LINEAR POLYGRAPHS AND KOSZULITY OF ALGEBRAS

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Abstract – We define higher dimensional linear rewriting systems, called linear polygraphs, for presentations of associative algebras, generalizing the notion of noncommutative Gröbner bases. They are constructed on the notion of category enriched in higher-dimensional vector spaces. Linear polygraphs allow more possibilities of termination orders than those associated to Gröbner bases. We introduce polygraphic resolutions of algebras giving a description obtained by rewriting of higher-dimensional syzygies for presentations of algebras. We show how to compute polygraphic resolutions starting from a convergent presentation, and how to relate these resolutions with the Koszul property of algebras.

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1. Introduction

In homological algebra, several constructive methods based on noncommutative Gröbner bases were developed to compute projective resolutions for algebras. In particular, these methods lead to relate the Koszul property for an associative algebra to the existence of a quadratic Gröbner basis for its ideal of relations: an associative algebra having a presentation by a quadratic Gröbner basis is Koszul. In this article, we explain how these constructions can be interpreted from the point of view of higher-dimensional rewriting theory. Moreover, we use this setting to develop several improvements of these methods.

We define *linear polygraphs* as higher-dimensional linear rewriting systems for presentations of algebras, generalizing the notion of noncommutative Gröbner bases. Linear polygraphs allow more possibilities of termination orders than those associated to Gröbner bases, only based on monomial orders. Moreover, we introduce polygraphic resolutions of algebras giving a description obtained by rewriting of higher-dimensional syzygies for presentations of algebras. We show how to compute polygraphic resolutions starting from a convergent presentation, and how to relate these resolutions with the Koszul property.

An overview on rewriting and Koszulity

Linear rewriting and Gröbner bases. In order to effectively compute normal forms in algebras, to decide the word problem (ideal membership) or to construct bases (e.g., Poincaré-Birkhoff-Witt bases), Buchberger and Shirshov have independently introduced the notion of Gröbner bases for commutative and Lie algebras, respectively [13, 30]. Subsequently, Gröbner bases have been developed for other types of algebras, such as associative algebras by Bokut [11] and by Bergman [10]. The notion of Gröbner bases had already been introduced by Hironaka in [22], under the name of *standard bases* but without a constructive method for computing such bases.

Consider an algebra $\mathbf{A}$ presented by a set of generators X and a set of relations R, that is $\mathbf{A}$ is the quotient of the free algebra $\mathbb{K}\langle \mathsf{X}\rangle$ by the congruence generated by R. The elements of the free monoid X^* form a linear basis of the free algebra $\mathbb{K}\langle \mathsf{X}\rangle$. One main application of Gröbner bases is to explicitly find a basis of the algebra $\mathbf{A}$, in the form of a subset of X^*. This is based on a monomial order on the monoid X^* and the idea is to change the presentation of the ideal generated by R with respect to this order. The property that the new presentation has to satisfy is the algebraic counterpart of the confluence of a rewriting system. The central theorem for Gröbner basis is the counterpart of Newman's lemma. In particular, Buchberger's algorithm, producing Gröbner bases, is in essence the analogue of Knuth-Bendix's completion procedure in a linear setting. Several frameworks unify Buchberger and Knuth-Bendix algorithms, in particular a Gröbner basis corresponds to a convergent (i.e., confluent and terminating) presentation of an algebra, see [14]. This correspondence is well known in the case of associative and commutative algebras, as recalled in the papers of Bergman, Mora and Ufnarovski [10, 27, 34].

Gröbner bases and projectives resolutions. At the end of 1980s, through Anick's and Green's works [1, 2, 3, 19], non-commutative Gröbner bases have found new applications for the study of algebras as a constructive method to compute free resolutions. Their constructions provide small explicit resolutions to compute homological invariants (homology groups, Hilbert and Poincaré series) of algebras presented by generators and relations defined by a Gröbner basis. We refer the reader to [34] for a survey on Anick's resolution and to [5] for an implementation of the resolution. Nevertheless, the chains (given by some of the iterated overlaps of the leading terms of the Gröbner basis) and the differential in these resolutions are constructed recursively, which makes computations sometimes complicated.

Confluence and Koszulity. Recall that a connected graded algebra is called Koszul if it has a nice homological property, which can be defined in several equivalent ways. For instance, **A** is Koszul if the Tor groups $\mathrm{Tor}^{\mathbf{A}}_{k,(i)}(\mathbb{K},\mathbb{K})$ vanish for $i \neq k$ (where the first grading is the homological degree and the second grading corresponds to the internal grading of the algebra). The property can be also be stated in terms of existence of a linear minimal graded free resolution of $\mathbb{K}$ seen as a **A**-module. This notion was generalized by Berger in [8] to the case of N-homogeneous algebras, asking that $\mathrm{Tor}^{\mathbf{A}}_{k,(i)}(\mathbb{K},\mathbb{K})$ vanish for $i \neq \ell_N(k)$, where $\ell_N : \mathbb{N} \to \mathbb{N}$ is the function defined by

$$\ell_N(k) = \begin{cases} lN & \text{if } k = 2l, \\ lN + 1 & \text{if } k = 2l + 1, \end{cases}$$

for any integer k. In what follows, we will call Koszul algebras the generalized notion.

Anick's resolutions can be used to prove Koszulity of an algebra. Indeed, if an algebra **A** has a quadratic Gröbner basis, then Anick's resolution is concentrated in the right bidegree, and thus **A** is Koszul (see for instance Green and Huang [18, Theorem 9]). Another way to prove this result is that the existence of a quadratic Gröbner basis implies the existence of a Poincaré-Birkhoff-Witt basis of **A** (see Green [19, Proposition 2.14]). For the N-homogeneous case, a Gröbner basis concentrated in weight N is not enough to imply Koszulity: an extra condition has to be checked as shown by Berger in [8, Theorem 3.6]. When the algebra is monomial, this extra condition corresponds to the *overlap property* defined by Berger in [8, Proposition 3.8.]. This property consists in a combinatorial condition based on overlaps of the monomials of the relations.

Yet another method to prove Koszulity of algebras using a Gröbner basis can be found in the book of Loday and Vallette [25, Chap. 4]. The quadratic Gröbner basis method to prove Koszulity has been extended to the case of operads, see Dotsenko and Khoroshkin [16] or [25, Chap. 8].

All the constructions mentioned above rely on a monomial order, that is a well-founded total order of the monomials. The termination orders in linear polygraph introduced in this work are less restrictive.

Organisation and main results of the article

The next section consists in presenting the categorical background of our constructions. In Section 3, we develop the notion of linear rewriting system for algebras and we explain the links with Gröbner bases. In Section 4, we give a method to construct polygraphic resolutions for an algebra from a convergent presentation of these algebra. In the last section, we show that polygraphic resolutions induce free modules resolutions for algebroids. We deduce finiteness conditions and several sufficient conditions for an algebroid to be Koszul. We now give a detailed preview of the main construction and results of the article.

Higher-dimensional algebroids. In Section 2.1, we introduce the notion of *graded n-vector space* as an internal (strict globular) $(n-1)$-category in the category of non-negatively graded spaces GrVect. This definition extends in higher dimensions the notion of 2-vector space introduced by Baez and Crans in [6]. An 1-algebroid (which we call algebroid from now on) is an algebra with several objects, also called $\mathbb{K}$-category by Mitchell in [26]. We define in Section 2.2 a *graded n-algebroid* as a category enriched in graded n-vector spaces. Note that the Bourn's equivalence, [12, Theorem 3.3], states that the category of graded n-algebroids is equivalent to the category of graded chain complexes of length n.

Linear polygraphs. Higher-dimensional rewriting has unified several paradigms of rewriting. This approach is based on presentations by generators and relations of higher-dimensional categories, independently introduced by Burroni and Street under the respective names of *polygraphs* in [15] and *computads* in [32, 33]. The notion of *linear polygraph* extends this framework to a linear setting. A string (or path) rewriting system is a 2-*polygraph*. This is a data $(\Sigma_0, \Sigma_1, \Sigma_2)$ made of an oriented graph

$$\begin{tikzcd} \Sigma_0 & \Sigma_1 \arrow[l, shift right, "s_0"'] \arrow[l, shift left, "t_0"] \end{tikzcd}$$

where Σ_0 and Σ_1 denote respectively the sets of 0-cells, or objects, and of 1-cells, or arrows and s_0, t_0 denote the source and target maps, with a *cellular extension* Σ_2 of the free category Σ_1^*, that is a set of globular 2-cells relating parallel 1-cells:

$$\begin{tikzcd} p \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\varphi"] & q \end{tikzcd}$$

A *linear* 2-*polygraph* corresponds to the notion of a linear rewriting system for presentations of algebras. It is constructed in the same manner as a 2-polygraph, but the cellular extension is linear. This means it is defined as a family of vector spaces $\big(\Lambda_2(p, q)\big)_{p,q\in\Sigma_0}$ where each $\Lambda_2(p, q)$ is a space of 2-cells relating parallel 1-cells of the free algebroid Σ_1^ℓ on the graph (Σ_0, Σ_1). In the free 2-algebroid Λ_2^ℓ, any 2-cell is invertible, i.e., it is a $(2, 1)$-category. As a consequence, the notion of rewriting step induced by a linear polygraph needs to be defined with attention as it is done in Section 3.2.1. Then we develop properties of linear rewriting such as termination, confluence and local confluence in Section 3.2. We state the Newman's Lemma, also called Diamond Lemma, for linear 2-polygraphs, Proposition 3.2.12. In Section 3.3 we recover the notion of Poincaré-Birkhoff-Witt bases in term of family of irreducible monomial of convergent 2-polygraphs. We also recover the Gröbner bases as a special case of convergent linear 2-polygraphs in Proposition 3.4.6.

Polygraphic resolutions of algebroids. In Section 4, we define a *polygraphic resolution* for an algebroid **A** as an acyclic polygraphic extension of a presentation of **A**, that is a linear ∞-polygraph, which satisfies an acyclicity condition. A method to construct such a polygraphic resolution is to consider a *normalisation strategy*, inducing a notion of normal form in every dimension, together with a homotopically coherent reduction of every cell to its normal form. This notion was introduced in [20] for presentations of categories. In Section 4.1, we develop this notion for algebroids. We prove that a polygraphic resolution of **A** is equivalent to the data of a polygraph whose underlying 2-polygraph is a presentation of **A** and equipped with a normalisation strategy, Proposition 4.1.9.

In Section 4.2, we show how to construct a polygraphic resolution for an algebroid **A** from a convergent presentation of **A**. Our construction consists in extending by induction a reduced monic linear 2-polygraph Λ into a polygraphic resolution of the presented algebroid, whose generating n-cells are indexed by the $(n-1)$-fold critical branchings of Λ, that is the iterated overlaps of leading terms of relations:

> **Theorem 4.2.10.** *Any convergent linear 2-polygraph Λ extends to an acyclic linear ∞-polygraph $\mathcal{C}_\infty(\Lambda)$, presenting the same algebroid, and whose n-cells, for $n \geq 3$, are indexed by the critical $(n-1)$-fold branchings.*

From this point of view, this resolution is similar to the Anick's resolution associated with a Gröbner basis. The acyclicity condition is obtained by the construction explicitly of a homotopy, via a normalisation strategy, as in [20].

Free resolutions of algebroids. In the last section, we show how a polygraphic resolution of an algebroid **A** induces free resolutions in categories of modules over **A**. Given a function $\omega : \mathbb{N} \to \mathbb{N}$, we call a polygraphic resolution ω-*concentrated* when for any integer k, all k-cells are concentrated in degree

$\omega(k)$. Similarly, a free resolution $P_\bullet$ of **A**-modules is ω-concentrated when for any integer k, the **A**-module P_k is generated in degree $\omega(k)$. Given a linear ∞-polygraph Λ whose underlying 2-polygraph is presentation of **A**. In Section 5.1.2, we construct a complex of **A**-bimodules, denoted by $\mathbf{A}^e[\Lambda]$, whose boundary maps are induced by the source and target maps of the polygraph. We prove that if the linear polygraph Λ is acyclic, then the complex is acyclic and thus it is a resolution of the **A**-bimodule **A**:

Theorem 5.1.3. *If Λ is a (finite) ω-concentrated polygraphic resolution of an algebroid* **A**, *then the complex* $\mathbf{A}^e[\Lambda]$ *is a (finite) ω-concentrated free resolution of the* **A***-bimodule* **A**.

In the same way, we construct in Theorem 5.1.5 such a resolution for the **A**-module $\mathbb{K}$.

Using these constructions, we deduce homological properties and Koszul property of an algebroid **A** from polygraphic resolutions of **A**.

Finiteness properties. In Section 5.2 we introduce the property of *finite n-derivation type* for an algebroid. Proposition 5.2.2 relates this finiteness condition for an algebroid **A** with the existence of a normalising polygraph whose underlying 2-polygraph is a presentation of **A**.

Finally, we prove that an algebroid **A** having a finite convergent presentation is of finite ∞-derivation type, Proposition 5.2.3, and thus of homological type FP_∞, Proposition 5.2.6.

Convergence and Koszulity. In Section 5.3, we apply our constructions to study Koszulity of some algebras. As a consequence of Theorem 5.1.3, we obtain the main result of this section:

Theorem 5.3.4. *Let* **A** *be an N-homogeneous algebroid. If* **A** *has a ℓ_N-concentrated polygraphic resolution, then* **A** *is right-Koszul (resp. left-Koszul, resp. bi-Koszul).*

As a consequence of Theorem 5.3.4, we have

Theorem 5.3.6. *Let* **A** *be an algebra presented by a quadratic convergent linear 2-polygraph Λ. Then Λ can be extended into a ℓ_2-concentrated polygraphic resolution. In particular, any algebra having a presentation by a quadratic convergent linear 2-polygraph is Koszul.*

This theorem generalizes for instance the criterion using a quadratic Gröbner basis We also show in Section 5.3.11 how it is possible in some cases to reduce the size of a polygraphic resolution. This method can be used to show Koszulity. We end this paper by discussing several examples were we apply rewriting methods to prove the Koszul property presented in this section.

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2. Linear polygraphs

Throughout this section, we denote by n either a natural number or ∞.

2.1. Higher-dimensional vector spaces

2.1.1. Notation. We denote by $\mathbb{K}$ the ground field. The category of vector spaces over the field $\mathbb{K}$ is denoted by Vect. We say space and map instead of vector space and linear map. The tensor product of two spaces V and W is denoted $V \otimes W$. The tensor product of n copies of V is denoted $V^{\otimes n} = V \otimes \ldots \otimes V$. We denote by GrVect the category of non-negatively graded spaces and of morphisms (of degree 0) of graded spaces.

2.1.2. Notation on n-categories. We denote by Cat_n the category of strict globular n-categories and n-functors. We refer the reader to the book of Leinster [24] for definitions on higher-dimensional categories. If $\mathcal{C}$ is an n-category, we denote by $\mathcal{C}_k$ the set (and the k-category) of k-cells of $\mathcal{C}$. If f is a k-cell of $\mathcal{C}$, then $s_l(f)$ and $t_l(f)$ respectively denote the l-source and l-target of f. The source and target maps

$$\mathcal{C}_l \underset{t_l}{\overset{s_l}{\leftleftarrows}} \mathcal{C}_{l+1}$$

satisfy the *globular relations*:

$$s_l \circ s_{l+1} \;=\; s_l \circ t_{l+1} \qquad \text{and} \qquad t_l \circ s_{l+1} \;=\; t_l \circ t_{l+1},$$

for any $0 \leq l \leq n-1$. We respectively denote by $f : u \to v$, $\;f : u \Rightarrow v\;$ or $\;f : u \Rrightarrow v\;$ a 1-cell, a 2-cell or a 3-cell f with source u and target v.

If f and g are l-composable k-cells, that is when $t_l(f) = s_l(g)$, we denote by $f \star_l g$ their l-composite; we simply use fg when $l = 0$. The compositions satisfy the *exchange relations* given, for every $l_1 \neq l_2$ and every possible cells f, f', g and g', by:

$$(f \star_{l_1} f') \star_{l_2} (g \star_{l_1} g') \;=\; (f \star_{l_2} g) \star_{l_1} (f' \star_{l_2} g').$$

If f is a k-cell, we denote by 1_f its identity $(k+1)$-cell. When 1_f is composed with cells of dimension $k+1$ or higher, we simply denote it by f in the composition. A k-cell f whose l-source and l-target are equal is called an l-*endo*-k-*cell*.

2.1.3. n-vector spaces. An internal n-category in Vect (resp. in GrVect) is a n-category whose each set of k-cells $\mathcal{V}_k$ forms a (resp. graded) space, in such a way that all the source and target maps, identity maps and the composition maps are (resp. graded) linear.

