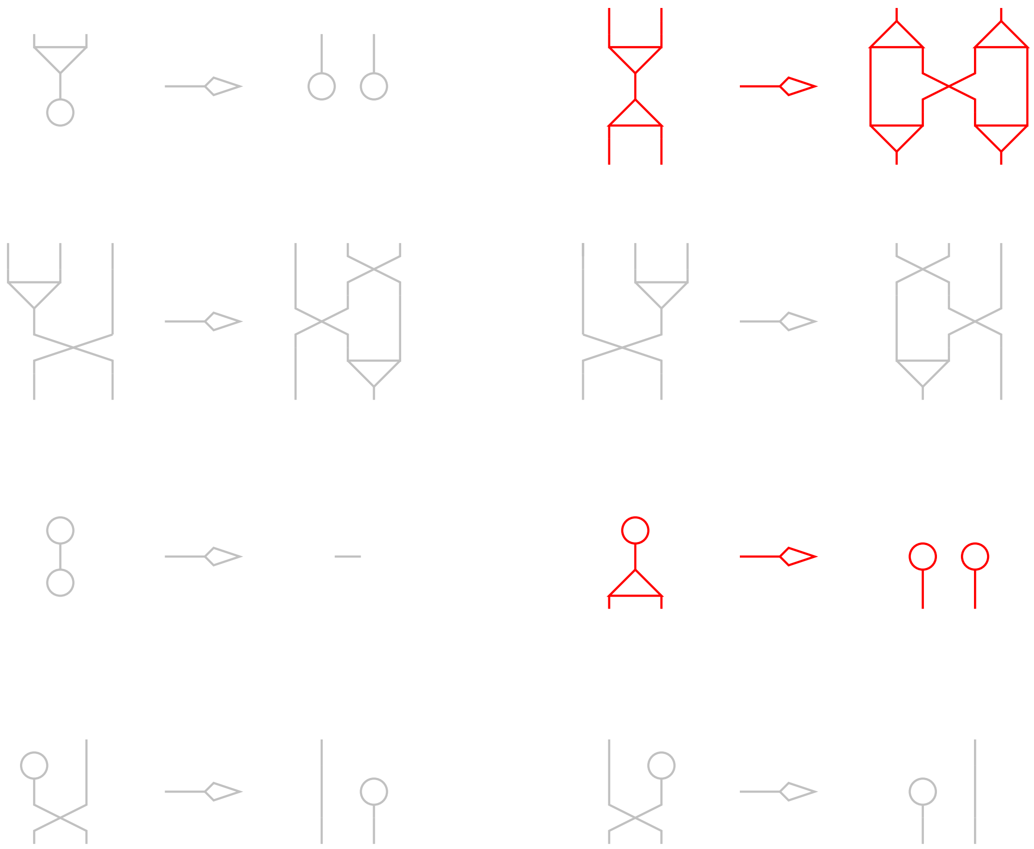
Additional rules ($R_{\Delta,\Sigma}$) :

- The second family (R_{Σ}) :



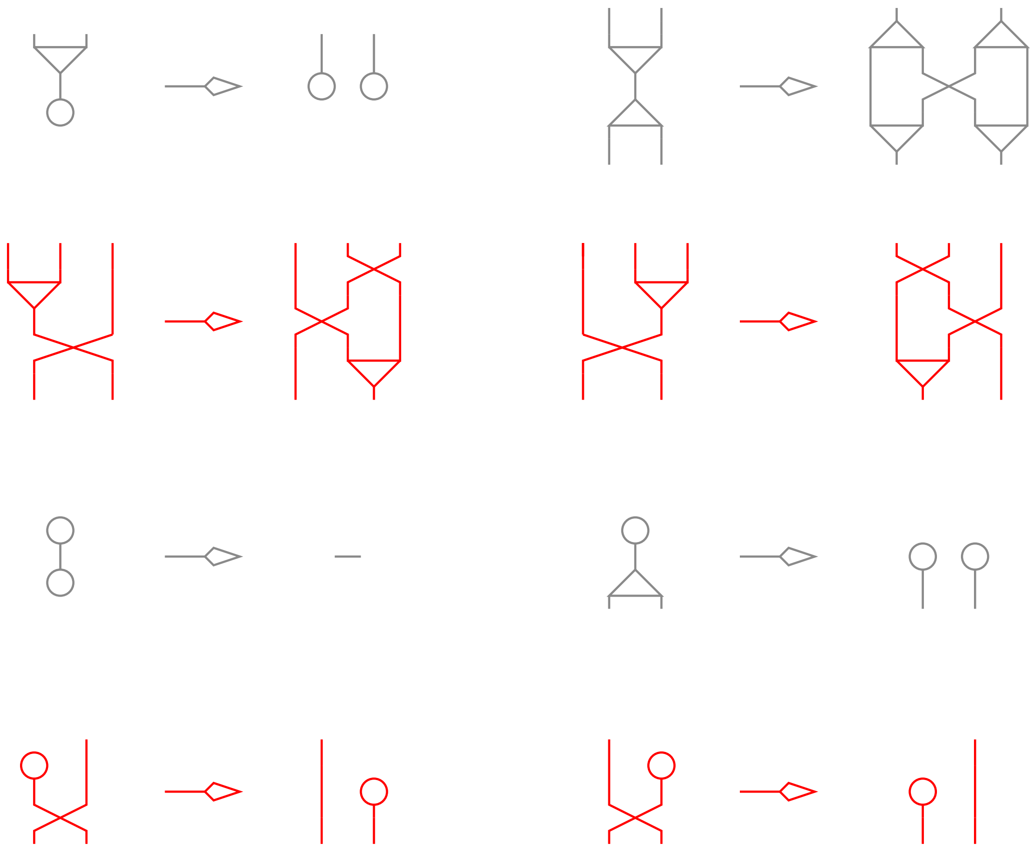
Additional rules ($R_{\Delta,\Sigma}$) :

- The second family (R_{Σ}) :



Additional rules $(R_{\Delta,\Sigma})$:

- The second family (R_{Σ}) :



From the term rewrite system (Σ, R_2), a presentation (Σ^c, R_2^c) has been built. It has 5 operators and 25 rules :

$$\Sigma^c = \Sigma \amalg \Delta \quad \text{and} \quad R_2^c = \Phi(R_2) \amalg R_{\Delta,\Sigma}.$$

From the term rewrite system (Σ, R_2), a presentation (Σ^c, R_2^c) has been built. It has 5 operators and 25 rules :

$$\Sigma^c = \Sigma \amalg \Delta \quad \text{and} \quad R_2^c = \Phi(R_2) \amalg R_{\Delta,\Sigma}.$$

Generalization :

From a term rewrite system (Ω, R) with m operators and n rules, one can build a presentation (Ω^c, R^c) with $m+3$ operators and $n + 12 + 2m$ rules.

Question number 1 :
Is there any link between the computational properties of (Ω, R) and the ones of (Ω^c, R^c) ?

Question number 1 :

Is there any link between the computational properties of (Ω, R) and the ones of (Ω^c, R^c) ?

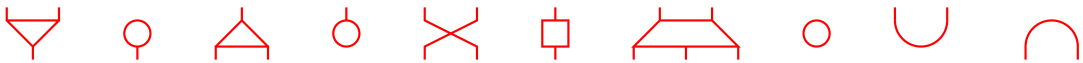
Question number 2 :

What is (Ω^c, R^c) ?