For $n \geq 1$, a n-*vector space* is an internal $(n-1)$-category in Vect. In a equivalent way, it can be defined as a vector space object in Cat_{n-1}. We will use the same notation for higher-dimensional vector spaces as for higher-dimensional categories in 2.1.2. For a n-vector space $\mathcal{V}$, we denote by $\mathcal{V}_k$ the vector space of k-cells of $\mathcal{V}$. The source, target, composition and identity maps are denoted as for n-categories.

We set that a 0-vector space is a set. Note that a 1-vector space is a space. Explicitly, for $n \geq 1$, a n-vector space is a $(n-1)$-category $\mathcal{V}$ whose k-cells form a vector space $\mathcal{V}_k$ in such a way that all the source and target maps s_k and t_k, the identity maps and the $\star_k$-composite maps are linear.

By linearity of the l-composition maps, for any l-composable pairs of k-cells $u \xrightarrow{f} v \xrightarrow{g} w$ and $u \xrightarrow{f'} v \xrightarrow{g'} w$ in $\mathcal{V}$, with $0 \leq l < k \leq n$, we have

$$(f + f') \star_l (g + g') = f \star_l g + f' \star_l g' = f \star_l g' + f' \star g.$$

A *linear* n-*functor* $\mathcal{V} \longrightarrow \mathcal{W}$ between n-vector spaces is an internal $(n-1)$-functor in Vect. The n-vector spaces and linear functors form a category denoted by Vect_n.

2.1.4. Graded n-vector spaces. We define a *graded* n-*vector space* $\mathcal{V}$ as an internal $(n-1)$-category in the category GrVect. Explicitly the k-cells in $\mathcal{V}$ form a graded space

$$\mathcal{V}_k = \bigoplus_{i \in \mathbb{N}} \mathcal{V}_k^{(i)}.$$

The k-cells in $\mathcal{V}_k^{(i)}$ are called *homogeneous* k-cells *of degree* i. For $l < k$, the identity maps, source maps and target maps s_l and t_l are graded: they send homogeneous k-cells of degree i on homogeneous l-cells of the same degree. The l-compositions $\star_l$ are also graded: if f and g are l-composable k-cells of degree i, then their l-composite $f \star_l g$ is homogeneous of degree i.

A *graded* n*-linear functor* between graded n-vector spaces is an internal $(n-1)$-functor in GrVect. The graded n-vector spaces and graded linear functors form a category denoted by GrVect_n. A *trivially graded* n-vector space $\mathcal{V}$ satisfies $\mathcal{V}_k^{(i)} = 0$ for any $k \geq 0$ and any $i \geq 1$.

2.1.5. The arrow part. We define the arrow part of a k-cell of a graded n-vector space as in the case of 2-vector spaces by Baez and Crans, [6]. Let $\mathcal{V}$ be a graded n-vector space. The *arrow part* of a k-cell f, for $k \geq 1$, is the k-cell $\overrightarrow{f}$ defined by

$$\overrightarrow{f} = f - s_{k-1}(f).$$

We have

$$s_{k-1}(\overrightarrow{f}) = 0 \quad \text{and} \quad t_{k-1}(\overrightarrow{f}) = t_{k-1}(f) - s_{k-1}(f).$$

The arrow part

$$\overrightarrow{f} : 0 \to t_{k-1}(f) - s_{k-1}(f)$$

corresponds to a 'translation to the origin' of the k-cell $f : s_{k-1}(f) \to t_{k-1}(f)$. In particular, the arrow part of an identity is zero: we have $\overrightarrow{1_u} = 0$, for any $(k-1)$-cell u. Any k-cell f is the sum of its source and its arrow part: $f = \overrightarrow{f} + s_{k-1}(f)$. We will use the notation of [6], where a k-cell $f : u \to v$ is identified with the pair $(u, \overrightarrow{f})$.

Baez and Crans showed that the structure of 2-vector space $\mathcal{V}$ is entirely determined by the vector spaces structure on the set of cells and the source, target and identity maps, [6, Lemma 6]. The composition maps can be expressed using these maps together with the addition in vector spaces. For n-vector spaces, we have

2.1.6. Proposition. *Let $\mathcal{V}$ be a graded n-vector space and let $1 \leq k \leq n$. For any $0 \leq l \leq k-1$, any l-composables k-cells f and g satisfy the following properties:*

i) $f \star_l g = f + g - s_l(g)$*, that is* $\overrightarrow{f \star_l g} = \overrightarrow{f} + \overrightarrow{g}$*;*

ii) $f \star_l g = g \star_l f$*, if f and g are l-endo-k-cells with same l-source.*

Proof. We prove the assertion **i)**. The assertion **ii)** is an immediate consequence of **i)**.
Let $u \xrightarrow{f} v$ and $v \xrightarrow{g} w$ be l-composable pairs of k-cells in $\mathcal{V}$. By linearity of the source and target maps s_l and t_l, the k-cells $u + v \xrightarrow{f+v} 2v$ and $2v \xrightarrow{g+v} w + v$ are l-composable, and by linearity of the l-composition, their l-composition is given by

$$(f + v) \star_l (g + v) = f + g.$$

Hence,

$$(u + v, \overrightarrow{f}) \star_l (v + v, \overrightarrow{g}) = (u + v, \overrightarrow{f} + \overrightarrow{g}).$$

That is $f \star_l g = (u, \overrightarrow{f}) \star_l (v, \overrightarrow{g}) = (u, \overrightarrow{f} + \overrightarrow{g})$, hence $f \star_l g = f + g - s_l(g)$. □

The second part of the previous proposition applies in particular to composable endo-k-cells.

2.1.7. Invertible cells. A k-cell f of a graded n-vector space $\mathcal{V}$, with $(k-1)$-source u and $(k-1)$-target v, is *invertible* when there exists a (necessarily unique) k-cell denoted by f^- in $\mathcal{V}$, with $(k-1)$-source v and $(k-1)$-target u, called the *inverse of* f, that satisfies

$$f \star_{k-1} f^- = 1_u \qquad \text{and} \qquad f^- \star_{k-1} f = 1_v.$$

As a consequence of the Proposition 2.1.6, we have

2.1.8. Proposition. *Let $\mathcal{V}$ be a graded n-vector space and let $k \geq 1$. Then any k-cell f in $\mathcal{V}$ is invertible with inverse $f^- = -f + s_{k-1}(f) + t_{k-1}(f)$, that is*

$$\overrightarrow{f^-} = -\overrightarrow{f};$$

2.1.9. Bilinear n-functor. Given two graded n-vector spaces $\mathcal{V}$ and $\mathcal{W}$, we define their *biproduct* as the graded n-vector space, denoted by $\mathcal{V} \times \mathcal{W}$, and defined by

$$(\mathcal{V} \times \mathcal{W})_k = \mathcal{V}_k \times \mathcal{W}_k,$$

for any $k \leq n$. Source, target, identity and composition maps are defined in the obvious way. The inclusion and projection maps

$$\iota_1 : \mathcal{V} \to \mathcal{V} \times \mathcal{W}, \quad \iota_2 : \mathcal{W} \to \mathcal{V} \times \mathcal{W}, \qquad \pi_1 : \mathcal{V} \times \mathcal{W} \to \mathcal{V}, \quad \pi_2 : \mathcal{V} \times \mathcal{W} \to \mathcal{W},$$

are defined in the obvious way. Given graded n-vector spaces $\mathcal{V}$, $\mathcal{V}'$ and $\mathcal{W}$, a n-functor $F : \mathcal{V} \times \mathcal{V}' \to \mathcal{W}$ is said to be *bilinear* if the maps $F_i : \mathcal{V}_i \times \mathcal{V}'_i \to \mathcal{W}_i$ are bilinear for any $i \leq n$.

2.1.10. Tensor product. Given two n-vector spaces $\mathcal{V}$ and $\mathcal{W}$, we define their *tensor product* as the n-vector space, denoted by $\mathcal{V} \otimes \mathcal{W}$, and defined by

$$(\mathcal{V} \otimes \mathcal{W})_k = \mathcal{V}_k \otimes \mathcal{W}_k,$$

for any $k \leq n$. The l-source $s_l^{\mathcal{V}\otimes\mathcal{W}}$ and l-target map $t_l^{\mathcal{V}\otimes\mathcal{W}}$ are defined by

$$s_l^{\mathcal{V}\otimes\mathcal{W}} = s_l^{\mathcal{V}} \otimes s_l^{\mathcal{W}}, \qquad t_l^{\mathcal{V}\otimes\mathcal{W}} = t_l^{\mathcal{V}} \otimes t_l^{\mathcal{W}}.$$

For a k-cell f of $\mathcal{V}$ and a k-cell g of $\mathcal{W}$, the identity $(k+1)$-cell $1_{f\otimes g}$ is defined by $1_{f\otimes g} = 1_f \otimes 1_g$. The l-composition is defined by

$$(f \otimes g) \star_l (f' \otimes g') = (f \star_l f') \otimes (g \star_l g'),$$

for any l-composable k-cells f and g in $\mathcal{V}$ and f' and g' in $\mathcal{W}$.

The tensor product $\mathcal{V} \otimes \mathcal{W}$ satisfies the following universal property: For any bilinear n-functor F on $\mathcal{V} \times \mathcal{W}$ with values in a n-vector space $\mathcal{U}$, there exists a unique linear n-functor $G : \mathcal{V} \otimes \mathcal{W} \to \mathcal{U}$ such that the following diagram commutes

$$\begin{tikzcd} \mathcal{V} \times \mathcal{W} \arrow[r, "I"] \arrow[d, "F"'] & \mathcal{V} \otimes \mathcal{W} \arrow[dl, "G"] \\ \mathcal{U} & \end{tikzcd}$$

where I is the bilinear n-functor defined by $I(f, g) = f \otimes g$, for any k-cells f in $\mathcal{V}$ and g in $\mathcal{W}$.

Define the n-vector space $\mathcal{K}$ where $\mathcal{K}_i$ is the ground field $\mathbb{K}$, for any $i \leq k$ and the source, target, identity and composition maps are the identities on $\mathbb{K}$. For any n-vector space $\mathcal{V}$, we have isomorphisms

$$l_{\mathcal{V}} : \mathcal{K} \otimes \mathcal{V} \xrightarrow{\simeq} \mathcal{V}, \qquad r_{\mathcal{V}} : \mathcal{V} \otimes \mathcal{K} \xrightarrow{\simeq} \mathcal{V},$$

given by $l_{\mathcal{V}}(\lambda \otimes f) = \lambda f$ and $r_{\mathcal{V}}(f \otimes \lambda) = \lambda f$, for any k-cell f in $\mathcal{V}$ and $\lambda \in \mathbb{K}$.

If the n-vector spaces $\mathcal{V}$ and $\mathcal{W}$ are graded, we define their *graded tensor product*, also denoted by $\mathcal{V} \otimes \mathcal{W}$, by

$$(\mathcal{V} \otimes \mathcal{W})_0^{(i)} = \bigoplus_{i_1+i_2=i} \mathcal{V}_0^{(i_1)} \otimes \mathcal{W}_0^{(i_2)},$$

and, for $1 \leq k \leq n$, by

$$(\mathcal{V} \otimes \mathcal{W})_k^{(i)} = \mathcal{V}_k^{(i)} \otimes \mathcal{W}_k^{(i)}.$$

2.1.11. Higher-dimensional vector spaces and complexes. Note that Bourn shown that the category of chain complexes in an abelian category $\mathbb{V}$ is equivalent to the category of internal ∞-categories in $\mathbb{V}$, [12, Theorem 3.3]. When $\mathbb{V}$ is the category of graded vector spaces GrVect, the Bourn's correspondence can be stated as follows. There is an equivalence between the category $\mathsf{GrCh}_n(\mathbb{K})$ of positively graded chains complexes of length n and the category GrVect_n, which preserves quasi-isomorphisms and weak equivalences.

2.2. Higher-dimensional algebroids

2.2.1. Higher-dimensional algebroids. A (resp. graded) n-*algebroid* is a category enriched in (resp. graded) n-vector spaces, with the latter equipped with their (resp. graded) tensor product defined in 2.1.10. In details, a (resp. graded) n-algebroid $\mathcal{A}$ is specified by the following data:

– a set $\mathcal{A}_0$, whose elements are called the 0-*cells of* $\mathcal{A}$,

– for every 0-cells p and q, a (resp. graded) n-vector space $\mathcal{A}(p, q)$, the set of all k-cells of all the $\mathcal{A}(p, q)$ being called the $(k+1)$-*cells of* $\mathcal{A}$,

– for every 0-cells p, q and r, a morphism of (resp. graded) n-vector spaces

$$\mathcal{A}(p, q) \otimes \mathcal{A}(q, r) \longrightarrow \mathcal{A}(p, r)$$

called *the* 0-*composition of* $\mathcal{A}$ and whose image on (f, g) is denoted by $f \star_0 g$ or just fg, which is associative:

$$(f \star_0 g) \star_0 h = f \star_0 (g \star_0 h),$$

– for every 0-cell p, a specified 1-cell 1_p of $\mathcal{A}(p, p)$, called the *identity of* p, such that for any k-cell f in $\mathcal{A}(p, q)$:

$$1_p \star_0 f = f = f \star_0 1_q.$$

In the graded case, note that the morphism of n-vector spaces $\mathcal{A}(p, q) \otimes \mathcal{A}(q, r) \longrightarrow \mathcal{A}(p, r)$ looks differently for 0-cells and for k-cells when $1 \leq k \leq n$:

$$\mathcal{A}(p, q)_0^{(i_1)} \otimes \mathcal{A}(q, r)_0^{(i_2)} \longrightarrow \mathcal{A}(p, r)_0^{(i_1+i_2)}$$

$$\mathcal{A}(p, q)_k^{(i)} \otimes \mathcal{A}(q, r)_k^{(i)} \longrightarrow \mathcal{A}(p, r)_k^{(i)}$$

because of the two different formulas for the tensor product.

In particular, a graded 0-algebroid is a graded 1-category, a graded 1-algebroid is a category enriched in graded vector spaces and graded 1-algebroids with exactly one 0-cell coincide with graded associative algebras. The notion of an 1-algebroid, simply called algebroid if there is no possible confusion, corresponds to the notion of a $\mathbb{K}$-category studied by Mitchell in [26, Section 11.]. In the rest of the paper, we impose the following conditions on the graded n-algebroids:

$$\mathcal{A}(p, q)_k^{(0)} = \begin{cases} \mathbb{K} & \text{if } q = p, \\ \{0\} & \text{if } q \neq p, \end{cases} \qquad \text{for any } 0 \leq k \leq n.$$

For the case of an algebroid $\mathcal{A}$ with a single 0-cell, these conditions imply exactly that $\mathcal{A}$ is a connected associative algebra.

2.2.2. Category of n-algebroids. The enriched (resp. graded) functors corresponding to n-algebroids are called (resp. *graded*) *linear n-functors*. We denote by (resp. GrAlg_n) Alg_n the category of (resp. graded) n-algebroids and (resp. graded) linear n-functors.

2.2.3. n-algebroids and $(n,1)$-categories. A graded n-algebroid $\mathcal{A}$ inherits a structure of n-category

$$\mathcal{A}_0 \underset{t_0}{\overset{s_0}{\Longleftarrow}} \mathcal{A}_1 \underset{t_1}{\overset{s_1}{\Longleftarrow}} \mathcal{A}_2 \Longleftarrow (\cdots) \Longleftarrow \mathcal{A}_k \underset{t_k}{\overset{s_k}{\Longleftarrow}} \mathcal{A}_{k+1} \Longleftarrow (\cdots)$$

where $\mathcal{A}_k$ is the set of k-cells of $\mathcal{A}$. For every $k \geq 1$ and every 0-cells p and q of $\mathcal{A}$, the set $\mathcal{A}(p,q)_k$ of k-cells of $\mathcal{A}(p,q)$ is also equipped with a structure of graded space and the restriction of source and target maps to this space are graded linear. By Proposition 2.1.8, in a graded n-algebroid, for $2 \leq k \leq n$, every k-cell f is invertible, with its inverse defined by

$$f^- \;=\; -f + s_{k-1}(f) + t_{k-1}(f).$$

And for $2 \leq k \leq n$, for any composable endo-k-cell f and g, we have $f \star_{k-1} g = g \star_{k-1} f$.

Recall that an $(n,1)$-*category* $\mathcal{C}$ is an n-category whose k-cells are invertible for every $2 \leq k \leq n$. When $n < \infty$, this is a 1-category enriched in $(n-1)$-groupoids and, when $n = \infty$, a 1-category enriched in ∞-groupoids, see [20]. For $n \geq 2$, a $(n,1)$-category $\mathcal{C}$ is said to be *abelian* if for any composable endo-k-cells f and g in $\mathcal{C}$, where $2 \leq k \leq n$, the relation $f \star_{k-1} g = g \star_{k-1} f$ holds. Our previous observations prove that for $n \geq 2$, any n-algebroid has a structure of an abelian $(n,1)$-category whose underlying 1-category is an algebroid.