Operad presentations :

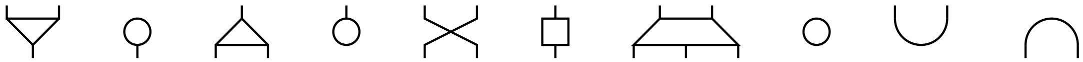
Operad presentations :

- Examples of generators (operators) :

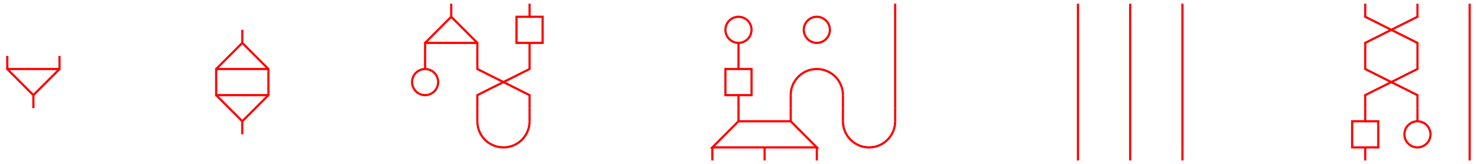


Operad presentations :

- Examples of generators (operators) :



- Examples of "terms" (diagrams, circuits, arrows) :



Operad presentations :

- Rules :

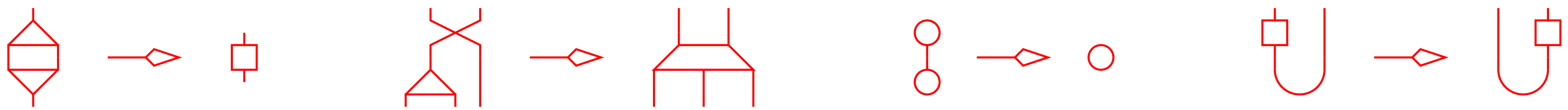
  A *rule* is a pair $f \to g$, where f and g are parallel arrows.

Operad presentations :

- Rules :

  A *rule* is a pair $f \to g$, where f and g are parallel arrows.

- Rules examples :

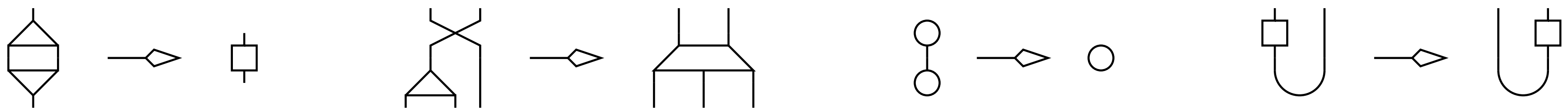


Operad presentations :

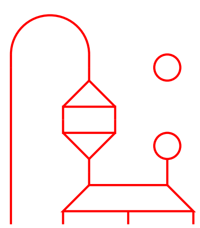
- Rules :

  A *rule* is a pair $f \to g$, where f and g are parallel arrows.

- Rules examples :



- Reduction example :

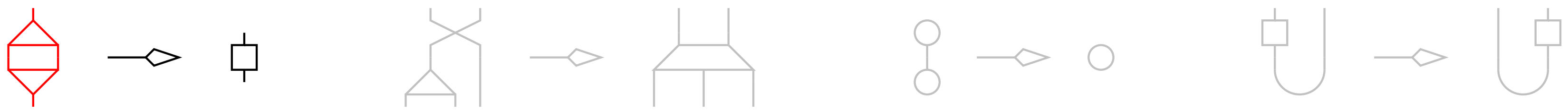


Operad presentations :

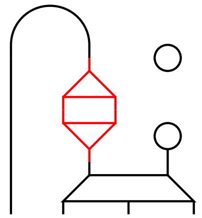
- Rules :

  A *rule* is a pair $f \to g$, where f and g are parallel arrows.

- Rules examples :



- Reduction example :

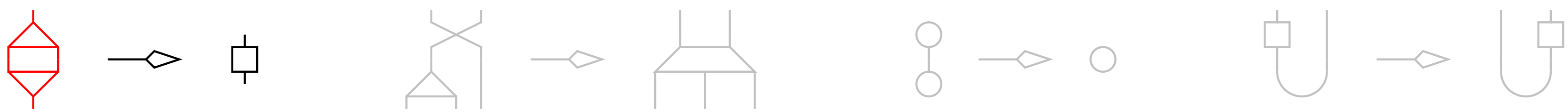


Operad presentations :

- Rules :

  A *rule* is a pair $f \to g$, where f and g are parallel arrows.

- Rules examples :



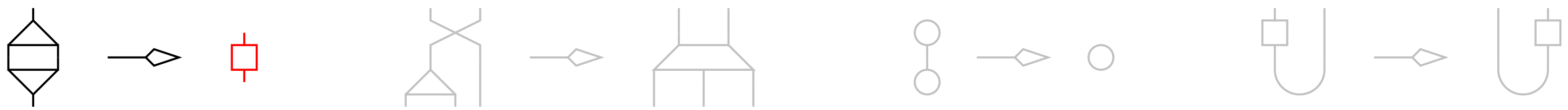
- Reduction example :

Operad presentations :

- Rules :

  A *rule* is a pair $f \to g$, where f and g are parallel arrows.

- Rules examples :



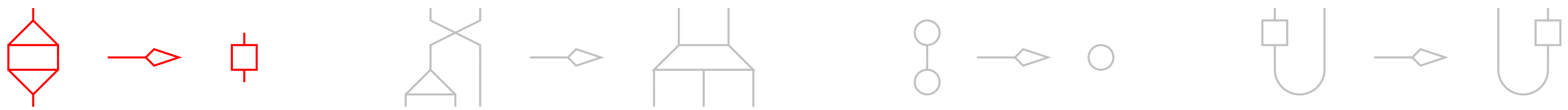
- Reduction example :

Operad presentations :

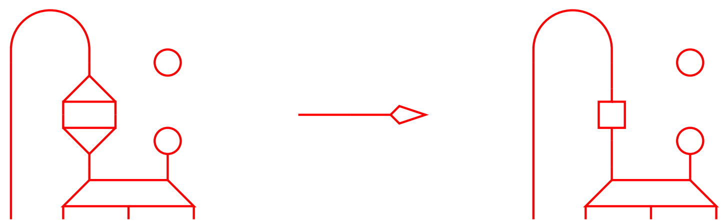
- Rules :

  A *rule* is a pair $f \to g$, where f and g are parallel arrows.

- Rules examples :



- Reduction example :



Theorem number 1 :

Let (Σ, R) be any term rewrite system. Then :

Theorem number 1 :

Let (Σ, R) be any term rewrite system. Then :

- The operad presentation $(\Sigma^c, R_{\Delta,\Sigma})$ is convergent.

Theorem number 1 :

Let (Σ, R) be any term rewrite system. Then :

- The operad presentation $(\Sigma^c, R_{\Delta,\Sigma})$ is convergent.

Furthermore, if (Σ, R) is left linear :

Theorem number 1 :

Let (Σ, R) be any term rewrite system. Then :

- The operad presentation $(\Sigma^c, R_{\Delta,\Sigma})$ is convergent.

Furthermore, if (Σ, R) is left linear :

- The translation Φ preserves reduction steps :

$$\text{if} \quad u \rightarrow_\alpha v \quad \text{then} \quad \Phi(u) \rightarrow_{\Phi(\alpha)} f \twoheadrightarrow_{R_{\Delta,\Sigma}} \Phi^u(v).$$

Theorem number 1 :

Let (Σ, R) be any term rewrite system. Then :

- The operad presentation $(\Sigma^c, R_{\Delta,\Sigma})$ is convergent.

Furthermore, if (Σ, R) is left linear :

- The translation Φ preserves reduction steps :

$$\text{if} \quad u \rightarrow_\alpha v \quad \text{then} \quad \Phi(u) \rightarrow_{\Phi(\alpha)} f \twoheadrightarrow_{R_{\Delta,\Sigma}} \Phi^u(v).$$

- The operad presentation (Σ^c, R^c) terminates if and only if the term rewrite system (Σ, R) terminates.

Theorem number 1 :

Let (Σ, R) be any term rewrite system. Then :

- The operad presentation $(\Sigma^c, R_{\Delta,\Sigma})$ is convergent.

Furthermore, if (Σ, R) is left linear :

- The translation Φ preserves reduction steps :

$$\text{if} \quad u \to_\alpha v \quad \text{then} \quad \Phi(u) \to_{\Phi(\alpha)} f \twoheadrightarrow_{R_{\Delta,\Sigma}} \Phi^u(v).$$

- The operad presentation (Σ^c, R^c) terminates if and only if the term rewrite system (Σ, R) terminates.
- The operad presentation (Σ^c, R^c) is confluent if and only if the term rewrite system (Σ, R) is confluent.