2.2.4. Distributivity. The structures of n-category and of vector space satisfy the following compatibility relations, whenever they have a meaning:

$$(\lambda f + \mu g) \star_0 (\lambda' f' + \mu' g') \;=\; \lambda\lambda'(f \star_0 f') + \lambda\mu'(f \star_0 g') + \mu\lambda'(g \star_0 f') + \mu\mu'(g \star_0 g'),$$

and for $1 \leq l \leq n$:

$$(\lambda f + \mu g) \star_l (\lambda f' + \mu g') \;=\; \lambda(f \star_l f') + \mu(g \star_l g'),$$

for any k-cells f, f', g, g', for $l \leq k \leq n$, and scalars $\lambda, \lambda', \mu, \mu'$ in $\mathbb{K}$. The first relation is given by the linearity of the 0-composition and the second relation corresponds to the exchange relation.

2.2.5. Spheres in higher-dimensional algebroid. Let $\mathcal{A}$ be an n-algebroid. A 0-*sphere of* $\mathcal{A}$ is a pair $\gamma = (f,g)$ of 0-cells of $\mathcal{A}$ and, for $1 \leq k \leq n$, a k-*sphere of* $\mathcal{A}$ is a pair $\gamma = (f,g)$ of parallel k-cells of $\mathcal{A}$, *i.e.*, with $s_{k-1}(f) = s_{k-1}(g)$ and $t_{k-1}(f) = t_{k-1}(g)$; we call f the *source* of γ and g its *target*. If f is a k-cell of $\mathcal{A}$, for $1 \leq k \leq n$, the *boundary of* f is the $(k-1)$-sphere $(s_{k-1}(f), t_{k-1}(f))$.

Let p and q be 0-cells in $\mathcal{A}$. For any $1 \leq k \leq n$, the k-spheres in $\mathcal{A}(p,q)$ form a space defined in the following natural way: for any k-sphere (f,g) and (f',g') in $\mathcal{A}(p,q)$ and scalar λ in $\mathbb{K}$, we have

$$(f,g) + (f',g') = (f+f', g+g'), \qquad \lambda(f,g) = (\lambda f, \lambda g).$$

For the remaining of the section 2.2, we suppose that n is finite.

2.2.6. Linear cellular extensions. Let $\mathcal{A}$ be an n-algebroid, with $n \geq 1$. A *linear cellular extension of* $\mathcal{A}$ is a pair (Γ, ∂) where $\Gamma = (\Gamma(p,q))_{p,q \in \mathcal{A}_0}$ is a family of spaces and $\partial \;=\; (\partial_{p,q})_{p,q \in \mathcal{A}_0}$ is a collection of maps, where each $\partial_{p,q}$ goes from $\Gamma(p,q)$ to the space of n-spheres of $\mathcal{A}(p,q)$. The image of an element $\gamma \in \Gamma$ is then a pair (f, g) of parallel n-cells, which can be intuitively thought as the source and the target of γ.

Given an n-algebroid $\mathcal{A}$ and a cellular extension Γ of $\mathcal{A}$, we define $\mathcal{A}[\Gamma]$ as the $(n+1)$-algebroid whose k-cells, for $0 \leq k \leq n$, are the ones of $\mathcal{A}$ and whose $(n+1)$-cells are all the linear combinations of formal compositions of elements of $\mathcal{A}$ with at least one element of Γ, seen as $(n+1)$-cells with source and target in $\mathcal{A}$, considered up to the exchange relations.

More explicitly, by the exchange relations between the different compositions and the linear structures of an $(n+1)$-algebroid, the $(n+1)$-cells of $\mathcal{A}[\Gamma]$ are equivalently defined as the formal n-composites of elements with shape

$$\lambda f + 1_u,$$

where f is an $(n+1)$-cell of the free $(n+1)$-category generated by $\mathcal{A}$ and Γ, u is an n-cell of $\mathcal{A}$ and λ is a scalar. An $(n+1)$-cell of the form $\lambda f + 1_u$ has source $\lambda s_n(f) + u$ and target $\lambda t_n(f) + u$. The n-composites of $(n+1)$-cells of the form $\lambda f + 1_u$ are considered up to the exchange relations:

$$(\lambda f + 1_{\mu s_n(g)}) \star_n (\mu g + 1_{\lambda t_n(f)}) \;=\; (\mu g + 1_{\lambda s_n(f)}) \star_n (\lambda f + 1_{\mu t_n(g)}),$$

for any cell f and g and scalar λ and μ.

2.2.7. Quotient algebroids. Given an n-algebroid $\mathcal{A}$ and a linear cellular extension Γ of $\mathcal{A}$, the *quotient of $\mathcal{A}$ by* Γ, denoted by $\mathcal{A}/\Gamma$, is the n-algebroid obtained by identifying in $\mathcal{A}$ the n-cells $s_n(\gamma)$ and $t_n(\gamma)$ for every n-sphere γ in the image of Γ by the maps ∂. Equivalently, it is the quotient of the n-algebroid $\mathcal{A}$ by the congruence relation generated by the $(n+1)$-cells of $\mathcal{A}[\Gamma]$.

2.2.8. Asphericity and homotopy bases. An n-algebroid $\mathcal{A}$ is *aspherical* when the source and the target of each n-sphere of $\mathcal{A}$ coincide, *i.e.*, when every n-sphere of $\mathcal{A}$ has shape (f, f) for some $(n-1)$-cell f of $\mathcal{A}$. A *homotopy basis* of $\mathcal{A}$ is a linear cellular extension Γ of $\mathcal{A}$ such that the n-algebroid $\mathcal{A}/\Gamma$ is aspherical. In other words, a linear cellular extension Γ of $\mathcal{A}$ is a homotopy basif if, for every n-sphere γ of $\mathcal{A}$, there exists an $(n+1)$-cell in $\mathcal{A}[\Gamma]$ with boundary γ.

2.3. Graded linear polygraphs

2.3.1. Linear polygraphs. *Linear n-polygraphs* and the *free n-algebroid* functor are defined by mutual induction as follows. A *linear 1-polygraph* is a 1-polygraph, that is a data Σ made of a set Σ_0 and a cellular extension Σ_1 of Σ_0. The *free algebroid over* Σ, denoted by Σ_1^ℓ, can be obtained as the algebroid $\mathbb{K}\Sigma_1^*$ spanned by the free 1-category Σ_1^*, that is, for any 0-cells p and q, $\mathbb{K}\Sigma_1^*(p, q)$ is the free vector space on $\Sigma_1^*(p, q)$. The maps s_0 and t_0 from Σ_1 to Σ_0 can be extended into maps from Σ_1^ℓ to Σ_0.

For $n \geq 1$, provided linear n-polygraphs and free n-algebroids have been defined, a *linear $(n+1)$ - polygraph* is a data $\Lambda = (\Lambda_n, \Lambda_{n+1})$ made of

i) a linear n-polygraph Λ_n,

ii) a linear cellular extension Λ_{n+1} of the free n-algebroid Λ_n^ℓ:

$$\Lambda_n^\ell = \Sigma_1^\ell[\Lambda_2] \cdots [\Lambda_n].$$

As in the set-theoretic case, we abusively use the same notation Λ_k for the collection of k-cells of a linear n-polygraph and for its underlying linear k-polygraph, so that a linear n-polygraph Λ is usually defined by its collections of k-cells in every dimension:

$$\Lambda \;=\; (\Sigma_0, \Sigma_1, \Lambda_2 \ldots, \Lambda_n).$$

The *free $(n+1)$-algebroid over* Λ is defined as $\Lambda_{n+1}^\ell = \Lambda_n^\ell[\Lambda_{n+1}]$. An element of Λ_k is called a k-*cell of* Λ and Λ is called *finite* when the space of k-cells if finite dimensional for all $A \leq k \leq n$.

2.3.2. Remark. A linear $(n+1)$-polygraph yields the following diagram

$$\begin{tikzcd}
\Sigma_0 & \Sigma_1^{\ell} \arrow[l, shift right, "s_0"'] \arrow[l, shift left, "t_0"] & \Lambda_2^{\ell} \arrow[l, shift right, "s_1"'] \arrow[l, shift left, "t_1"] & (\cdots) \arrow[l, shift right, "s_2"'] \arrow[l, shift left, "t_2"] & \Lambda_n^{\ell} \arrow[l, shift right, "s_{n-1}"'] \arrow[l, shift left, "t_{n-1}"] & \\
& \Sigma_1 \arrow[u, hook] \arrow[ul, shift left, "s_0"] \arrow[ul, shift right, "t_0"'] & \Lambda_2 \arrow[u, hook] \arrow[ul, shift left, "s_1"] \arrow[ul, shift right, "t_1"'] & (\cdots) \arrow[ul, shift left, "s_2"] \arrow[ul, shift right, "t_2"'] & \Lambda_n \arrow[u, hook] \arrow[ul, shift left, "s_{n-1}"] \arrow[ul, shift right, "t_{n-1}"'] & \Lambda_{n+1} \arrow[ul, shift left, "s_n"] \arrow[ul, shift right, "t_n"']
\end{tikzcd}$$

This diagram contains the source and target attachment maps of generating $(k+1)$-cells on composite k-cells, their extension to composite $(k+1)$-cells and the inclusion of generating k-cells into composite k-cells. The source and target maps $s_k, t_k : \Lambda_k \to \Lambda_{k-1}^{\ell}$ are uniquely extended to Λ_k^{ℓ} into maps, also denoted s_k and t_k.

2.3.3. Bases. Alternatively, we often fix bases of all the involved vector spaces and we specify such a Λ by listing all the basis elements dimension after dimension. We call a *basis* of a n-polygraph Λ a sequence $\Sigma = (\Sigma_1, \ldots, \Sigma_n)$ such that, for any $i \leq n$, Σ_i is a basis of Λ_i. In that case, Λ will be denoted $\mathbb{K}\Sigma$.

2.3.4. Asphericity and acyclicity. A linear n-polygraph Λ is *aspherical* when the free n-algebroid Λ^{ℓ} is aspherical. A linear n-polygraph Λ is *acyclic* when, for any $k \leq n-1$, the cellular extension Λ_{k+1} is an homotopy basis of the k-algebroid Λ_k^{ℓ}. That is, for any $k \leq n-1$, for every pair of parallel k-cells u and v in Λ^{ℓ}, there exists an $(k+1)$-cell from u to v.

2.3.5. Graded linear polygraphs. Let n be a natural number. *Graded* linear n-polygraphs are defined inductively in the same way as linear n-polygraphs. A *graded linear* 1*-polygraph* is a 1-polygraph Σ made of a set Σ_0 and a graded cellular extension Σ_1 of Σ_0, that is, Σ_1 is a family $(\Sigma_1^{(i)})_{i\geq 0}$. A 1-cell x in $\Sigma_1^{(i)}$ is said *homogeneous of degree* $|x| = i$.

The *free graded algebroid over* Σ is denoted by Σ_1^{ℓ}, is defined as follows. For any distinct 0-cells p and q,

$$\Sigma_1^{\ell}(p,q) = \mathbb{K}\Sigma_1(p,q) \oplus \left(\bigoplus_{\substack{n\geq 2 \\ p_i \in \Sigma_0}} \mathbb{K}\Sigma_1(p,p_1) \otimes \ldots \otimes \mathbb{K}\Sigma_1(p_{n-1},q) \right).$$

For a 0-cell p,

$$\Sigma_1^{\ell}(p,p) = \mathbb{K} \oplus \mathbb{K}\Sigma_1(p,p) \oplus \left(\bigoplus_{\substack{n\geq 2 \\ p_i \in \Sigma_0}} \mathbb{K}\Sigma_1(p,p_1) \otimes \ldots \otimes \mathbb{K}\Sigma_1(p_{n-1},p) \right).$$

The additional summand in the second case corresponds to the 1-dimensional space generated by the identity.

An element $u = x_1 \otimes \ldots \otimes x_n$ in $\mathbb{K}\Sigma_1(p,p_1) \otimes \ldots \otimes \mathbb{K}\Sigma_1(p_{n-1},q)$ has a *degree* $|u| = |x_1| + \ldots + |x_n|$ and a *weight* equals to n. The homogeneous component of $\Sigma_1^{\ell}(p,q)^{(i)}$ of degree i of $\Sigma_1^{\ell}(p,q)$ is

$$\Sigma_1^{\ell}(p,q)^{(i)} = \mathbb{K}\Sigma_1(p,q)^{(i)} \oplus \left(\bigoplus_{j\geq 2} \bigoplus_{\substack{i_1+\ldots+i_j=i \\ p_i \in \Sigma_0}} \mathbb{K}\Sigma_1(p,p_1)^{i_1} \otimes \ldots \otimes \mathbb{K}\Sigma_1(p_{j-1},q)^{i_j} \right).$$

For $n \geq 1$, provided that graded linear n-polygraphs and free graded n-algebroids have been defined, a *graded linear* $(n+1)$*-polygraph* is a data $\Lambda = (\Lambda_n, \Lambda_{n+1})$ made of

i) a graded linear n-polygraph $\Lambda_n = \big(\Sigma_0, (\Sigma_1^{(i)})_{i\geq 0}, \ldots, (\Lambda_n^{(i)})_{i\geq 0}\big)$,

ii) a linear cellular extension $\Lambda_{n+1} = (\Lambda_{n+1}^{(i)})_{i\geq 0}$ of the free graded n-algebroid Λ_n^ℓ:

$$\Lambda_n^\ell = \Sigma_1^\ell[\Lambda_2]\cdots[\Lambda_n].$$

The *free graded* $(n+1)$*-algebroid over* Λ is defined as $\Lambda_{n+1}^\ell = \Lambda_n^\ell[\Lambda_{n+1}]$.

Note that, as the source and target maps are graded, any k-sphere (f, g) is homogeneous, in the sense that the k-cells f and g have the same degree. It follows that the linear cellular extensions have a natural induced grading; an element (f, g) of Λ_k such that f and g are of degree i is a called a *homogeneous* k-cell of Λ of *degree* i.

When Σ_1 is concentrated in degree 1, then the notions of degree and weight coincide. Unless it is specified, we will suppose that the 1-cells in Σ_1 are concentrated in degree 1.

2.3.6. Homogeneous polygraphs. Let $\omega : \mathbb{N} \to \mathbb{N}$ be a (*degree*) function. We said that a graded linear n-polygraph Λ is ω*-concentrated* if for any $1 \leq k \leq n$, Λ_k is concentrated in degree $\omega(k)$, that is for any k-cell f, $|s_{k-1}(f)| = |t_{k-1}(f)| = \omega(k)$. We will use the degree function ℓ_N, where $N \geq 2$ is an integer, defined by

$$\ell_N(k) = \begin{cases} lN & \text{if } k = 2l, \\ lN + 1 & \text{if } k = 2l+1, \end{cases}$$

for any integer $k \geq 0$. A n-polygraph is said to be N*-homogeneous*, or N*-diagonal*, if it is ℓ_N-concentrated. In particular, an N-homogeneous linear 2-polygraph is a graded linear 2-polygraph, such that any 2-cell has the form

$$\sum_{j\in J} \lambda_j m_j \Rightarrow \sum_{i\in I} \lambda_i m_i,$$

where $|m_j| = |m_i| = N$, for any i and j. We will say *quadratic* (resp. *cubical*) for 2-homogeneous (resp. 3-homogeneous) 2-polygraph.

2.3.7. Presentations of algebroids. A *presentation* of an algebroid $\mathbf{A}$ is a linear 2-polygraph Λ such that $\mathbf{A}$ is isomorphic to the quotient algebroid Σ_1^ℓ/Λ_2. In the case where $\mathbf{A}$ has exactly one 0-cell, the notion coincides with the usual notion of presentation of $\mathbf{A}$, as the free algebra generated by the set Σ_1 quotiented by the space of relations Λ_2. We will denote by $\overline{\Lambda}$ the algebroid presented by a linear 2-polygraph Λ. We denote by $\overline{u}$ the image of a 1-cell u in Σ_1^ℓ by the canonical projection $\Sigma_1^\ell \to \overline{\Lambda}$. An algebroid is said to be N*-homogeneous* if it is presented by a N-homogeneous linear 2-polygraph Λ, that is, for any 2-cell f in Λ, we have $|s_1(f)| = |t_1(f)| = N$. The relations of $\mathbf{A}$ being N-homogeneous, the algebroid $\mathbf{A}$ is equipped with a degree grading. The usual homological notions related to the algebroid $\mathbf{A}$ can be equipped with this additional degree grading.

2.3.8. Tietze equivalence. Two linear n-polygraphs Λ and Δ are said to be *Tietze equivalent* if they present the same algebroid, that is, there is an isomorphism of algebroids $\overline{\Lambda} \simeq \overline{\Delta}$.

3. Two-dimensional linear rewriting systems

3.1. Linear 2-polygraph

3.1.1. Linear 2-polygraph. Recall from 2.3.1 that a *linear* 2*-polygraph* is a data made of a 1-polygraph (Σ_0, Σ_1) and a linear cellular extension Λ_2 of the free algebroid Σ_1^ℓ:

$$\Sigma_0 \; \underset{t_0}{\overset{s_0}{\leftleftarrows}} \; \Sigma_1^\ell \; \underset{t_1}{\overset{s_1}{\leftleftarrows}} \; \Lambda_2.$$

Suppose that Σ_1 is a finite set $\{x_1, \ldots, x_k\}$. The 1-cells in the free 1-category Σ_1^* are called *monomial* 1-cells in the variables $x_1, \ldots, x_k$. The 1-cells in the free algebroid Σ_1^ℓ are *polynomials* 1-cells in the variables $x_1, \ldots, x_k$. Any 1-cell f in Σ_1^ℓ can be written uniquely as a linear sum:

$$f = \sum_{i \in I} \lambda_i m_i,$$

where, for any $i \in I$, λ_i are non-zero scalars and the m_i are pairwise distinct non-zero monomial 1-cells. Such a decomposition is called a *reduced expression* of the 1-cell f with respect to the basis Σ_1.

3.1.2. Monic linear 2-polygraph. A linear 2-polygraph Λ is said to be *monic* if it has a basis $(\Sigma_0, \Sigma_1, \Sigma_2)$ such that any 2-cell in Σ_2 has a non-zero monomial source. That is, any 2-cell in Σ_2 has the form

$$\alpha : m \Rightarrow \sum_{i \in I} \lambda_i m_i,$$

where m and the m_i's, for any $i \in I$, are non-zero monomial 1-cells. Obviously, any linear 2-polygraph is Tietze equivalent to a monic linear 2-polygraph. Note that any 2-polygraph can be viewed as a monic linear 2-polygraph for which the target of any 2-cell is also monomial.

3.2. Rewriting properties of linear 2-polygraphs

In this section, Λ denotes a monic linear 2-polygraph with basis $(\Sigma_0, \Sigma_1, \Sigma_2)$.

The notion of rewriting step induced by Λ needs to be defined with attention owing to the invertibility of 2-cells in the free algebroid Λ_2^ℓ. Indeed, given a rule $\varphi : m \Rightarrow h$ in Λ_2, we have in the 2-algebroid Λ_2^ℓ the 2-cell $-\varphi : -m \Rightarrow -h$ hence the 2-cell $-\varphi + (m + h) : h \Rightarrow m$. It is useless to hope for termination if we consider all the 2-cells of Λ_2^ℓ as rewriting sequences. We define a rewriting step as the application of a rule on a reduced 1-cell, eg. $-m + (m+h)$ is not reduced, thus $-\varphi + (m+h)$ will not considered as a rewriting step.

3.2.1. Rewriting step. A *rewriting step* is a 2-cell in Λ_2^ℓ with the shape $\alpha = \lambda m_1 \varphi m_2 + g$:

$$\lambda\left(\begin{tikzcd} p \arrow[r, "m_1"] & q \arrow[r, bend left, "m"{name=U}] \arrow[r, bend right, "h"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\varphi"] & q' \arrow[r, "m_2"] & p' \end{tikzcd}\right) + \begin{tikzcd} p \arrow[r, "g"] & p' \end{tikzcd}$$

where λ is a non-zero scalar, m_1 and m_2 are non-zero monomial 1-cells in Σ_1^ℓ, $\varphi : m \Rightarrow h$ is a monic rule in Λ_2 and g a 1-cell in Σ_1^ℓ such that the monomial $m_1 m m_2$ does not appear in the basis decomposition of g. A rewriting step α from f to f' will be denoted by

$$\alpha : f \Rightarrow^+_\Lambda f',$$

or $\alpha : f \Rightarrow^+ f'$ if there is no ambiguity. There is such a rewriting step if f has a non-zero term $\lambda m_1 m m_2$, where $\lambda \in \mathbb{K} - \{0\}$, m_1, m_2 are monomial 1-cells in Σ_1^ℓ and there is a rule $m \Rightarrow h$ in Λ_2 such that

$$f' = f - \lambda m_1 (m - h) m_2.$$

The relation $\Rightarrow^+$ is called the *reduction relation* induced by Λ. A *rewriting sequence* of Λ is a finite or an infinite sequence

$$f_1 \Rightarrow^+ f_2 \Rightarrow^+ f_3 \Rightarrow^+ \cdots \Rightarrow^+ f_n \Rightarrow^+ \cdots$$

of rewriting steps. If there is a non-empty rewriting sequence from f to g, we say that f *rewrites* into g and we denote $f \Rrightarrow^*_\Lambda g$, or $f \Rrightarrow^* g$ if there is no confusion.

We denote by Λ_2^+ (resp. Λ_2^{+f}) the set of (resp. finite) rewriting sequences of Λ, also called *positive 2-cells* of the linear 2-polygraph Λ. Note that the free 2-groupoid on Λ_2^{+f} is the 2-algebroid Λ_2^ℓ.

We denote $f \Leftrightarrow^*_\Lambda g$, or $f \Leftrightarrow^* g$, when there exists a finite zigzag of rewriting steps between f and g, that is when there exist 1-cells $f_1, \ldots, f_p$ such that $f_i \Rrightarrow^+ f_{i+1}$ or $f_{i+1} \Rrightarrow^+ f_i$ for $1 \leq i < p$, with $f_1 = f$ and $f_p = g$.

3.2.2. Ideal generated by a linear 2-polygraph. We denote by $I(\Lambda)$ the two-sided ideal of the algebroid Σ_1^ℓ generated by the set

$$\{ m - h \mid m \Rightarrow h \in \Lambda_2 \}.$$

Given 1-cells f and f' in Σ_1^ℓ, there is a 2-cell $f \Rightarrow f'$ in Λ_2^ℓ if and only if $f - f' \in I(\Lambda)$. In particular, for a 1-cell f in Σ_1^ℓ, we have

$$f \Leftrightarrow^* 0 \quad \text{if and only if} \quad f \in I(\Lambda).$$

3.2.3. Normal forms. A 1-cell f of Σ_1^ℓ is *irreducible* when there is no rewriting step of Λ with source f. In particular a zero 1-cell is irreducible. A *normal form* of f is an irreducible 1-cell g such that f rewrites into g. A 1-cell in Σ_1^ℓ is *reducible* if it is not irreducible. We denote by $\mathrm{ir}(\Lambda)$ (resp. $\mathrm{ir}_m(\Lambda)$) the set of irreducible polynomial (resp. monomial) 1-cells for Λ. The set $\mathrm{ir}(\Lambda)$ forms a vector space ; the polygraph Λ being monic, it is generated by $\mathrm{ir}_m(\Lambda)$.

3.2.4. Termination. We say that Λ is *terminating* when it has no infinite rewriting sequence. In that case, every 1-cell in Σ_1^ℓ has at least one normal form. Moreover, Noetherian induction, also called well-founded induction, allows definitions and proofs of properties of 1-cells by induction on the number of rewriting steps reducing a 1-cell to a normal form.

When Λ is terminating, as a vector space, the algebroid Σ_1^ℓ has the following decomposition

$$\Sigma_1^\ell = \mathrm{ir}(\Lambda) + I(\Lambda).$$

This decomposition is proved by induction on monomial 1-cells. Let m be a monomial 1-cell in Σ_1^ℓ. If m is irreducible, then $m = m + 0$, else it can be written $m = m_1 m' m_2$, where m' is the source of a 2-cell $m' \Rightarrow h$ in Λ_2. The polynomial $f = m_1(m' - h)m_2$ is in $I(\Lambda)$ and we have $f = m - m_1 h m_2$. By induction, there is a decomposition $m_1 h m_2 = h_{\mathrm{ir}} + h_I$, with h_{ir} irreducible and h_I in $I(\Lambda)$. Then $m = h_{\mathrm{ir}} + (f + h_I)$. This proves the decomposition.

3.2.5. Methods to prove termination. One idea to prove the termination of a linear 2-polygraph is to associate a 2-polygraph whose termination can be proven using usual techniques on 2-polygraphs. Given a basis of Λ_2 by cells of the form

$$\alpha : m \Rightarrow \sum_{i \in I_m} \lambda_i m_i$$

where the λ_i are non-zero scalars, the associated 2-polygraph $\mathcal{T}(\Lambda)$ is defined by $\mathcal{T}(\Lambda) = (\Sigma_0, \Sigma_1, \mathcal{T}(\Lambda)_2)$ where $\mathcal{T}(\Lambda)_2$ is defined by

$$\bigcup_m \{m \Rightarrow m_i, i \in I_m\}.$$

We suppose moreover that any monomial m is the source of a finite number of 2-cells.

3.2.6. Proposition. *If $\mathcal{T}(\Lambda)$ is terminating, then Λ is terminating.*

Proof. To any rewriting sequence in Λ starting with a monomial m in Σ_1^ℓ, we associate a tree labelled by monomials in Σ_1^ℓ as follows:

- The root vertex of the tree is m.
- If a vertex v is at some point during the rewriting sequence, rewritten into a linear combination of monomials v_i's, then in the tree, the vertices labelled by v have outgoing edges to vertices labelled by v_i.

Note that in this tree, every edge corresponds to a rewriting step in $\mathcal{T}(\Lambda)$. Suppose now that Λ is not terminating, that is there exists an infinite rewriting sequence in Λ. Then the associated tree has an infinite number of vertices, and therefore has a monotonous path of infinite length starting at the root. This implies that $\mathcal{T}(\Lambda)$ is not terminating. □

Proposition 3.2.6 implies that the usual methods to prove termination in the usual context can be used to prove termination in the linear context.

We now study the notion of confluence of linear 2-polygraphs, to study what happens when a monomial 1-cell is the source of more than one rewriting step.

3.2.7. Branchings. A *branching of* Λ is a pair (α, β) of 2-cells of Λ_2^+ with a common source, as in the following diagram

$$\begin{tikzcd}[row sep=small] & g \\ f \arrow[ur, Rightarrow, bend left, "\alpha"] \arrow[dr, Rightarrow, bend right, "\beta"'] & \\ & g' \end{tikzcd}$$

The 1-cell f is the *source* of this branching and the pair (g, g') is its *target*. We do not distinguish the branchings (α, β) and (β, α).

A branching (α, β) is *local* when α and β are rewriting steps. Local branchings belong to one of the four following families:

– *aspherical* branchings have the following shape

$$\begin{tikzcd} f \arrow[r, Rightarrow, bend left, "\alpha"] \arrow[r, Rightarrow, bend right, "\alpha"'] & g \end{tikzcd}$$

with $\alpha : f \Rightarrow g$ a rewriting step of Λ,

– *Peiffer* branchings and *additive Peiffer* having respectively the following shapes

$$\begin{tikzcd}[row sep=small] & f'g \\ fg \arrow[ur, Rightarrow, bend left, "\alpha g"] \arrow[dr, Rightarrow, bend right, "f\beta"'] & \\ & fg' \end{tikzcd} \qquad \begin{tikzcd}[row sep=small] & f'+g \\ f+g \arrow[ur, Rightarrow, bend left, "\alpha+g"] \arrow[dr, Rightarrow, bend right, "f+\beta"'] & \\ & f+g' \end{tikzcd}$$

with $\alpha : f \Rightarrow f'$ and $\beta : g \Rightarrow g'$ rewriting steps of Λ,

– *overlapping* branchings are the remaining local branchings.

The local branchings are compared by the strict order $\prec$ generated by

$$(\alpha, \beta) \prec (\lambda m \alpha m' + g, \lambda m \beta m' + g)$$

for any local branching (α, β), where

1. λ is in $\mathbb{K} \setminus \{0\}$,
2. m and m' are monomial 1-cells such that $m\alpha m'$ exists (and, thus, so does $m\beta m'$),
3. g is in Σ_1^ℓ,
4. no monomial in the basis decomposition of g appears in the basis decomposition of $ms(\alpha)m'$,
5. and at least one of the two following conditions is satisfied
   (i) either m or m' is not an identity monomial.
   (ii) g is not zero.

An overlapping local branching that is minimal for the order $\prec$ is called a *critical branching*. If (α, β) is a critical branching, the difference $t_1(\alpha) - t_1(\beta)$ is called the S*-polynomial* of the critical branching (α, β). Note that a critical branching has a monomial source.

3.2.8. Confluence. A branching (α, β) is *confluent* when there exists a pair (α', β') of 2-cells of Λ_2^+ with the following shape:

$$\begin{tikzcd}
 & g \arrow[dr, Rightarrow, "\alpha'"] & \\
f \arrow[ur, Rightarrow, "\alpha"] \arrow[dr, Rightarrow, "\beta"'] & & f' \\
 & g' \arrow[ur, Rightarrow, "\beta'"'] &
\end{tikzcd}$$

When there exists such a pair of reduction sequences to a common 1-cell, we denote $g \Downarrow_\Lambda g'$, or $g \Downarrow g'$. We say that Λ is *confluent* when all of its branchings are confluent. In a confluent linear 2-polygraph, every 1-cell has at most one normal form. For a rewriting system in general, the confluence property is equivalent to the *Church-Rosser* property, that is

$$f \Leftrightarrow^* g \quad \text{implies} \quad f \Downarrow g.$$

For linear rewriting systems, this equivalence can be stated as follows:

3.2.9. Proposition. *A linear 2-polygraph Λ is confluent if and only if*

$$f \in I(\Lambda) \quad \textit{implies} \quad f \Rightarrow^* 0.$$

3.2.10. Local confluence. We say that Λ is *locally confluent* when all of its local branchings are confluent. The *critical pairs lemma* holds for linear rewriting systems:

3.2.11. Proposition. *A linear 2-polygraph is locally confluent if and only if all its critical branchings are confluent.*

The proof can be adapted from the same result for 2-polygraphs in [21, 3.1.5.] with the consideration of additive Peiffer branchings.

The fundamental Newman's Lemma [28, Theorem 3] can be stated as follows for linear 2-polygraphs.

3.2.12. Proposition. *For terminating linear 2-polygraphs, local confluence and confluence are equivalent properties.*

Thus, by Proposition 3.2.11, for a terminating linear 2-polygraph, the confluence can be proved by checking the confluence of each critical branching.

3.2.13. Convergence. A linear 2-polygraph is said to be *convergent* when it terminates and is confluent. In that case, every 1-cell f has a unique normal form, denoted $\widehat{f}$. Such a Λ is called a *convergent presentation* of the algebroid $\overline{\Lambda}$ presented by Λ. In that case, there is a canonical section $\overline{\Lambda} \to \Sigma_1^\ell$ sending f to its normal form $\widehat{f}$, so that $\widehat{f} = \widehat{g}$ holds in Σ_1^ℓ if, and only if, we have $\overline{f} = \overline{g}$ in $\overline{\Lambda}$. As a consequence, a finite and convergent linear 2-polygraph Λ yields generators for the 1-cells of the 1-algebroid $\overline{\Lambda}$, together with a decision procedure for the corresponding word problem. The finiteness is used to effectively check that a given 1-cell is a normal form.

We end this section with a criterion to prove confluence of terminating polygraphs, like the Buchberger criterion for Gröbner bases.

3.2.14. Proposition. *Let Λ be a terminating linear 2-polygraph.*

i) *For any 1-cells g and g', if $g - g' \Rightarrow^* 0$, then $g \Downarrow g'$.*

ii) *Λ is confluent if and only if the S-polynomial of every critical branching is reduced to 0.*

Proof. Let Λ be terminating. The proof of **i)** is made by Noetherian induction, as in [4, Lemma 8.3.3] or [23, Lemma 2.2.].

Prove **ii)**. Suppose that Λ is confluent, then any critical branching (α, β) is confluent. Thus there exists reductions $\alpha' : t_1(\alpha) \Rightarrow^* f'$ and $\beta' : t_1(\beta) \Rightarrow^* f'$, hence the S-polynomial $t_1(\alpha) - t_1(\beta)$ is reduced to 0. Conversely, suppose that the S-polynomial of every critical branching is reduced to 0. By Propositions 3.2.12 and 3.2.11, it suffices to prove that every critical branching in Λ is confluent. Let (α, β) be a critical branching of source f, with $t_1(\alpha) = g$ and $t_1(\beta) = g'$. We have $g - g' \Rightarrow^* 0$, and we conclude by **i)**. $\square$

3.3. The basis of irreducibles

In this section, Λ denotes a monic linear 2-polygraph $(\Sigma_0, \Sigma_1, \Lambda_2)$.