Application to the rewrite system (Σ, R_0) presenting the theory of monoids :

Application to the rewrite system (Σ, R_0) presenting the theory of monoids :

It is a left linear, convergent rewrite system.

Application to the rewrite system (Σ, R_0) presenting the theory of monoids :

It is a left linear, convergent rewrite system.

Hence (Σ^c, R_0^c) is a convergent presentation of the theory of monoids *with explicit resource management.*

Application to the rewrite system (Σ, R_1) presenting the theory of commutative monoids :

Application to the rewrite system (Σ, R_1) presenting the theory of commutative monoids :

It is a left linear, non terminating rewrite system.

Application to the rewrite system (Σ, R_1) presenting the theory of commutative monoids :

It is a left linear, non terminating rewrite system.

Hence (Σ^c, R_1^c) is a non terminating presentation of the theory of commutative monoids, with explicit resource management.

Application to the rewrite system (Σ, R_2) presenting the theory of $\mathbb{Z}/2\mathbb{Z}$-vector spaces :

Application to the rewrite system (Σ, R_2) presenting the theory of $\mathbb{Z}/2\mathbb{Z}$-vector spaces :

This rewrite system is not left linear.

Application to the rewrite system (Σ, R_2) presenting the theory of $\mathbb{Z}/2\mathbb{Z}$-vector spaces :

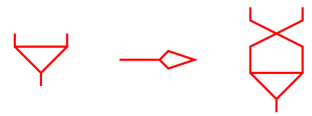
This rewrite system is not left linear.

Theorem 1 says nothing about the computational properties of (Σ^c, R_2^c).

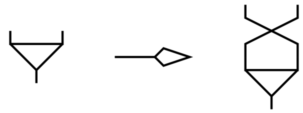
The operad presentation $L(\mathbb{Z}_2)$

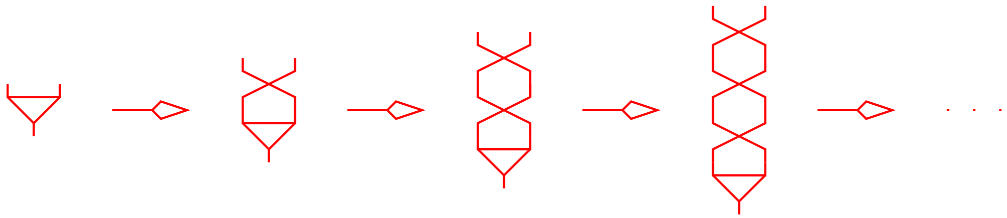
Problem number 1 : the presentation (Σ^c, R_2^c) does not terminate.

Problem number 1 : the presentation (Σ^c, R_2^c) does not terminate.



Problem number 1 : the presentation (Σ^c, R_2^c) does not terminate.

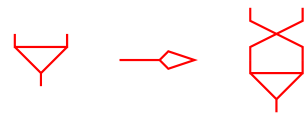




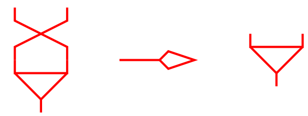
First change :

First change :

The rule :

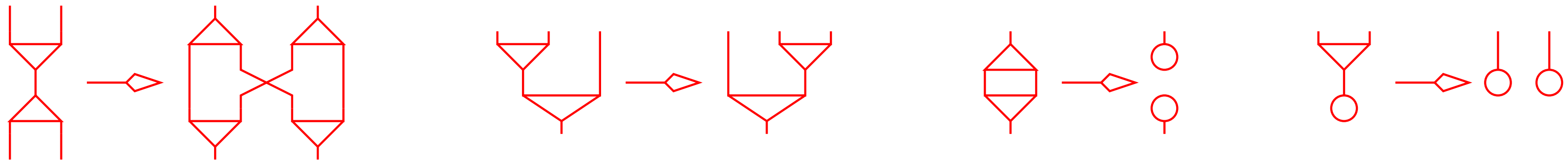


is replaced by :

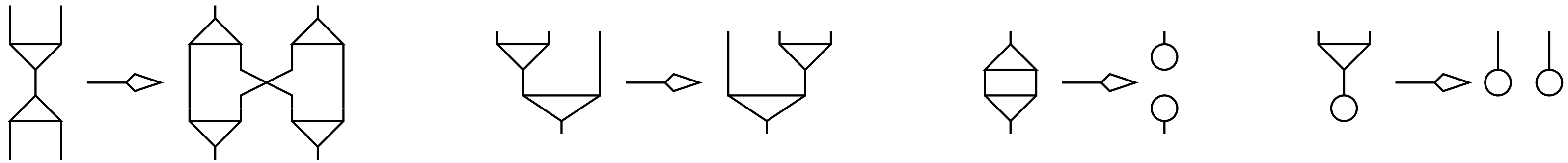


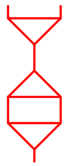
Problem number 2 : the presentation (Σ^c, R_2^c) is not confluent.

Problem number 2 : the presentation (Σ^c, R_2^c) is not confluent.

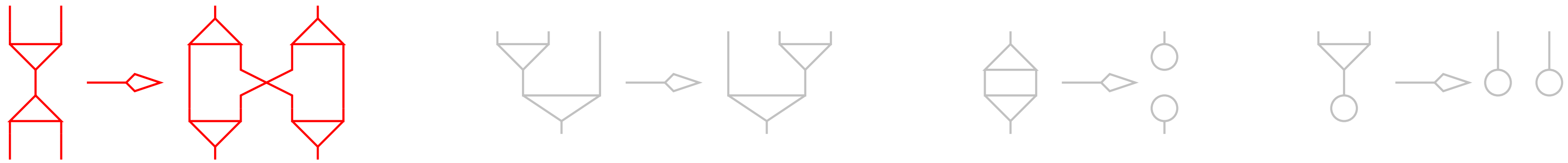


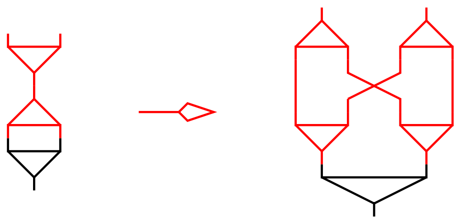
Problem number 2 : the presentation (Σ^c, R_2^c) is not confluent.



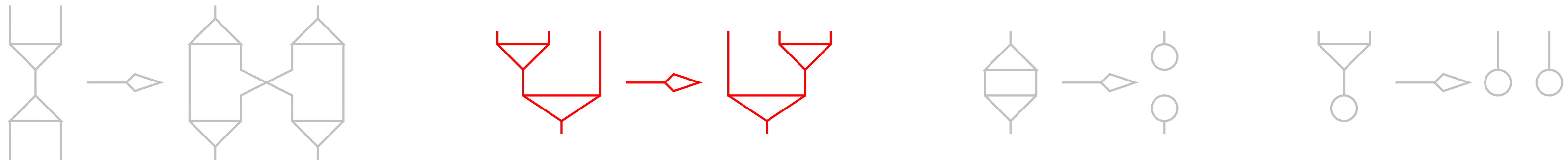


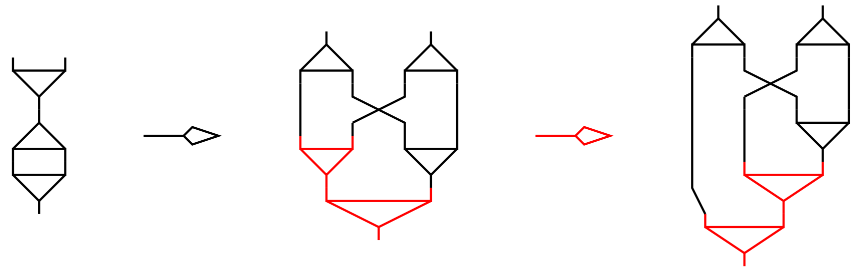
Problem number 2 : the presentation (Σ^c, R_2^c) is not confluent.