3.3.1. Bases of irreducibles. The decomposition $\Sigma_1^\ell = \mathrm{ir}(\Lambda) + I(\Lambda)$ obtained in 3.2.4 is not direct in general. Suppose that Λ is convergent and consider the projection

$$\pi : \Sigma_1^\ell \twoheadrightarrow \mathrm{ir}(\Lambda)$$

sending a polynomial f on its unique normal form. By Proposition 3.2.9, we have $\pi(f) = 0$ if and only if f is in $I(\Lambda)$. Thus we have a family of exact sequences of vector spaces

$$0 \longrightarrow I(\Lambda)_{(p,q)} \hookrightarrow \Sigma_1{}^\ell_{(p,q)} \xrightarrow{\pi_{(p,q)}} \mathrm{ir}(\Lambda)_{(p,q)} \longrightarrow 0$$

indexed by 0-cells p, q in $\mathcal{A}_0$. Thus the maps $\pi_{(p,q)}$ induce an isomorphism of vector spaces from the algebroid $\overline{\Lambda}$ to $\mathrm{ir}(\Lambda)$. In this situation, we call the map π a *linear isomorphism*, that is a map of algebroids which is an isomorphism of vector spaces.

As a consequence, we have

3.3.2. Proposition. *Let Λ be a terminating linear 2-polygraph. Then Λ is confluent if and only if the decomposition is direct:*

$$\Sigma_1^\ell = ir(\Lambda) \oplus I(\Lambda).$$

Proof. Suppose Λ terminating, by 3.2.4, we have the decomposition $\Sigma_1^\ell = \mathrm{ir}(\Lambda) + I(\Lambda)$. By Proposition 3.2.9, the polygraph Λ is confluent if and only if, for any $f \in I(\Lambda)$, $f \Rightarrow^* 0$. It follows that Λ is confluent if and only if $\mathrm{ir}(\Lambda) \cap I(\Lambda) = \{0\}$, hence the decomposition is direct. $\square$

3.3.3. Standard bases. As a consequence, when Λ is convergent, the set of irreducible monomials $\mathrm{ir}_m(\Lambda)$ forms a $\mathbb{K}$-linear basis of the algebroid $\overline{\Lambda}$ via the canonical map $\mathrm{ir}(\Lambda) \longrightarrow \overline{\Lambda}$, called a *standard basis* of **A**. Moreover, with the multiplication on $\mathrm{ir}(\Lambda)$ defined by

$$f \cdot g = \pi(fg),$$

for any f and g in $\mathrm{ir}(\Lambda)$, then $\mathrm{ir}(\Lambda)$ is isomorphic to the algebroid $\overline{\Lambda}$.

3.3.4. Example. Consider the algebra $\mathbf{A}\langle\ x, y \mid xy = x^2\ \rangle$. The presentation by the polygraph Λ defined by the 2-cell $xy \Rightarrow x^2$ is confluent, because there is no critical branching. Hence, the set of normal forms $\mathrm{ir}_m(\Lambda) = \{y^i x^j \mid i, j \in \mathbb{N}\}$ forms a basis of algebra **A**. However, the polygraph Λ' with the 2-cell $x^2 \Rightarrow xy$ is a non convergent presentation of **A** ; there is a non-confluent critical branching:

$$\begin{tikzcd}[row sep=small] & xyx & \\ x^3 \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \\ & x^2y \arrow[r, Rightarrow] & xy^2 \end{tikzcd}$$

The monomials xyx and xy^2 in $\mathrm{ir}_m(\Lambda')$ are equal in **A**, thus are not linearly independant.

3.3.5. Monomial algebras. We associate to a monic linear 2-polygraph Λ the linear 2-polygraph $\mathcal{M}(\Lambda) = (\Sigma_0, \Sigma_1, \mathcal{M}(\Lambda)_2)$, whose 2-cells are defined by

$$\mathcal{M}(\Lambda)_2 = \{\ s_1(\alpha) \Rightarrow 0 \mid \alpha \in \Lambda_2\ \}.$$

Following 3.3.2, if Λ is convergent, there is a linear isomorphism $\overline{\Lambda} \simeq \overline{\mathcal{M}(\Lambda)}$. A linear basis of the algebroid $\overline{\mathcal{M}(\Lambda)}$ is given by the monomial 1-cells in Σ_1^ℓ not reducible by a 2-cell in Λ_2.

3.3.6. Poincaré-Birkhoff-Witt bases. Let **A** be an N-homogeneous algebroid and let $\Lambda = (\Sigma_0, \Sigma_1, \Lambda_2)$ be a monic N-homogeneous presentation of **A**. A set Ξ_1 of 1-cells in Σ_1^* is called a *Poincaré-Birkhoff-Witt basis*, PBW for short, of **A** if the three following conditions are satisfied:

i) For all 0-cells p and q, there is an isomorphism of vector spaces $\mathbb{K}\Xi_1(p, q) \simeq \mathbf{A}(p, q)$.

ii) For 0-composable 1-cells u and v in Ξ, the 0-composition uv is either in Ξ_1 or reducible by Λ_2.

iii) For any natural number p and any 0-composable 1-cells $v_1, \ldots, v_p$, the 0-composition $v_1 \ldots v_p$ is in Ξ_1 if, and only if, for all $1 \leq k \leq p - N + 1$, the 0-composition $v_k \ldots v_{k+N-1}$ is in Ξ_1.

When Λ is convergent, the associated standard basis is a PBW basis.

3.3.7. Proposition. *If Λ is N-homogeneous and convergent, then the standard basis $\mathrm{ir}_m(\Lambda)$ is a PBW basis of the algebroid* **A**.

Conversely, suppose that an N-homogeneous algebroid **A** presented by $(\Sigma_0, \Sigma_1, \Lambda_2)$ admits a PBW basis Ξ_1. For any 0-composable 1-cells u and v in Ξ_1, we denote by $[uv]_{\Xi_1}$ the linear decomposition of the 1-cell uv in the basis Ξ_1. Let us define

$$\Xi_2 = \{\ uv \Rightarrow [uv]_{\Xi_1} \mid u, v \text{ in } \Xi_1 \text{ and } |uv| = N\ \}.$$

Let Ξ be the 2-linear polygraph Ξ defined by (Σ_0, Ξ_1, Ξ_2).

3.3.8. Proposition. *If Ξ terminates, the 2-linear polygraph Ξ is a convergent presentation of* **A**.

Proof. First, note that any monomial 1-cell of degree N can be expressed as the 0-composite of two monomials of degree smaller than N, thus which are in Ξ_1. This observation implies that all relations in Λ_2 are in Ξ_2. Thus the algebroid presented by Ξ surjects (as an algebroid) to **A** . Moreover, by definition, Ξ and Λ have the same irreducible monomials. As Ξ terminates, these monomials form the standard basis of the algebroid presented by Ξ. This implies that Ξ is a presentation of **A**.

Let us show now that Ξ is convergent. It is enough to prove that for any monomials u and v in Σ_1^ℓ such that $\overline{u} = \overline{v}$ in A, u and v can be rewritten using Ξ to the same w in $\mathbb{K}\Xi_1$. Suppose that u is not in Ξ_1. Then by the third property of a PBW basis, we can find a subword u' of u of degree N that is not in Ξ_1. This subword can be expressed as the 0-composite of two monomials of degree smaller than N, thus which are in Ξ_1. Thus, by the second property of a PBW basis, u' is reducible by Ξ_2. Therefore $u \Rightarrow_\Xi \sum \lambda_i u_i$ such that $\overline{u} = \sum \lambda_i \overline{u}_i$. If some u_i is not in Ξ_1, we can iterate this process until we obtain $u \Rightarrow^*_\Xi w_u$ where w_u is in $\mathbb{K}\Xi_1$ and $\overline{u} = \overline{w}_u$. Note that this process is finite as Ξ terminates. Similarly, we obtain that $v \Rightarrow^*_\Xi w_v$ where w_v is in $\mathbb{K}\Xi_1$ and $\overline{v} = \overline{w}_v$. But as $\overline{v} = \overline{u}$, we obtain $\overline{w}_u = \overline{w}_v$, which implies $w_u = w_v$ as Ξ_1 is a basis of **A**. □

3.4. Associative Gröbner bases

In this section, we show how the Gröbner bases correspond to convergent linear 2-polygraphs. Throughout this section, (Σ_0, Σ_1) denotes a 1-polygraph.

3.4.1. Monomial order. Let us fix a *monomial order* $\prec$ on Σ_1^*, that is a well-order compatible with the associative product. Explicitly, it is a strict total order $\prec$ on Σ_1^* such that there is no infinite decreasing sequence and $m_1 \prec m_2$ implies $mm_1n \prec mm_2n$, for any monomials m and n in Σ_1^*. Any non-zero 1-cell f in Σ_1^ℓ can be uniquely written

$$f = \lambda_1 m_1 + \ldots + \lambda_p m_p,$$

where $m_1, \ldots, m_p$ are pairwise distinct monomials 1-cells and $\lambda_i \in \mathbb{K} - \{0\}$, for any $i \in I = \{1, \ldots, p\}$. The *leading term* of f, denoted $\mathrm{lt}(f)$, is the polynomial $\lambda_j m_j$, such that $m_i \prec m_j$, for any $i \in I - \{j\}$. Then we say that λ_j is the *leading coefficient of* f, denoted $\mathrm{lc}(f)$ and m_j is the *leading monomial of* f, denoted $\mathrm{lm}(f)$. We also define $\mathrm{lt}(0) = \mathrm{lc}(0) = \mathrm{lm}(0) = 0$.

3.4.2. Well-ordered linear 2-polygraphs. A monomial order $\prec$ on Σ_1^* induces a partial order on the free algebroid Σ_1^ℓ, also denoted by $\prec$, defined by

- **i)** for any non-zero 1-cell f, $0 \prec f$,
- **ii)** for non-zero 1-cells f and g, define $g \prec f$ if, and only if, either $\mathrm{lm}(g) \prec \mathrm{lm}(f)$ or ($\mathrm{lm}(g) = \mathrm{lm}(f)$ and $g - \mathrm{lt}(g) \prec f - \mathrm{lt}(f)$).

In this way, the partial order $\prec$ on Σ_1^ℓ is well-founded and compatible with associative product.

A linear cellular extension Λ_2 of Σ_1^ℓ is said to be *compatible with the monomial order* $\prec$, if for any 2-cell $m \Rightarrow u$ in Λ_2, we have $u \prec m$. A *well-ordered* linear 2-polygraph is a linear 2-polygraph $(\Sigma_0, \Sigma_1, \Lambda_2)$ together with a monomial order on Σ_1^* and whose cellular extension Λ_2 is compatible with this monomial order. Note that a well-ordered linear 2-polygraph is always terminating.

3.4.3. Polynomial 2-linear polygraphs. Let $\prec$ be a monomial order on Σ_1^*. Given a non-zero 1-cell g in Λ_1^ℓ. The *polynomial reduction by* g is the monic 2-cell defined by

$$\alpha_g : \mathrm{lm}(g) \Rightarrow \mathrm{lm}(g) - \frac{1}{\mathrm{lc}(g)} g.$$

For a set of non-zero 1-cells G in Σ_1^ℓ, we denote by $\Lambda(G)$ the linear 2-polygraph whose generating 2-cells are the 2-cells α_g, for g in G. Note that the reduction in $\Lambda(G)$ is by definition compatible with the monomial order $\prec$, hence the linear polygraph $\Lambda(G)$ is terminating.

3.4.4. Gröbner bases. Given a monomial order on Σ_1^* and I an ideal in Σ_1^ℓ, a *Gröbner basis* of I is a subset G of I such that, for any 1-cell f in I, there exists g in G such that $\mathrm{lt}(f) = m\mathrm{lt}(g)m'$, where m and m' are non-zero monomial 1-cells. That is, the two-sided ideal generated by the leading terms of 1-cells in I coincide with the two-sided ideal generated by the leading terms of elements in G:

$$\langle\ \mathrm{lt}(I)\ \rangle = \langle\ \mathrm{lt}(G)\ \rangle.$$

3.4.5. S-polynomials. Suppose that g_1 and g_2 are two non-zero 1-cells in Σ_1^ℓ such that their leading monomials have a small common multiple h, *i.e.* such that there are monomial 1-cells m and m' such that $h = \mathrm{lm}(g_1)m = m'\mathrm{lm}(g_2)$. We define the *S-polynomial* of the polynomial 1-cells g_1 and g_2 as the S-polynomial of the critical branching $(\alpha_{g_1}, \alpha_{g_2})$, that is

$$S(g_1, g_2) = \frac{1}{\mathrm{lc}(g_1)} g_1 m - \frac{1}{\mathrm{lc}(g_2)} m' g_2.$$

By construction, we have $\mathrm{lm}(S(g_1, g_2)) < \mathrm{lm}(h) = h$.

We recall here the Buchberger criterion for Gröbner bases.

3.4.6. Proposition. *Let Σ_1 be a 1-polygraph and let $\prec$ be a monomial order on Σ_1^*. Let $G = \{g_1, \ldots, g_k\}$ be a finite set of polynomials and let $I(G)$ be the ideal of Σ_1^ℓ generated by G. The following conditions are equivalents:*

i) *G is a Gröbner basis of $I(G)$,*

ii) *for any g_1 and g_2 in G, $S(g_1, g_2) \Rightarrow^*_{\Lambda(G)} 0$,*

iii) *the linear 2-polygraph $\Lambda(G)$ is confluent.*

Proof. The equivalence **i) <=>ii)** is standard, see for instance [19, Theorem 2.3]. Note that the critical pairs of $\Lambda(G)$ are of the form $(\alpha_{g_1}, \alpha_{g_2})$ such that the monomials $\mathrm{lm}(g_1)$ and $\mathrm{lm}(g_2)$ overlap as in 3.4.5. By **ii)** of Proposition 3.2.14, the polygraph $\Lambda(G)$ is confluent if and only if the S-polynomial of each such a critical pair is reduced to 0. This proves the equivalence between **ii)** and **iii)**. □

3.4.7. Example. Consider the algebra given by the following presentation

$$\mathbf{A}\langle\ x, y, z \ \mid\ xyz = x^3 + y^3 + z^3\ \rangle.$$

With the lexicographic order induced by $x < y < z$, the ideal generated by the relation admits the Gröbner basis with two elements corresponding to the following two rules:

$$z^3 \overset{\alpha}{\Longrightarrow} xyz - x^3 - y^3, \qquad zy^3 \overset{\beta}{\Longrightarrow} zxyz - zx^3 - xyz^2 + x^3z + y^3z.$$

With such a presentation, Anick's resolution is infinite ; the Anick's chains are of the form z^n and z^ny^3, with $n \geq 0$. It is possible to consider another presentation of the algebra $\mathbf{A}$, with the rule

$$xyz \overset{\gamma}{\Longrightarrow} x^3 + y^3 + z^3.$$

This presentation is confluent, as it has no critical branching. Moreover, it is terminating, as for each monomial, the quantity $3A + B$ decreases when we apply the rewriting rule, where A is the number of occurrences of the product xyz and B the number of occurences of y.

4. POLYGRAPHIC RESOLUTIONS OF ALGEBROIDS

In this section, we introduce the notion of polygraphic resolution for algebroids. We show how to construct such a resolution from a convergent presentation. Our construction consists in extending a reduced monic linear 2-polygraph Λ into a polygraphic resolution of the presented algebroid, whose generating n-cells, for $n \geq 3$, are indexed by the $(n-1)$-fold critical branchings of Λ.

Throughout this section, **A** denotes an algebroid.

4.1. Polygraphic resolutions of algebroids and normalisation strategies

4.1.1. Polygraphic resolutions of algebroids. A *polygraphic resolution (resp. graded polygraphic resolution)* of **A** is an acyclic linear ∞-polygraph (resp. graded acyclic linear ∞-polygraph) Λ, whose underlying linear 2-polygraph Λ_2 is a presentation of **A**. For a natural integer n, such that $2 \leq n < \infty$, a *partial polygraphic resolution of length* n of **A** is an acyclic linear n-polygraph Λ, whose underlying linear 2-polygraph is a presentation of **A**.

When f is a k-cell of Λ, for $k \geq 2$, we will denote by $\overline{f}$ the 1-cell $\overline{s_1(f)} = \overline{t_1(f)}$ in **A**.

4.1.2. N-homogeneous resolution. Given a degree function $\omega : \mathbb{N} \to \mathbb{N}$, a graded polygraphic resolution Λ is ω*-concentrated* (resp. N*-homogeneous*) when the linear polygraph Λ is ω-concentrated (resp. ℓ_N - concentrated).

4.1.3. Sections. Let us fix $n \geq 2$ either a natural number or ∞. Let $\Lambda = (\Sigma_0, \Sigma_1, \Lambda_2, \ldots, \Lambda_n)$ be a linear n-polygraph, whose underlying linear 2-polygraph is a presentation of **A**.

A *section of* Λ is a choice of a representative 1-cell $\widehat{u} : p \to q$ in Σ_1^ℓ, for every 1-cell $u : p \to q$ of **A**, such that $\widehat{1_p} = 1_p$ holds for every 0-cell p of **A**. Such an assignment is not assumed to be functorial with respect to the 0-composition, but linear, that is for any 1-cells u and v and scalar λ in $\mathbb{K}$, the section satisfies:

$$\widehat{u+v} = \widehat{u} + \widehat{v}, \qquad \widehat{\lambda u} = \lambda \widehat{u}.$$

The assignment $\widehat{\cdot} : u \mapsto \widehat{u}$ is extended in a unique way by precomposition with the canonical projection $\Sigma_1^\ell \twoheadrightarrow \mathbf{A}$, into a map

$$\widehat{\cdot} : \Sigma_1^\ell \longrightarrow \Sigma_1^\ell$$

mapping each 1-cell u in Σ_1^ℓ to a parallel 1-cell $\widehat{u}$ in Σ_1^ℓ, in such a way that the equality $\overline{u} = \overline{v}$ holds in **A** if, and only if, we have $\widehat{u} = \widehat{v}$ in Σ_1^ℓ.

4.1.4. Normalisation strategies. A *normalisation strategy* for the linear n-polygraph Λ is a mapping σ of every k-cell f of Λ_k^ℓ, with $1 \leq k < n$, to a $(k+1)$-cell

$$f \xrightarrow{\sigma_f} \widehat{f}$$

in Λ_{k+1}^ℓ, where, for $k \geq 2$, the notation $\widehat{f}$ stands for the k-cell $\sigma_{s_{k-1}(f)} \star_{k-1} \sigma^-_{t_{k-1}(f)}$, such that the following properties are satisfied, for $1 \leq k < n$,

i) for every k-cell f, $\sigma_{\widehat{f}} = 1_{\widehat{f}}$,

ii) for every k-cells f and g, $\sigma_{f+g} = \sigma_f + \sigma_g$:

$$\begin{tikzcd} u+u' \arrow[rr, Rightarrow, "f+g"] \arrow[dr, Rightarrow, "\sigma_{u+u'}"'] \arrow[rr, phantom, shift right=4ex, "\Downarrow \sigma_{f+g}"] & & v+v' \\ & \widehat{u+u'} \arrow[ur, Rightarrow, "\sigma^-_{v+v'}"'] & \end{tikzcd} = \begin{tikzcd} u+u' \arrow[rr, Rightarrow, "f+g"] \arrow[dr, Rightarrow, "\sigma_u + \sigma_{u'}"'] \arrow[rr, phantom, shift right=4ex, "\Downarrow \sigma_f + \sigma_g"] & & v+v' \\ & \widehat{u}+\widehat{u}' \arrow[ur, Rightarrow, "\sigma^-_v + \sigma^-_{v'}"'] & \end{tikzcd}$$

iii) for every k-cell f and any λ in $\mathbb{K}$, $\sigma_{\lambda f} = \lambda\sigma_f$:

$$
\begin{tikzcd}
\lambda u \arrow[rr, Rightarrow, "\lambda f"] \arrow[dr, Rightarrow, "\sigma_{\lambda u}"'] \arrow[rr, phantom, shift right=4ex, "\Downarrow\sigma_{\lambda f}"] & & \lambda v \\
& \widehat{\lambda u} \arrow[ur, Rightarrow, "\sigma^-_{\widehat{\lambda v}}"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\lambda u \arrow[rr, Rightarrow, "\lambda f"] \arrow[dr, Rightarrow, "\lambda\sigma_u"'] \arrow[rr, phantom, shift right=4ex, "\Downarrow\lambda\sigma_f"] & & \lambda v \\
& \lambda\widehat{u} \arrow[ur, Rightarrow, "\lambda\sigma^-_v"'] &
\end{tikzcd}
$$

A linear n-polygraph is *normalising* when it admits a normalisation strategy. This property is independent of the chosen section, see [20, 3.2.2.].

4.1.5. Lemma. *Let Λ be a linear n-polygraph with a chosen section $\widehat{\cdot}$ and let σ be a normalisation strategy for Λ.*

i) *For every k-cell f, with $0 \le k < n-1$, we have $\sigma_{1_f} = 1_{1_f}$:*

$$
\begin{tikzcd}
f \arrow[rr, Rightarrow, "1_f"] \arrow[dr, Rightarrow, "\sigma_f"'] \arrow[rr, phantom, shift right=4ex, "\Downarrow\sigma_{1_f}"] & & f \\
& \widehat{f} \arrow[ur, Rightarrow, "\sigma^-_f"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
f \arrow[r, Rightarrow, bend left, "1_f"] \arrow[r, Rightarrow, bend right, "1_f"'] \arrow[r, phantom, "\Downarrow 1_{1_f}"] & f
\end{tikzcd}
$$

ii) *For every k-cell f, with $1 \le k < n-1$, we have $\sigma_{\sigma_f} = 1_{\sigma_f}$:*

$$
\begin{tikzcd}
f \arrow[rr, Rightarrow, "\sigma_f"] \arrow[dr, Rightarrow, "\sigma_f"'] \arrow[rr, phantom, shift right=4ex, "\Downarrow\sigma_{\sigma_f}"] & & \widehat{f} \\
& \widehat{f} \arrow[ur, Rightarrow, "1_{\widehat{f}}"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
f \arrow[r, Rightarrow, bend left, "\sigma_f"] \arrow[r, Rightarrow, bend right, "\sigma_f"'] \arrow[r, phantom, "\Downarrow 1_{\sigma_f}"] & \widehat{f}
\end{tikzcd}
$$

iii) *For every k-cell $f : u \Rightarrow v$, with $1 < k < n$, we have $\sigma_{f^-} = f^- \star_{k-1} \sigma^-_f \star_{k-1} (\widehat{f})^-$:*

$$
\begin{tikzcd}
v \arrow[rr, Rightarrow, "f^-"] \arrow[dr, Rightarrow, "\sigma_v"'] \arrow[rr, phantom, shift right=4ex, "\Downarrow\sigma_{f^-}"] & & u \\
& \widehat{u} \arrow[ur, Rightarrow, "\sigma^-_u"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& & \widehat{u} \arrow[dr, Rightarrow, "\sigma^-_v"] \arrow[d, phantom, "\Downarrow\sigma^-_f"] & & & \\
v \arrow[r, Rightarrow, "f^-"] & u \arrow[ur, Rightarrow, "\sigma_u"] \arrow[rr, Rightarrow, "f"'] & {} & v \arrow[r, Rightarrow, "\sigma_v"] & \widehat{u} \arrow[r, Rightarrow, "\sigma^-_u"] & u
\end{tikzcd}
$$

iv) *for every pair (f, g) of l-composable k-cells, with $1 \le l < k < n$, $\sigma_{f \star_l g} = \sigma_f \star_l \sigma_g$:*

$$
\begin{tikzcd}
u \arrow[rr, Rightarrow, "f \star_l g"] \arrow[dr, Rightarrow, "\sigma_u"'] \arrow[rr, phantom, shift right=4ex, "\Downarrow\sigma_{f \star_l g}"] & & w \\
& \widehat{u} \arrow[ur, Rightarrow, "\sigma^-_w"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
u \arrow[rr, Rightarrow, "f"] \arrow[dr, Rightarrow, "\sigma_u"'] \arrow[rr, phantom, shift right=4ex, "\sigma_f\Downarrow"] & & v \arrow[rr, Rightarrow, "g"] \arrow[dr, Rightarrow, "\sigma_v"] \arrow[d, phantom, "\copyright"] \arrow[rr, phantom, shift right=4ex, "\Downarrow\sigma_g"] & & w \\
& \widehat{u} \arrow[ur, Rightarrow, "\sigma^-_v"] \arrow[rr, equal] & {} & \widehat{u} \arrow[ur, Rightarrow, "\sigma^-_w"'] &
\end{tikzcd}
$$

Proof. For **i) - iii)**, the proof is the same as in the case of polygraphs, see [20, Lemma 3.2.3.]. For **iv)**, by Proposition 2.1.6 **i)**, for any $1 \le l \le k-1$, we have $f \star_l g = f + g - s_l(g)$, hence

$$\sigma_{f \star_l g} = \sigma_f + \sigma_g - \sigma_{1_{s_l(g)}}.$$

As $\sigma_{1_{s_l(g)}} = \sigma_{1_{s_l(\sigma_g)}} = 1_{1_{s_l(g)}}$, it follows that $\sigma_{f \star_l g} = \sigma_f \star_l \sigma_g$. $\square$

4.1.6. The star notation. Given a normalisation strategy for Λ, we define, for every 1-cell f of Σ_1^ℓ, the 1-cell f^* as f. And, by induction on the dimension, for every k-cell f in Λ_k^ℓ, with $2 \le k \le n$, the k-cell f^* in Λ^ℓ is given by

$$f^* \;=\; ((f \star_1 \sigma_{t_1(f)^*}) \star_2 \cdots) \star_{k-1} \sigma_{t_{k-1}(f)^*}\ .$$

For example, for a 2-cell $f : u \Rightarrow v$, the 2-cell f^* is $f \star_1 \sigma_v$:

$$u \overset{f}{\Longrightarrow} v \overset{\sigma_v}{\Longrightarrow} \widehat{u}$$

and, for a 3-cell $A : f \Rrightarrow g : u \Rightarrow v$, the 3-cell A^* is $(A \star_1 \sigma_v) \star_2 \sigma_{g^*}$:

$$\begin{tikzcd}
 & v \arrow[dr, Rightarrow, "\sigma_v"] \arrow[d, phantom, "\Downarrow\sigma_{g^*}"] & \\
u \arrow[ur, Rightarrow, bend left=60, "f"] \arrow[ur, Rightarrow, "g"'] \arrow[rr, Rightarrow, "\sigma_u"'] & {} & \widehat{u}\,.
\end{tikzcd}$$

For any k-cell f, with $k \ge 2$, the k-cell f^* has source $s_{k-1}(f)^*$ and target $\widehat{t_{k-1}(f)}^{\,*}$. Moreover, we have $(\widehat{f})^* = \widehat{f^*}$, which implies $\sigma_{f^*} = \sigma_f^*$.

Since every k-cell of Λ_k^ℓ is invertible for $k \ge 2$, one can recover σ from σ^*, in a unique way, so that the normalisation strategy σ is uniquely and entirely determined by the values

$$\sigma_m^* \;=\; \sigma_m \;:\; m \;\Rightarrow\; \widehat{m}$$

for every monomial 1-cell m with $m \neq \widehat{m}$ and

$$\sigma^*_{m_1\varphi m_2} \;:\; (m_1\varphi m_2)^* \;\to\; \widehat{m_1\varphi m_2}^{\,*}$$

for every k-cell φ in Λ_k, with $1 < k < n$, and every monomial 1-cells m_1 and m_2 in Σ_1^ℓ 0-composable with φ.

4.1.7. Right normalisation strategies. A normalisation strategy σ for the linear n-polygraph Λ is said to be *right* when it satisfies the following properties:

i) for every 0-composable monomials 1-cells m_1 and m_2 of Σ_1^ℓ, we have $\sigma_{m_1m_2} \;=\; m_1\sigma_{m_2} \star_1 \sigma_{m_1\widehat{m}_2}$:

$$\begin{tikzcd}
m_1m_2 \arrow[rr, Rightarrow, "\sigma_{m_1m_2}"] \arrow[dr, Rightarrow, "m_1\sigma_{m_2}"'] & \text{©} & \widehat{m_1m_2} \\
 & m_1\widehat{m}_2 \arrow[ur, Rightarrow, "\sigma_{m_1\widehat{m}_2}"'] &
\end{tikzcd}$$

ii) for every 0-composable k-cells f and g of Λ_k^ℓ, with $2 \le k \le n$, we have

$$\sigma_{fg} \;=\; s_1(f)\sigma_g \star_1 \sigma_{ft_1(g)}.$$

In particular, for 0-composable 2-cells $f : u \Rightarrow u'$ and $g : v \Rightarrow v'$, we have $\sigma_{fg} = u\sigma_g \star_1 \sigma_{fv'}$:

$$\begin{tikzcd}
uv \arrow[rr, Rightarrow, "fg"] \arrow[dr, Rightarrow, "\sigma_{uv}"'] & \arrow[d, phantom, "\Downarrow\sigma_{fg}"] & u'v' \\
 & \widehat{uv} \arrow[ur, Rightarrow, "\sigma^-_{u'v'}"'] &
\end{tikzcd}
\;=\;
\begin{tikzcd}
 & & uv' \arrow[dr, Rightarrow, "\sigma_{uv'}"] \arrow[rrd, Rightarrow, bend left, "fv'"] & & \\
uv \arrow[r, Rightarrow, "u\sigma_v"] \arrow[urr, Rightarrow, bend left, "ug"] \arrow[rrr, Rightarrow, bend right=40, "\sigma_{uv}"'] & u\widehat{v} \arrow[ur, Rightarrow, "u\sigma^-_{v'}"] \arrow[rr, Rightarrow, "\sigma_{u\widehat{v}}"] \arrow[u, phantom, "\Downarrow u\sigma_g"] \arrow[r, phantom, "\text{©}"'] & {} & \widehat{uv} \arrow[r, Rightarrow, "\sigma^-_{u'v'}"'] \arrow[u, phantom, "\sigma_{fv'}\Downarrow"] & u'v'
\end{tikzcd}$$

Note that by the additivity property of strategies, we deduce from **i)** that for every 0-composable 1-cells f and g of Σ_1^ℓ, we have $\sigma_{fg} = f\sigma_g \star_1 \sigma_{f\widehat{g}}$.

A linear n-polygraph is *right normalising* when it admits a right normalisation strategy. Normalising and right normalising properties for linear polygraphs correspond to the same properties for $(n,1)$-polygraphs which are studied in [20, 3.2.]. In particular, we have

4.1.8. Proposition ([20, Corollary 3.3.5.]). *Let Λ be a linear n-polygraph. Right normalisation strategies on Λ are in bijective correspondence with the families*

$$\sigma_{\varphi\widehat{m}} \;:\; \varphi\widehat{m} \;\to\; \widehat{\varphi m}$$

and with the families

$$\sigma^*_{\varphi\widehat{m}} \;:\; (\varphi\widehat{m})^* \;\to\; \widehat{\varphi m}^*$$

of $(k+1)$-cells, indexed by k-cells φ of Λ_k, for $1 \leq k \leq n-1$, and by monomial 1-cells m of Σ_1^ℓ which are 0-composable with φ.

4.1.9. Proposition. *Let Λ be a linear n-polygraph. Then Λ is acyclic if and only if Λ is right normalising.*

Proof. Let us recall the proof from [20, Theorem 3.3.6.]. Suppose that Λ admits a right normalisation strategy σ. We consider a k-cell f in Λ_k^ℓ, for some $1 < k < n$. By definition, there is a $(k+1)$-cell $\sigma_f : f \Rrightarrow \widehat{f}$. If g is a k-cell which is parallel to f, recall that $\widehat{f} = \widehat{g}$. Thus the $(k+1)$-cell $\sigma_f \star_k \sigma_g^-$ of Λ_{k+1}^ℓ has source f and target g:

$$\begin{tikzcd}[column sep=huge]
u \arrow[rr, Rightarrow, bend left=70, "f"{name=F}] \arrow[rr, Rightarrow, "\widehat{f}"{name=H, description}] \arrow[rr, Rightarrow, bend right=70, "g"'{name=G}] & & v
\arrow[from=F, to=H, Rrightarrow, "\sigma_f"]
\arrow[from=H, to=G, Rrightarrow, "\sigma_g^-"]
\end{tikzcd}$$

This proves that Λ_{k+1} forms a homotopy basis of Λ_k^ℓ. Hence Λ is acyclic.

Conversely, suppose that the linear polygraph Λ is acyclic and let us define a right normalisation strategy σ for Λ. We choose a 2-cell

$$\sigma_{x\widehat{m}} : x\widehat{m} \Rightarrow \widehat{xm}$$

for every 1-cell x in Σ_1 and every monomial 1-cell m in Σ_1^ℓ such that $x\widehat{m}$ is defined. Then, for $2 \leq k < n$, the polygraph Λ being acyclic, Λ_{k+1} is a homotopy basis of Λ_k^ℓ, there is a $(k+1)$-cell

$$\sigma_{\varphi\widehat{m}} \;:\; \varphi\widehat{m} \;\longrightarrow\; \widehat{\varphi m}$$

for every k-cell φ in Λ_k and every monomial 1-cell m in Σ_1^ℓ, such that $\varphi\widehat{m}$ is defined. The Proposition 4.1.8 concludes. $\square$

4.2. Polygraphic resolutions from convergence

4.2.1. Reduced linear 2-polygraphs. Let Λ be a linear 2-polygraph with basis Σ. We say that Λ is *left-reduced* when, for every 2-cell $\varphi : m \Rightarrow f$ in Σ_2, the 1-cell m is a normal form for $\Sigma_2 \setminus \{\varphi\}$. We say that Λ is *right-reduced* when for every 2-cell $\varphi : m \Rightarrow f$ in Σ_2, the 1-cell f is a normal form for Σ_2. The linear polygraph Λ is said to be *reduced* when it is both left and right reduced.