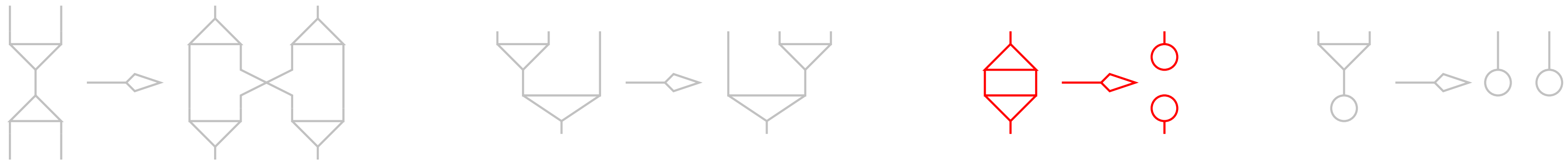


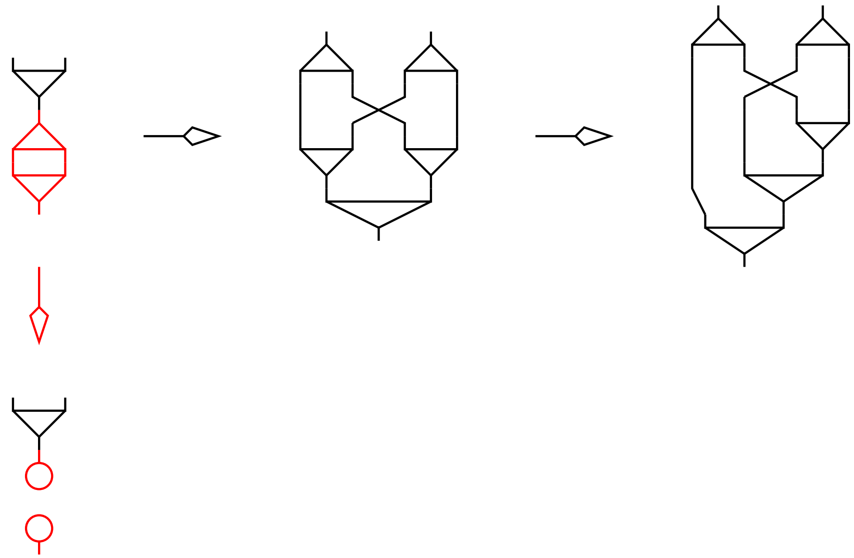
Problem number 2 : the presentation (Σ^c, R_2^c) is not confluent.



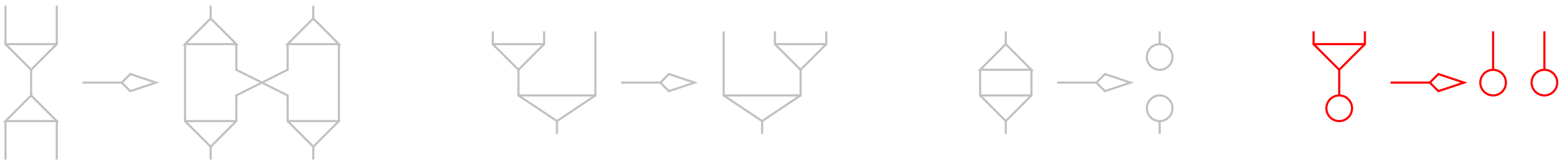


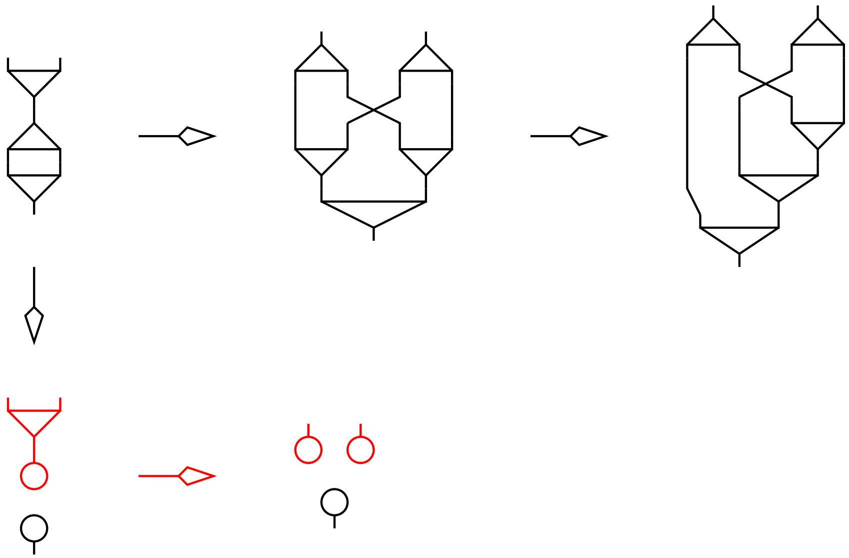
Problem number 2 : the presentation (Σ^c, R_2^c) is not confluent.



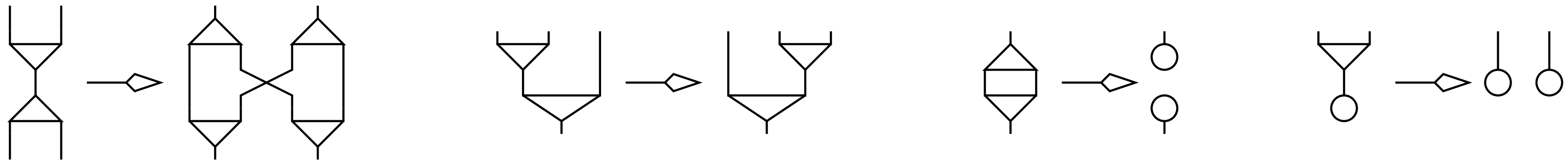


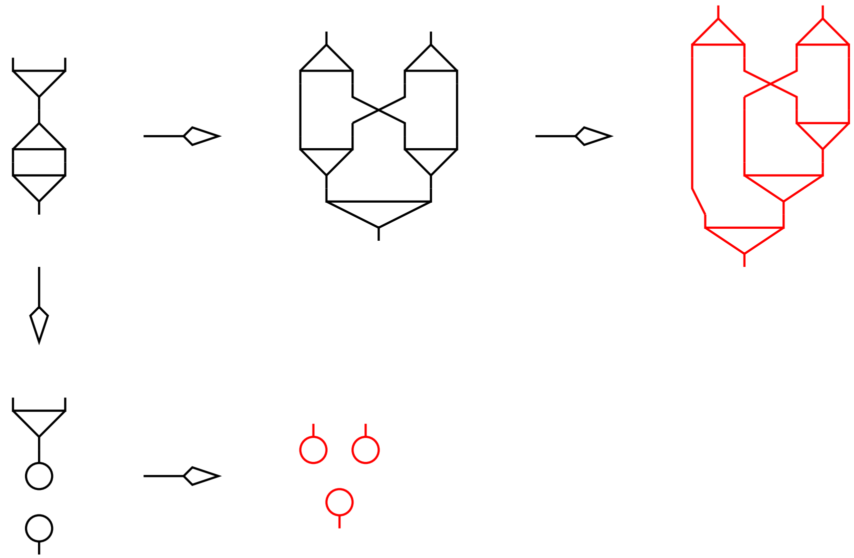
Problem number 2 : the presentation (Σ^c, R_2^c) is not confluent.





Problem number 2 : the presentation (Σ^c, R_2^c) is not confluent.

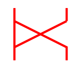




Second change :

Second change :

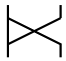
A new, superfluous operator is added :



κ

Second change :

A new, superfluous operator is added :

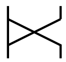


κ

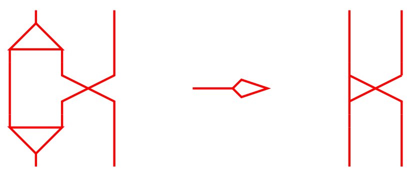
Interpretation : $\kappa(x, y) = (\mu(x, y), x)$.