4.2.2. The rightmost normalisation strategy. Let m be a monomial 1-cell in Σ_1^ℓ. We define a relation $\preceq$ on rewriting steps with source m as follows. If φ and ψ are 2-cells in Λ and if $f = m_1\varphi m_2$ and $g = m_1'\psi m_2'$ have common source m, then we write $f \preceq g$ when $|m_1| \leq |m_1'|$. As for 2-polygraphs, see [20, Lemma 4.2.2.], the relation $\preceq$ induces a total ordering on the rewriting steps of Λ with source m.

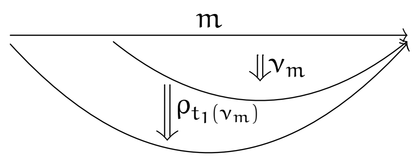
Let m be a reducible monomial 1-cell of Σ_1^ℓ. The *rightmost* rewriting step on m is denoted by ν_m and defined as the maximum elements for $\preceq$ of the (finite, non-empty) set of rewriting steps of Λ with source m. If m and m' are reducible 0-composable monomial 1-cells of Σ_1^ℓ, then we have:

$$\nu_{mm'} = m\nu_{m'}.$$

Suppose that the linear 2-polygraph Λ terminates. The *rightmost normalisation strategy of* Λ is the normalisation strategy ρ defined by induction as follows:

i) on a irreducible monomial 1-cells m, it is given by $\rho_m = 1_m$,

ii) on a reducible monomial 1-cells m, it is given by $\rho_m = \nu_m \star_1 \rho_{t_1(\nu_m)}$:

$\rho_m =$ 

iii) on a 1-cell $f = \sum\limits_{i\in I} \lambda_i m_i$, it is given by

$$\rho_f = \sum_{i\in I} \lambda_i \rho_{m_i}.$$

4.2.3. Proposition. *The rightmost normalisation strategy* ρ *is a right normalisation strategy for the linear* 2*-polygraph* Λ.

Proof. Prove by induction that, for every 0-composable monomial 1-cells m_1 and m_2 of Σ_1^ℓ, we have:

$$\rho_{m_1m_2} = m_1\rho_{m_2} \star_1 \rho_{m_1\widehat{m}_2}.$$

If $\widehat{m}_2$ is an irreducible monomial 1-cell, then $\rho_{\widehat{m}_2} = 1_{\widehat{m}_2}$ and $\rho_{m_1\widehat{m}_2} = \rho_{m_1m_2}$, so that the relation is satisfied. Otherwise, we have, using the definition of ρ and the properties of ν:

$$\rho_{m_1m_2} = \nu_{m_1m_2} \star_1 \rho_{t(\nu_{m_1m_2})} = m_1\nu_{m_2} \star_1 \rho_{t(m_1\nu_{m_2})}.$$

We apply the induction hypothesis to $t(m_1\nu_{m_2}) = m_1t(\nu_{m_2})$ to get:

$$\rho_{m_1m_2} = m_1\nu_{m_2} \star_1 m_1\rho_{t(\nu_{m_2})} \star_1 \rho_{m_1\widehat{m}_2} = m_1\rho_{m_2} \star_1 \rho_{m_1\widehat{m}_2}.$$

$\square$

Throughout the rest of this section, we suppose that Λ is a reduced monic convergent linear 2-polygraph with basis Σ and equipped with its rightmost normalisation strategy, denoted by ρ.

4.2.4. Critical branchings. In a monic linear 2-polygraph Λ, by case analysis on the source of critical branchings, they must have one of the following two shapes

$$
\begin{tikzcd}
\cdot \arrow[r, "m_1"'] \arrow[rrr, bend left=50, ""{name=T}] & \cdot \arrow[r, "m_1'"{name=M}] \arrow[r, bend right=60, ""'{name=B}] & \cdot \arrow[r, "m"'] & \cdot
\arrow[from=M, to=T, phantom, "\varphi\Uparrow"]
\arrow[from=M, to=B, phantom, "\psi\Downarrow"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\cdot \arrow[r, "m_1"'] \arrow[rr, bend left=50, ""{name=T}] & \cdot \arrow[r, "m_1'"{name=M}] \arrow[rr, bend right=50, ""'{name=B}] & \cdot \arrow[r, "m"] & \cdot
\arrow[from=M, to=T, phantom, "\varphi\Uparrow"]
\arrow[from=M, to=B, phantom, "\Downarrow\psi"]
\end{tikzcd}
$$

where φ, ψ are rules in Σ_2. As the linear 2-polygraph Λ is reduced, the first case cannot occur since, otherwise, the source of φ would be reducible by ψ. Moreover, the monomial 1-cells m_1, m_1' and m are normal forms and cannot be identities or null. Indeed, they are normal forms since, otherwise, at least one of the sources of φ and of ψ would be reducible by another 2-cell, preventing Λ from being reduced. If m_1' was an identity, then the branching would be Peiffer. Moreover, if m_1 (resp. m) was an identity, then the source of ψ (resp. φ) would be reducible by φ (resp. ψ).

The polygraph Λ being equipped with its rightmost normalisation strategy, any critical branching has the shape: $(\varphi\widehat{m}, \nu_{m_1\widehat{m}})$:

$$
\begin{tikzcd}
\cdot \arrow[r, "m_1"{name=M}] \arrow[r, bend left=60, "\widehat{m}_1"{name=T}] \arrow[rr, bend right=50, "t_1(\nu_{m_1\widehat{m}})"'{name=B}] & \cdot \arrow[r, "\widehat{m}"] & \cdot
\arrow[from=M, to=T, phantom, "\varphi\Uparrow"]
\arrow[from=M, to=B, phantom, "\Downarrow\nu_{m_1\widehat{m}}"]
\end{tikzcd}
$$

where m_1 and $\widehat{m}$ are 0-composable monomial 1-cells and φ a rule in Σ_2 with source m_1.

4.2.5. The basis of generating confluences. The *basis of generating confluences of the polygraph* Λ is the linear cellular extension $\mathcal{C}_3(\Lambda)$ of the free 2-algebroid Λ_2^ℓ made of one 3-cell

$$
\begin{tikzcd}
& \widehat{m}_1\widehat{m} \arrow[dr, Rightarrow, "\rho_{\widehat{m}_1\widehat{m}}"] & \\
m_1\widehat{m} \arrow[ur, Rightarrow, "\varphi\widehat{m}"] \arrow[rr, Rightarrow, bend right, "\rho_{m_1\widehat{m}}"'] \arrow[rr, phantom, "\Downarrow\omega_b"] & & \widehat{m_1\widehat{m}}
\end{tikzcd}
$$

for every critical branching $b = (\varphi\widehat{m}, \nu_{m_1\widehat{m}})$ of Λ. With the star notation given in Section 4.1.6, the 3-cell ω_b is equivalently denoted by:

$$
\begin{tikzcd}
m_1\widehat{m} \arrow[rr, Rightarrow, bend left, "(\varphi\widehat{m})^*"] \arrow[rr, Rightarrow, bend right, "\widehat{\varphi\widehat{m}}^*"'] \arrow[rr, phantom, "\Downarrow\omega_b"] & & \widehat{m_1\widehat{m}}
\end{tikzcd}
$$

4.2.6. Proposition. *The linear* 3*-polygraph* $\mathcal{C}_3(\Lambda)$ *is acyclic.*

Proof. We prove that the rightmost normalisation strategy ρ of Λ extends to a right normalisation strategy of the linear 3-polygraph $\mathcal{C}_3(\Lambda)$. From Proposition 4.1.8, it is sufficient to define a 3-cell

$$
\rho^*_{\varphi\widehat{m}} \,:\, (\varphi\widehat{m})^* \Rrightarrow \widehat{\varphi\widehat{m}}^*
$$

of the 3-algebroid $\mathcal{C}_3(\Lambda)^{\ell}$ for every 2-cell $\varphi : m_1 \Rightarrow \widehat{m}_1$ of Λ_2 and every monomial non-zero 1-cell m in Σ_1^{ℓ} 0-composable with φ. By definition, we have:

$$(\varphi\widehat{m})^* \;=\; \varphi\widehat{m} \star_1 \rho_{\widehat{m}_1\widehat{m}}$$

and

$$\widehat{\varphi m}^* \;=\; \rho_{m_1\widehat{m}} \;=\; \nu_{m_1\widehat{m}} \star_1 \rho_{t(\nu_{m_1\widehat{m}})}.$$

Let us proceed by case analysis on the type of the local branching $b = (\varphi\widehat{m}, \nu_{m_1\widehat{m}})$.

- If b is aspherical, then $\nu_{m_1\widehat{m}} = \varphi\widehat{m}$. In that case, we define $\rho^*_{\varphi\widehat{m}} \;=\; 1_{(\varphi\widehat{m})^*}$.
- The branching b cannot be a Peiffer branching. Indeed, the rewriting step $\nu_{m_1\widehat{m}}$ cannot reduce the normal form $\widehat{m}$.
- Otherwise, we have $\widehat{m} \;=\; \widehat{m}_2\widehat{m}_3$ and $b_1 \;=\; (\varphi\widehat{m}_2, \nu_{m_1\widehat{m}_2})$ is a critical branching of Λ. In that case, we define $\rho^*_{\varphi\widehat{m}}$ by induction as the composite 3-cell

$$\begin{tikzcd}[column sep=large]
 & \widehat{m}_1\widehat{m} \arrow[dr, Rightarrow, "\rho_{\widehat{m}_1\widehat{m}_2}\widehat{m}_3"'] \arrow[d, phantom, "\Downarrow\,\omega_{b_1}\widehat{m}_3"] \arrow[rrd, Rightarrow, bend left=20, "\rho_{\widehat{m}_1\widehat{m}}"] & \arrow[d, phantom, "\Downarrow\,(\rho^*_{\rho_{\widehat{m}_1\widehat{m}_2}\widehat{m}_3})^-"] & \\
m_1\widehat{m} \arrow[ur, Rightarrow, "\varphi\widehat{m}"] \arrow[dr, Rightarrow, "\nu_{m_1\widehat{m}}"'] & {} & \widehat{m_1m_2}\widehat{m}_3 \arrow[r, Rightarrow, "\rho_{\widehat{m_1m_2}\widehat{m}_3}"] \arrow[d, phantom, "\Downarrow\,\rho^*_{\rho_{m'}\widehat{m}_3}"] & \widehat{m_1\widehat{m}} \\
 & m'\widehat{m}_3 \arrow[ur, Rightarrow, "\rho_{m'}\widehat{m}_3"] \arrow[rru, Rightarrow, bend right=20, "\rho_{m'\widehat{m}_3}"'] & {} &
\end{tikzcd}$$

of the 3-algebroid $\mathcal{C}_3(\Lambda)^{\ell}$. □

4.2.7. Higher-dimensional branchings. An n*-fold branching* of Λ is a family $(\varphi_1, \ldots, \varphi_n)$ of Λ_2^+ with the same source and such that $\varphi_1 \preceq \cdots \preceq \varphi_n$. An n-fold branching $(\varphi_1, \ldots, \varphi_n)$ is *local* when $\varphi_1, \ldots, \varphi_n$ are rewriting steps. A local n-fold branching $(\varphi_1, \ldots, \varphi_n)$ is *aspherical* when there is $1 \leq i \leq n-1$ such that $(\varphi_i, \varphi_{i+1})$ is aspherical, (resp. *additive*) *Peiffer* when there is $1 \leq i \leq n-1$ such that $(\varphi_i, \varphi_{i+1})$ is (resp. additive) Peiffer. Otherwise, it is said *overlapping*.

The n-fold branchings are ordered as branchings, with the strict order $\prec$ generated by

$$(\varphi_1, \ldots, \varphi_n) \prec (\lambda m\varphi_1 m' + g, \;\ldots\; , \lambda m\varphi_n m' + g)$$

for any n-fold branching $(\varphi_1, \ldots, \varphi_n)$, λ in $\mathbb{K}-\{0\}$, g in Σ_1^{ℓ} and any monomial 1-cells m and m' in Σ_1^{ℓ}, such that m and m' are 0-composable with the φ_i, and either m or m' is not an identity monomial.

A *critical* n*-fold branching* is an overlapping local n-fold branching that is minimal for the order $\prec$. For instance, a 3-fold critical branching can have two different shapes, where φ_1, φ_2 and φ_3 are 2-cells in Λ_2:

$$\begin{tikzcd}
\cdot \arrow[r, "m_1"'] & \cdot \arrow[r, "m_2"] \arrow[rr, bend left=60, "\varphi_1\Uparrow"] \arrow[rrr, bend right=60, "\Downarrow\varphi_2"'] & \cdot \arrow[r, "m_3"] \arrow[rr, bend left=60, "\varphi_3\Uparrow"] & \cdot \arrow[r, "m_4"] & \cdot \arrow[r, "m"'] & \cdot
\end{tikzcd}
\quad\text{or}\quad
\begin{tikzcd}
\cdot \arrow[r, "m_1"'] & \cdot \arrow[r, "m_2"] \arrow[r, bend left=60, "\varphi_1\Uparrow"] \arrow[rrr, bend right=60, "\Downarrow\varphi_2"'] & \cdot \arrow[r, "m_3"] & \cdot \arrow[r, "m_4"] \arrow[r, bend left=60, "\varphi_3\Uparrow"] & \cdot \arrow[r, "m"'] & \cdot
\end{tikzcd}$$

For both shapes, the corresponding critical triple branching b can be written

$$b \;=\; \big(\, c\widehat{m},\; \nu_{m'\widehat{m}} \,\big) \;=\; \big(\, f\widehat{m},\; \nu_{m'}\widehat{m},\; \nu_{m'\widehat{m}} \,\big)$$

where $c = (f, \nu_{m'})$ is a critical branching of Λ with source $m' = m_1m_2m_3m_4$ and where $\rho_{m'} = m_1\psi$. More generally a critical n-fold branching b of Λ can be written

$$b \;=\; \big(\, c\widehat{m},\; \nu_{m'\widehat{m}} \,\big)$$

where c is a critical $(n-1)$-fold branching of Λ with source m'.

4.2.8. The basis of generating n-fold confluences. The *basis of generating n-fold confluences of Λ* is the linear cellular extension $\mathcal{C}_{n+1}(\Lambda)$ of the n-algebroid $\mathcal{C}_n(\Lambda)^\ell$ made of one $(n+1)$-cell

$$\begin{tikzcd}[column sep=large]
m'\widehat{m} \arrow[rr, bend left, Rightarrow, "(\varphi^*\widehat{m})^*"] \arrow[rr, bend right, Rightarrow, "(\widehat{\varphi m})^*"'] \arrow[rr, phantom, "\Downarrow (\omega_c\widehat{m})^*"] & & \widehat{m'm} \arrow[r, phantom, "\overset{\omega_b}{\Rrightarrow}"] & m'\widehat{m} \arrow[rr, bend left, Rightarrow, "(\varphi^*\widehat{m})^*"] \arrow[rr, bend right, Rightarrow, "(\widehat{\varphi m})^*"'] \arrow[rr, phantom, "\Downarrow \widehat{\omega_c m}^*"] & & \widehat{m'm}.
\end{tikzcd}$$

for every critical n-fold branching

$$b = (c\widehat{m}, \nu_{m'\widehat{m}})$$

of Λ, where c is a critical $(n-1)$-fold branching of Λ with source m' and φ is the first rewriting step of the critical branching c.

4.2.9. Lemma. *The rightmost normalisation strategy of the linear 2-polygraph Λ extends to a right normalisation strategy of the linear $(n+1)$-polygraph $\mathcal{C}_{n+1}(\Lambda)$.*

Proof. Let us define a $(n+1)$-cell

$$\sigma^*_{\omega_c\widehat{m}} \;:\; (\omega_c\widehat{m})^* \;\Rrightarrow\; \widehat{\omega_c m}^*$$

of the $(n+1)$-algebroid $\mathcal{C}_{n+1}(\Lambda)^\ell$ for every n-cell ω_c in $\mathcal{C}_n(\Lambda)$ and every monomial 1-cell m in Σ_1^ℓ, which is 0-composable with ω_c. Let us denote by m' the source of the critical $(n-1)$-fold branching c and denote by c' the critical $(n-2)$-fold branching in $\mathcal{C}_{n-2}(\Lambda)$ such that the critical branching c is $(c', \rho_{m'})$.