Second change :

A new, superfluous operator is added :

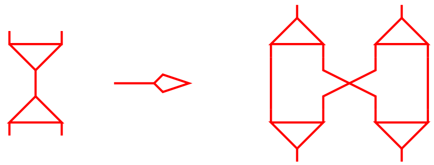


Interpretation : $\kappa(x,y) = (\mu(x,y), x)$. It is enforced by the new rule :

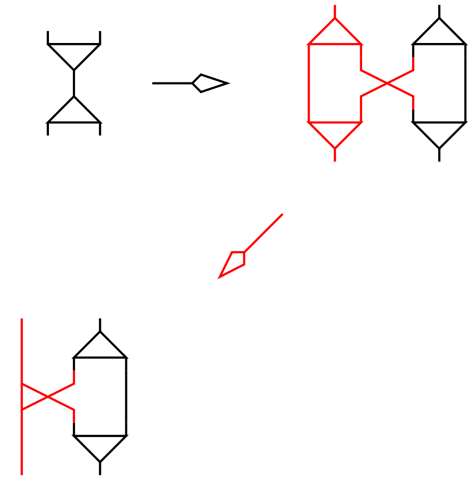


Third change :

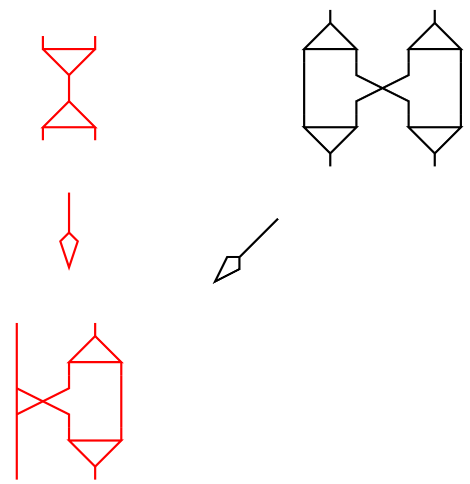
Third change :



Third change :



Third change :



After some more changes and rules additions, one gets the presentation denoted by $L(\mathbb{Z}_2)$:

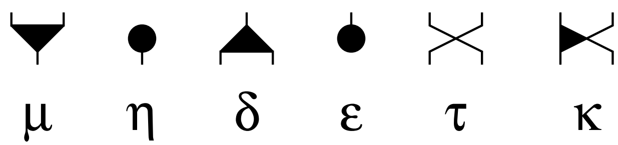
After some more changes and rules additions, one gets the presentation denoted by $L(\mathbb{Z}_2)$:

- Six operators :

μ η δ ε τ κ

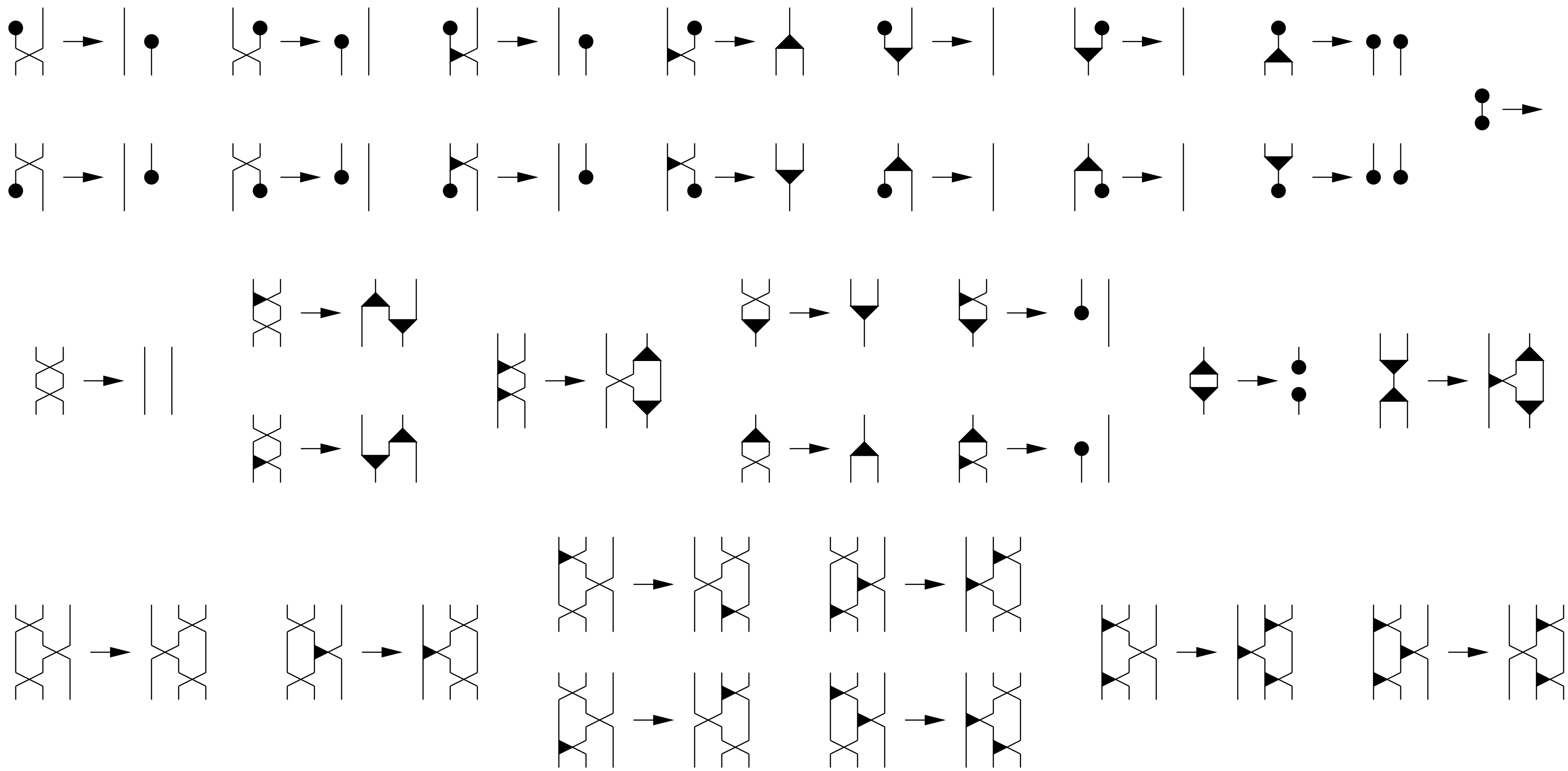
After some more changes and rules additions, one gets the presentation denoted by $L(\mathbb{Z}_2)$:

- Six operators :



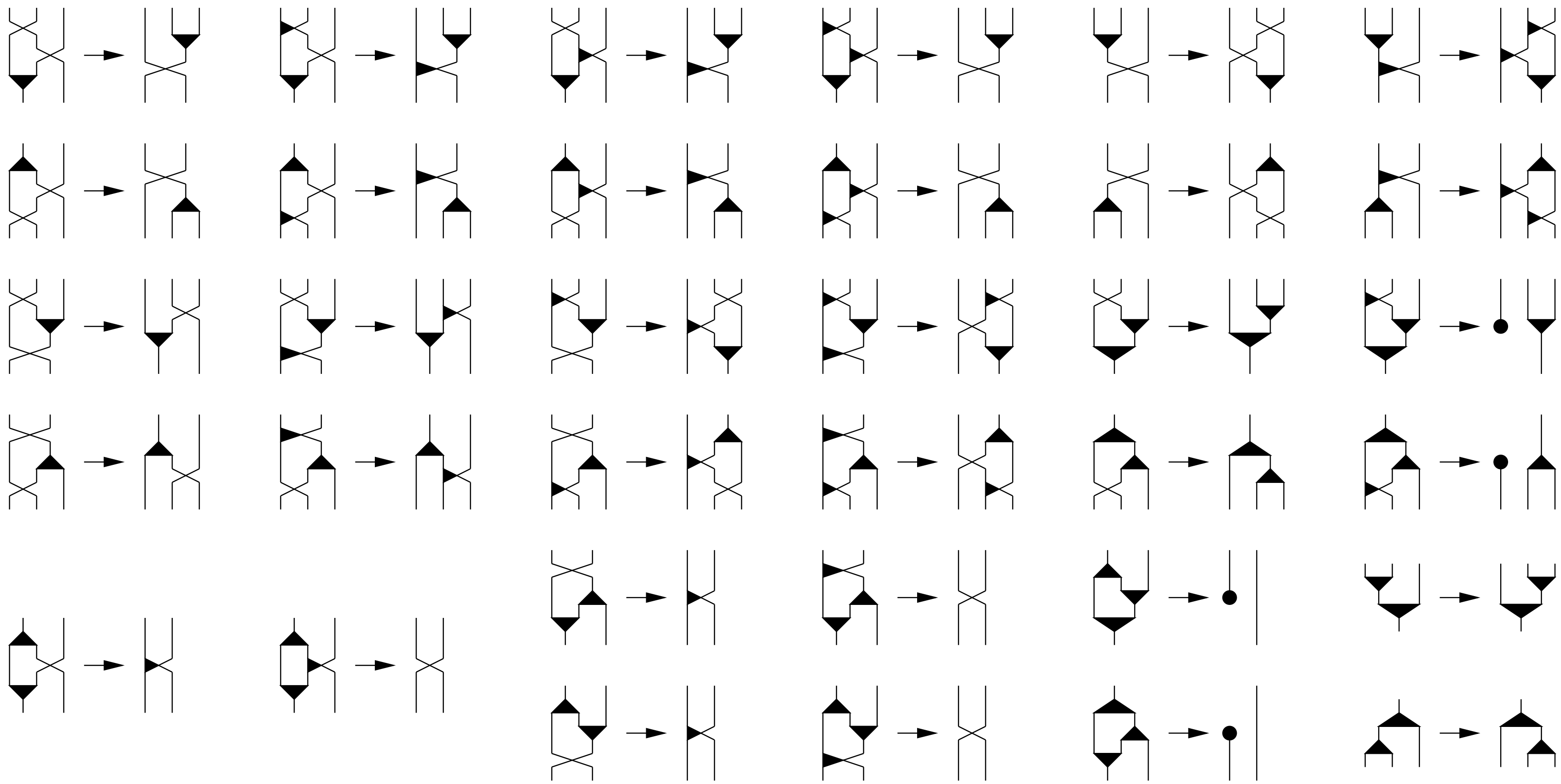
- Sixty-seven rules...

Thirty-three :



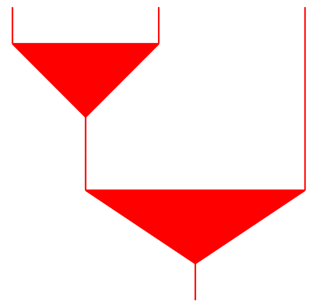
. . .

. . . plus thirty-four :

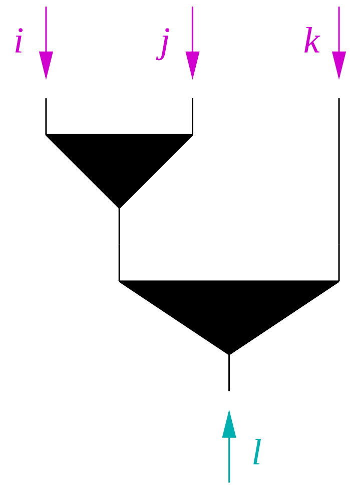


Termination of $L(\mathbb{Z}_2)$:

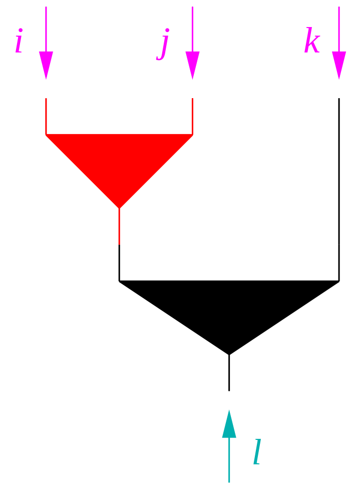
Termination of $\mathrm{L}(\mathbb{Z}_2)$:



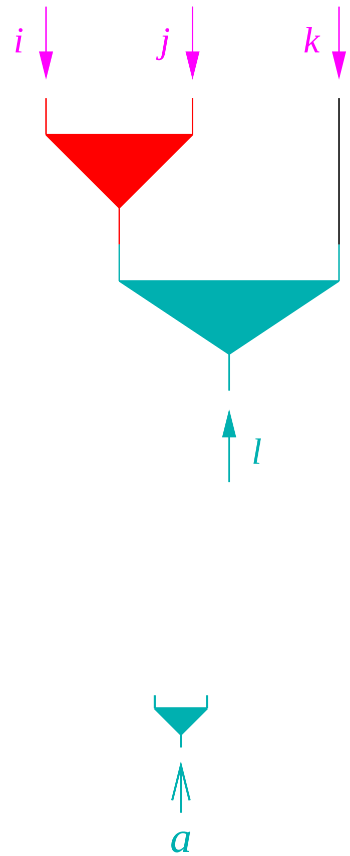
Termination of $\mathrm{L}(\mathbb{Z}_2)$:



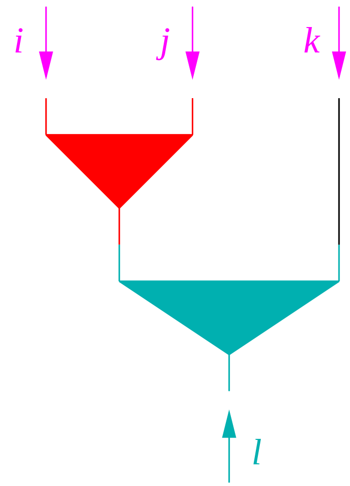
Termination of $\mathrm{L}(\mathbb{Z}_2)$:



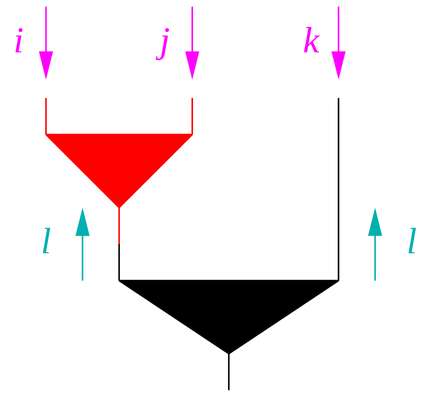
Termination of $\mathrm{L}(\mathbb{Z}_2)$:

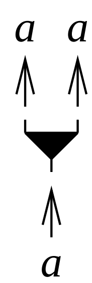


Termination of $L(\mathbb{Z}_2)$:

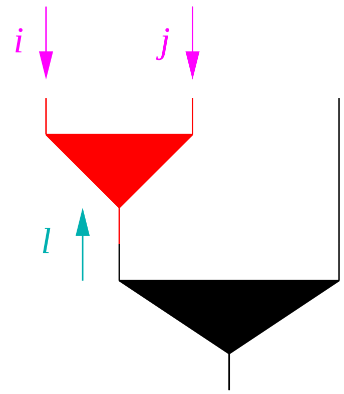


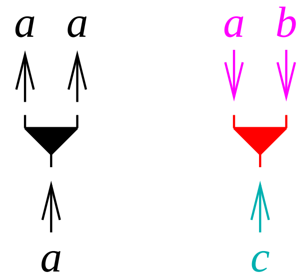
Termination of $L(\mathbb{Z}_2)$:



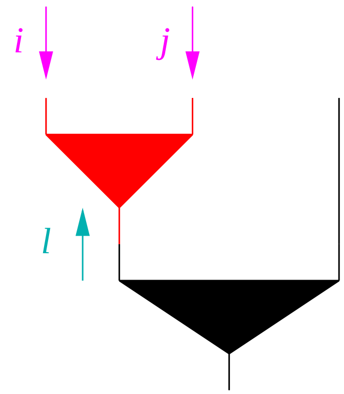


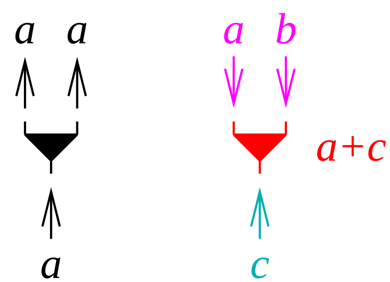
Termination of $L(\mathbb{Z}_2)$:



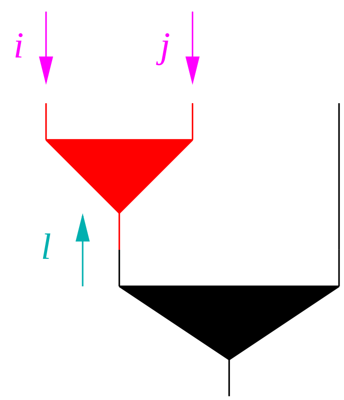


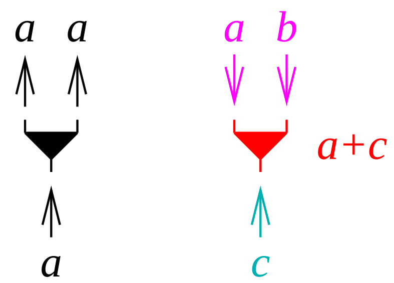
Termination of $L(\mathbb{Z}_2)$:





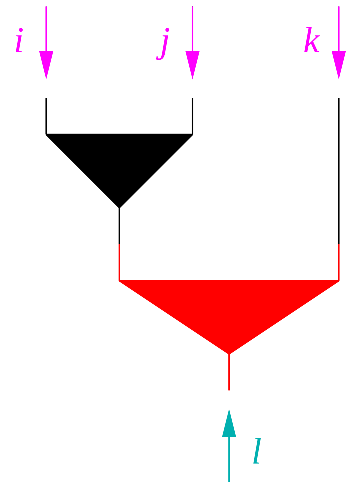
Termination of $\mathrm{L}(\mathbb{Z}_2)$:

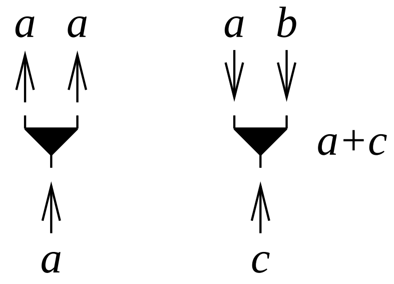




Produced heat : $i + l$.

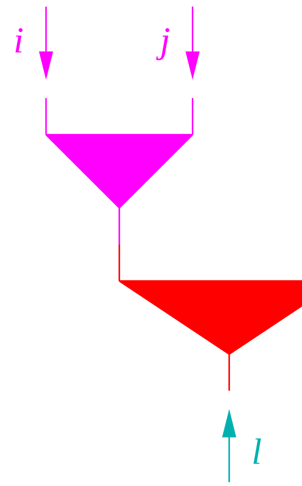
Termination of $\mathrm{L}(\mathbb{Z}_2)$:

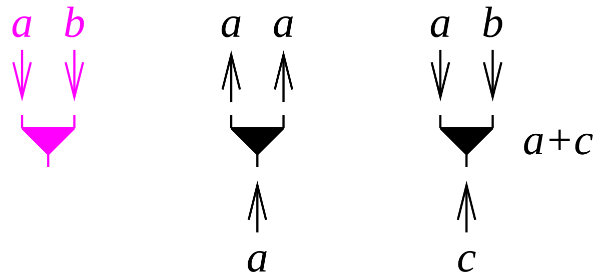




Produced heat : $i + l$.

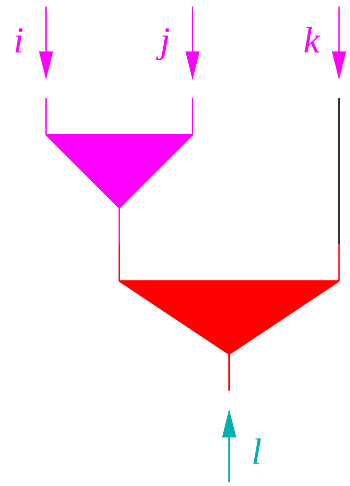
Termination of $L(\mathbb{Z}_2)$:

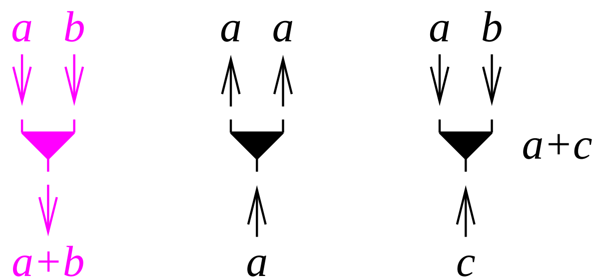




Produced heat : $i + l$.

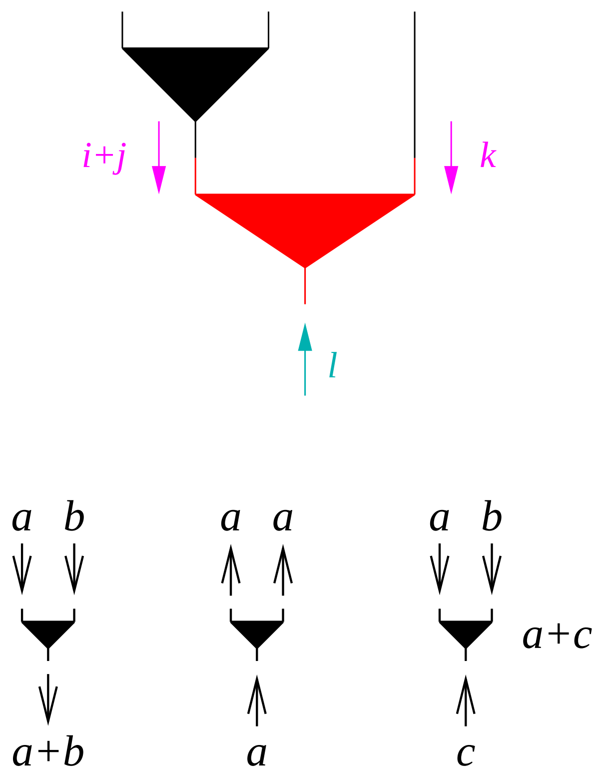
Termination of $\mathrm{L}(\mathbb{Z}_2)$:





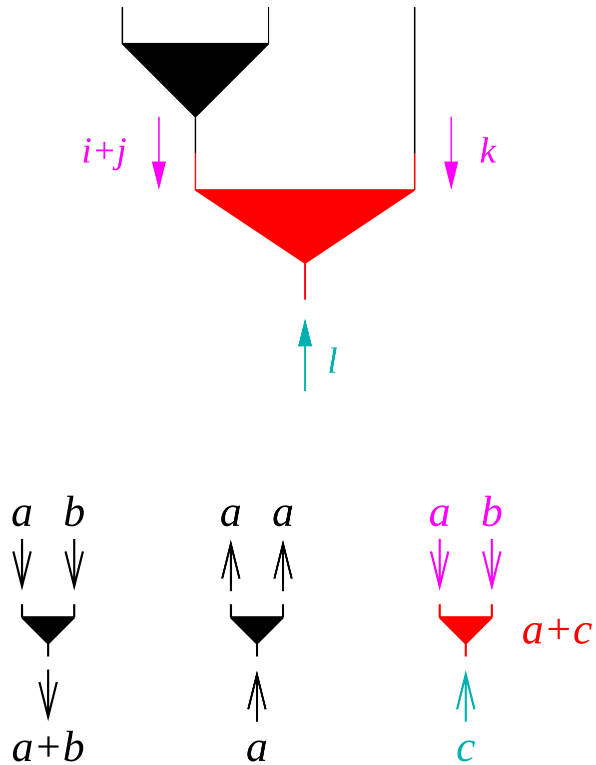
Produced heat : $i + l$.

Termination of $\mathrm{L}(\mathbb{Z}_2)$:



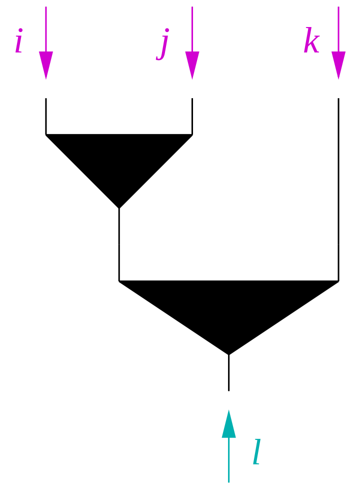
Produced heat : $i + l$.

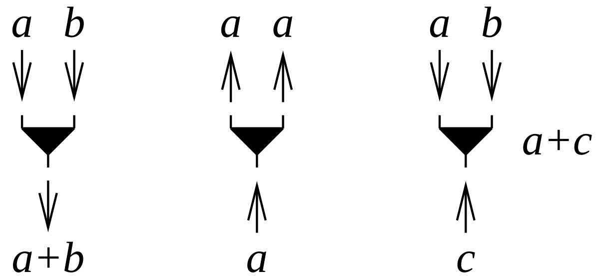
Termination of $\mathrm{L}(\mathbb{Z}_2)$:



Produced heat : $i + l$ and $i + j + l$.

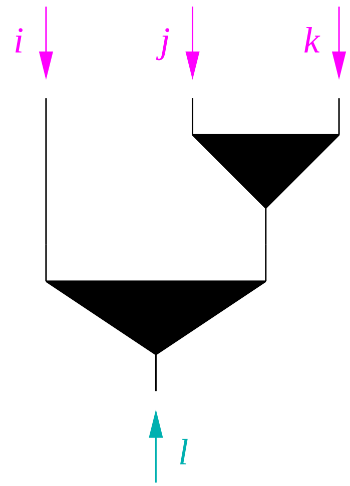
Termination of $\mathrm{L}(\mathbb{Z}_2)$:

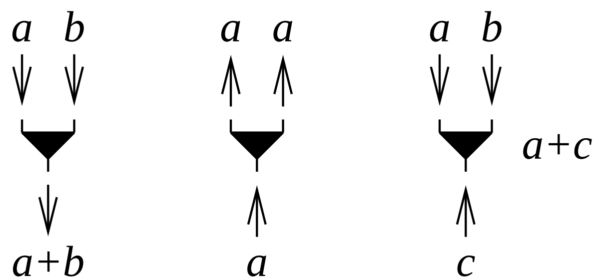




Produced heat : $2i + j + 2l$.

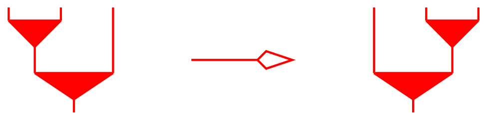
Termination of $\mathrm{L}(\mathbb{Z}_2)$:



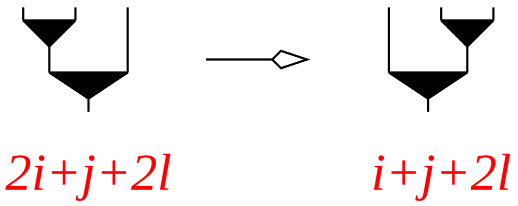


Produced heat : $i + j + 2l$.

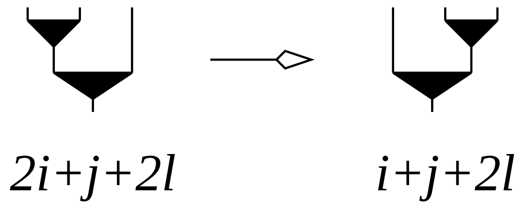
Termination of $\mathrm{L}(\mathbb{Z}_2)$:



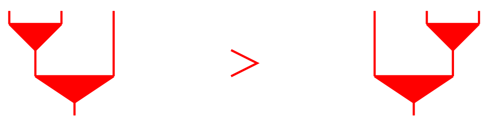
Termination of $L(\mathbb{Z}_2)$:



Termination of $L(\mathbb{Z}_2)$:



Which is translated into :



If the following conditions hold :

If the following conditions hold :

- $f \to g$ implies $f > g$,

If the following conditions hold :

- $f \longrightarrow g$ implies $f > g$,
- there is no infinite sequence : $f_0 > f_1 > f_2 > \cdots > f_k > f_{k+1} > \cdots$.

If the following conditions hold :

- $f \longrightarrow g$ implies $f > g$,
- there is no infinite sequence : $f_0 > f_1 > f_2 > \cdots > f_k > f_{k+1} > \cdots$.

Then the operad presentation terminates.

Theorem number 2 :

Theorem number 2 :

Let (Σ, R) be an operad presentation.

Theorem number 2 :

Let (Σ, R) be an operad presentation. If there exist :

Theorem number 2 :

Let (Σ, R) be an operad presentation. If there exist :

- two (non empty) ordered sets X and Y,

Theorem number 2 :

Let (Σ, R) be an operad presentation. If there exist :

- two (non empty) ordered sets X and Y,
- a commutative monoid M, equipped with a terminating, compatible with the sum strict order $>$,

Theorem number 2 :

Let (Σ, R) be an operad presentation. If there exist :

- two (non empty) ordered sets X and Y,
- a commutative monoid M, equipped with a terminating, compatible with the sum strict order $>$,
- for every $\varphi : m \to n$ in Σ, three non-decreasing maps :

$$\varphi_* : X^m \to X^n, \quad \varphi^* : Y^n \to Y^m, \quad [\varphi] : X^m \times Y^n \to M,$$

Theorem number 2 :

Let (Σ, R) be an operad presentation. If there exist :

- two (non empty) ordered sets X and Y,
- a commutative monoid M, equipped with a terminating, compatible with the sum strict order $>$,
- for every $\varphi : m \to n$ in Σ, three non-decreasing maps :

$$\varphi_* : X^m \to X^n, \quad \varphi^* : Y^n \to Y^m, \quad [\varphi] : X^m \times Y^n \to M,$$

such that, for every rule $f \to g$ in R, the three following inequalities hold :

$$f_* \geq g_*, \quad f^* \geq g^*, \quad [f] > [g],$$

Theorem number 2 :

Let (Σ, R) be an operad presentation. If there exist :

- two (non empty) ordered sets X and Y,
- a commutative monoid M, equipped with a terminating, compatible with the sum strict order $>$,
- for every $\varphi : m \to n$ in Σ, three non-decreasing maps :

$$\varphi_* : X^m \to X^n, \quad \varphi^* : Y^n \to Y^m, \quad [\varphi] : X^m \times Y^n \to M,$$

such that, for every rule $f \to g$ in R, the three following inequalities hold :

$$f_* \geq g_*, \quad f^* \geq g^*, \quad [f] > [g],$$

then : the operad presentation (Σ, R) terminates.

These termination orders have been successfully used to prove that the following operad presentations terminate :

These termination orders have been successfully used to prove that the following operad presentations terminate :

- The presentation $(\Sigma^c, R_{\Delta,\Sigma})$, for every algebraic signature Σ.

These termination orders have been successfully used to prove that the following operad presentations terminate :

- The presentation $(\Sigma^c, R_{\Delta,\Sigma})$, for every algebraic signature Σ.
- The presentation (Σ^c, R^c) for every terminating, left-linear term rewrite system (Σ, R).

These termination orders have been successfully used to prove that the following operad presentations terminate :

- The presentation $(\Sigma^c, R_{\Delta,\Sigma})$, for every algebraic signature Σ.
- The presentation (Σ^c, R^c) for every terminating, left-linear term rewrite system (Σ, R).
- The presentation $\mathrm{L}(\mathbb{Z}_2)$.

Future directions

Future directions :

Future directions :

- Confluence of operad presentations.

Future directions :

- Confluence of operad presentations.
- Calculi of structures and 2-operad presentations.

Future directions :

- Confluence of operad presentations.
- Calculi of structures and 2-operad presentations.
- Is the λ-calculus a n-operad presentation ?

Future directions :

- Confluence of operad presentations.
- Calculi of structures and 2-operad presentations.
- Is the λ-calculus a n-operad presentation ?
- Classical and quantum circuits.

Termination orders for operad presentations

Yves Guiraud

Institut de Mathématiques de Luminy
Université Aix-Marseille 2
guiraud@iml.univ-mrs.fr

Rencontres GÉOCAL : Complexité implicite et logique
Université Paris-Nord
6 et 7 septembre 2004