We proceed by case analysis on the type of the local n-fold branching

$$b \;=\; \big(c\widehat{m}, \rho_{m'\widehat{m}}\big) = (c'\widehat{m}, \rho_{m'}\widehat{m}, \rho_{m'\widehat{m}}).$$

- If b is aspherical, then $\rho_{m'\widehat{m}} = \rho_{m'}\widehat{m}$. In that case, we define $\sigma^*_{\omega_b\widehat{m}} \;=\; 1_{(\omega_b\widehat{m})^*}$.
- By hypothesis, the triple branching b cannot be a Peiffer one.
- Otherwise, there is a decomposition $\widehat{m} = \widehat{m}_1\widehat{m}_2$ such that

$$b_1 = (c\widehat{m}_1, \rho_{m'\widehat{m}_1}) = (c'\widehat{m}_1, \rho_{m'}\widehat{m}_1, \rho_{m'\widehat{m}_1}).$$

is a critical n-fold branching of Λ. We define the $(n+1)$-cell $\sigma^*_{\omega_c\widehat{m}}$ as the following composite

$$\begin{tikzcd}[column sep=large, row sep=large]
& m_l\widehat{m} \arrow[rr, Rightarrow, "\sigma_{m_l\widehat{w}_1}\widehat{m}_2"] \arrow[rrr, bend left, Rightarrow, "\sigma_{m''\widehat{m}}"] \arrow[d, phantom, "\overset{\omega_{b_1}\widehat{m}_2}{\Rrightarrow}"] & & \widehat{m'm_1}\widehat{m}_2 \arrow[r, Rightarrow, "\sigma_{\widehat{m'm_1}\widehat{m}_2}"] \arrow[r, phantom, "\Downarrow (\sigma^*_{\sigma_{m_l\widehat{m}_1}\widehat{m}_2})^-"', shift left=6ex] & \widehat{m'm} \\
m_r\widehat{m} \arrow[ur, Rightarrow, "\varphi\widehat{m}"] \arrow[dr, Rightarrow, "\rho_{\nu\widehat{m}_1}\widehat{m}_2"'] \arrow[r, phantom, "\Downarrow (\omega_c\widehat{m}_1)^*\widehat{m}_2"] & {} \arrow[r, phantom, "\Downarrow \widehat{\omega_c m_1}^*\widehat{m}_2"] & {} & & \\
& m_r\widehat{m}_2 \arrow[uurr, Rightarrow, "\sigma_{m_r}\widehat{m}_2"'] \arrow[uurrr, bend right, Rightarrow, "\sigma_{m_r\widehat{w}_2}"'] & & {} \arrow[u, phantom, "\Downarrow \sigma^*_{\sigma_{m_r}\widehat{m}_2}"] &
\end{tikzcd}$$

in the $(n+1)$-algebroid $\mathcal{C}_{n+1}(\Lambda)^{\ell}$, where φ is the first reducing step of the critical n-fold branching c.

We apply Proposition 4.1.8 to extend the family of $(n+1)$-cells we have defined to a right normalisation strategy of the linear $(n+1)$-polygraph $\mathcal{C}_{n+1}(\Sigma)$. $\square$

As a conclusion of this construction, we get that the $(n+1)$-polygraph $\mathcal{C}_{n+1}(\Lambda)$ is acyclic.

4.2.10. Theorem. *Any convergent linear 2-polygraph Λ extends to a Tietze-equivalent acyclic linear ∞-polygraph $\mathcal{C}_\infty(\Lambda)$, whose n-cells, for $n \geq 3$, are indexed by the critical $(n-1)$-fold branchings.*

5. Free resolutions of algebroids

5.1. Free modules resolution from polygraphic resolutions

5.1.1. Modules and bimodules over an algebroid. Given an algebroid $\mathbf{A}$, a *(left)* $\mathbf{A}$*-module* is a linear functor $M : \mathbf{A} \to \mathsf{Vect}$. Denote by $\mathbf{A}^o$ the *opposite* algebroid and denote by $\mathbf{A}^e$ the *enveloping* algebroid define by $\mathbf{A}^o \otimes \mathbf{A}$, where $\otimes$ denotes the natural tensor product on algebroids. We define a *right* $\mathbf{A}$*-module* (resp. $\mathbf{A}$*-bimodule*) as a $\mathbf{A}^o$-module (resp. $\mathbf{A}^e$-module). The $\mathbf{A}$-modules and their natural transformations form an Abelian category denoted by $\mathsf{Mod}(\mathbf{A})$. The free $\mathbf{A}$-modules are the coproducts of representable functors $\mathbf{A}(p,-)$, where p is a 0-cell of $\mathbf{A}$.

If M is an $\mathbf{A}$-module and x an element in $M(p)$, for p in $\mathbf{A}_0$, we simply say that x is an element of M. For u in $\mathbf{A}(p,q)$, we will denote ux for $M(u)(x)$. If $F : M \to N$ is a morphism of $\mathbf{A}$-modules, we denote $F(x)$ for $F(p)(x)$.

5.1.2. A free bimodules resolution. Let $\mathbf{A}$ be an algebroid and let Λ be a linear ∞-polygraph whose underlying 2-polygraph is a presentation of $\mathbf{A}$. We suppose that a basis Σ is fixed for Λ. We denote by $\mathbf{A}^e[\Lambda_k]$ the free $\mathbf{A}$-bimodule on Λ_k, defined by

$$\mathbf{A}^e[\Lambda_k] = \bigoplus_{f \in \Sigma_k} \mathbf{A}(-, s_0(f)) \otimes \mathbf{A}(t_0(f), -) \simeq \bigoplus_{f \in \Sigma_k} \mathbf{A}^e((s_0(f), t_0(f)), (-,-)).$$

Explicitly, for 0-cells p and q in $\mathbf{A}$, the value of the functor $\mathbf{A}^e[\Lambda_k]$ in (p,q) is the space given by the linear combinations of $u \otimes f \otimes v$, denoted by $u[f]v$, where u and v are 1-cells in $\mathbf{A}$ and f is a k-cell in Σ_k, such that the 0-composition $us_1(f)v$ is defined in $\mathbf{A}(p,q)$. When $k=0$, a triple $u[p_0]v$, such that $t_0(u) = p_0 = s_0(v)$ will be denoted by $u \otimes v$.

The mapping of every 1-cell x in Σ_1 to the element $[x]$ in $\mathbf{A}^e[\Lambda_1](s_0(x), t_0(x))$ is uniquely extended on a derivation, denoted by $[\cdot]$, from Λ_1^{ℓ} with values in the $\mathbf{A}$-bimodule $\mathbf{A}^e[\Lambda_1]$, sending a 1-cell u on the element $[u]$ in $\mathbf{A}^e[\Lambda_1](s_0(u), t_0(u))$, defined by induction on the weight of u by

$$[1_p] = 0, \qquad [uv] = [u]v + u[v],$$

for any 0-cell p of $\mathbf{A}$ and any 0-composable 1-cells u and v in Σ_1^{ℓ}.

We extend the bracket notation to $\mathbf{A}$-bimodules $\mathbf{A}^e[\Lambda_k]$, for $1 < k \leq n$ as follows. The mapping of every k-cell f of Σ_k to the element $[f]$ in $\mathbf{A}^e[\Lambda_k](s_0(f), t_0(f))$ is extended to any k-cell f of Λ_k^{ℓ} by induction on the size of f. For any $(k-1)$-cell u, any k-cells f and g and scalar λ, we set

$$[1_u] \;=\; 0, \qquad [f+g] \;=\; [f]+[g], \qquad [f \star_0 g] \;=\; [f]\overline{g} + \overline{f}[g], \qquad [\lambda f] \;=\; \lambda[f].$$

Note that using relations in 2.1.6, we deduce that

$$[f^-] \;=\; -[f], \qquad [f \star_l g] \;=\; [f]+[g],$$

for any l-composable k-cells f and g and $1 \leq l \leq k-1$. The bracket map $[\cdot]$ is well-defined, because it is compatible with the exchange relations, for every $0 \leq l_1 < l_2 \leq k$:

$$[(f \star_{l_1} g) \star_{l_2} (h \star_{l_1} k)] \;=\; [(f \star_{l_2} h) \star_{l_1} (g \star_{l_2} k)] \;=\begin{cases} [f]\overline{g} + \overline{f}[g] + [h]\overline{k} + \overline{h}[k] & \text{if } l_1 = 0 \\ [f] + [g] + [h] + [k] & \text{otherwise.} \end{cases}$$

To the linear ∞-polygraph Λ, we associate a complex of $\mathbf{A}$-bimodules

$$0 \longleftarrow \mathbf{A} \xleftarrow{\mu} \mathbf{A}^e[\Lambda_0] \xleftarrow{\delta_0} \mathbf{A}^e[\Lambda_1] \longleftarrow \;\ldots\; \longleftarrow \mathbf{A}^e[\Lambda_k] \xleftarrow{\delta_k} \mathbf{A}^e[\Lambda_{k+1}] \longleftarrow \;\ldots$$

where the boundary maps are the functors defined as follows. The maps μ is defined by $\mu(u \otimes v) = uv$, for any 0-composable pair $p \xrightarrow{u} p_0 \xrightarrow{v} q$ in $\mathbf{A}$. For any triple $u[x]v$ in $\mathbf{A}^e[\Lambda_1]$, we define

$$\delta_0(u[x]v) = u \otimes xv - ux \otimes v.$$

For $k \geq 1$, for any triple $u[f]v$ in $\mathbf{A}^e[\Lambda_{k+1}]$, we define

$$\delta_k(u[f]v) = u[s_k(f)]v - u[t_k(f)]v.$$

By induction, we prove that $\delta_0([w]) = 1 \otimes w - w \otimes 1$, for any 1-cell w in Λ_1^ℓ. We have $\mu\delta_0 = 0$. For any k-cell f, with $k \geq 2$, we have

$$\delta_{k-1}\delta_k[f] = [s_{k-1}s_k(f)] + [t_{k-1}s_k(f)] - [s_{k-1}t_k(f)] - [t_{k-1}t_k(f)].$$

It follows from the globular relations that $\delta_{k-1}\delta_k = 0$. Moreover, the acyclicity of the polygraph induces the acyclic of the complex as shown by following result.

5.1.3. Theorem. *If Λ is a (finite) ω-concentrated polygraphic resolution of an algebroid $\mathbf{A}$, then the complex $\mathbf{A}^e[\Lambda]$ is a (finite) ω-concentrated free resolution of the $\mathbf{A}$-bimodule $\mathbf{A}$.*

This bimodule resolution can be used to compute Hochschild homology, as in [8, Section 5].

Proof. Suppose that Λ is a polygraphic resolution of the algebroid $\mathbf{A}$. Fix a section $\widehat{\cdot}$ of Λ. The polygraph Λ being acyclic, by Proposition 4.1.9, it admits a right normalisation strategy σ. The strategy σ induces a contracting homotopy ι for the complex $\mathbf{A}^e[\Lambda]$ constructed as follows. We define the maps

$$\iota_0 : \mathbf{A} \to \mathbf{A}^e[\Lambda_0], \quad \iota_1 : \mathbf{A}^e[\Lambda_0] \to \mathbf{A}^e[\Lambda_1]$$

by

$$\iota_0(u) = u \otimes 1 \quad \text{and} \quad \iota_1(u \otimes v) = u[\widehat{v}].$$

We have $\iota_0\mu(u \otimes v) = uv \otimes 1$ and $\delta_0\iota_1(u \otimes v) = u \otimes v - uv \otimes 1$, thus $\iota_0\mu + \delta_0\iota_1 = \mathrm{Id}_{\mathbf{A}^e[\Lambda_0]}$. For $k \geq 2$, we define the map

$$\iota_k : \mathbf{A}^e[\Lambda_{k-1}] \to \mathbf{A}^e[\Lambda_k]$$

by

$$\iota_k(u[f]v) = u[\sigma_{f\widehat{v}}].$$

We prove by induction on the size that for any $(k-1)$-cell f in Λ_{k-1}^ℓ, $\iota_k(u[f]v) = u[\sigma_{f\widehat{v}}]$. If f is an identity map 1_w on some $(k-2)$-cell w, we have

$$\iota_k(u[1_w]v) = 0 = u[\sigma_{1_{w\widehat{v}}}].$$

Suppose that f is a 0-composite of non-identity cells of size lower than the size of f: $f = f_1 \star_0 f_2$, by induction we have

$$\begin{aligned}
\iota_k(u[f_1 \star_0 f_2]v) &= \iota_k(u[f_1]\bar{f}_2 v) + \iota_k(u\bar{f}_1[f_2]v),\\
&= u[\sigma_{f_1 \widehat{\bar{f}_2 v}}] + u\bar{f}_1[\sigma_{f_2\widehat{v}}]\\
&= u\big([\sigma_{f_1 \widehat{\bar{f}_2 v}}] + \bar{f}_1[\sigma_{f_2\widehat{v}}]\big)\\
&= u\big([\sigma_{f_1 \widehat{\bar{f}_2 v}}] + s_1(f_1)[\sigma_{f_2\widehat{v}}]\big)\\
&= u[\sigma_{f_1 f_2 \widehat{v}}].
\end{aligned}$$

Suppose that f is a l-composite of non-identity cells of size lower than the size of f: $f = f_1 \star_l f_2$, with $l \geq 1$, by induction we have

$$\begin{aligned}
\iota_k(u[f_1 \star_l f_2]v) &= \iota_k(u[f_1]v) + \iota_k(u[f_2]v),\\
&= u[\sigma_{f_1\widehat{v}}] + u[\sigma_{f_2\widehat{v}}]\\
&= u\big([\sigma_{f_1\widehat{v}}] + [\sigma_{f_2\widehat{v}}]\big)\\
&= u[\sigma_{f_1\widehat{v}} \star_l \sigma_{f_2\widehat{v}}]\\
&= u[\sigma_{(f_1\star_l f_2)\widehat{v}}].
\end{aligned}$$

We have $\iota_1\delta_0(u[x]v) = u[\widehat{xv}] - ux[\widehat{v}]$ and

$$\delta_1\iota_2(u[x]v) = \delta_1(u[\sigma_{x\widehat{v}}]) = u[x\widehat{v}] - u[\widehat{xv}] = ux[\widehat{v}] + u[x]v - u[\widehat{xv}].$$

Thus $\iota_1\delta_0 + \delta_1\iota_2 = \mathrm{Id}_{\mathbf{A}^e[\Lambda_1]}$. Let $k \geq 2$ and let $u[f]v$ in $\mathbf{A}^e[\Lambda_k]$, we have

$$\begin{aligned}
\delta_k\iota_{k+1}(u[f]v) = \delta_k(u[\sigma_{f\widehat{v}}]) &= u[f\widehat{v}] - u[\sigma_{s_{k-1}(f)\widehat{v}} \star_{k-1} \sigma^-_{t_{k-1}(f)\widehat{v}}],\\
&= u[f\widehat{v}] - u[\sigma_{s_{k-1}(f)\widehat{v}}] + u[\sigma^-_{t_{k-1}(f)\widehat{v}}],\\
&= \iota_k(u[s_{k-1}(f)]v) - \iota_k(u[t_{k-1}(f)]v),\\
&= \iota_k\delta_{k-1}(u[f]v).
\end{aligned}$$

Thus $\iota_k\delta_{k-1} + \delta_k\iota_{k+1} = \mathrm{Id}_{\mathbf{A}^e[\Lambda_k]}$. □

5.1.4. A free right modules resolution. Recall that $\mathbf{A}_k^{(0)}(p,p) = \mathbb{K}$ for any 0-cell p in Σ_0 and any $k \in \mathbb{N}$, where the 1-dimensional space is generated by the identity k-cell. Also, recall that $\mathbf{A}^{(0)}(p,q) = \{0\}$ for distinct 0-cells p and q in Σ_0.

When Λ is a polygraphic resolution of $\mathbf{A}$, the complex of $\mathbf{A}$-modules $\mathbf{A}^{(0)} \otimes_{\mathbf{A}} \mathbf{A}^e[\Lambda]$, whose boundary map defined by $\overline{\delta}_k = 1 \otimes \delta_k$, is a resolution of $\mathbf{A}^{(0)}$ as a right $\mathbf{A}$-module. However, in general this resolution is not homogeneous even if the polygraphic resolution is homogeneous. We construct a homogeneous resolution of $\mathbf{A}^{(0)}$ in the category of right $\mathbf{A}$-modules using a homogeneous polygraphic resolution as follows.

Let $\mathbf{A}$ be an algebroid and let Λ be a linear ∞-polygraph whose underlying 2-polygraph is a presentation of $\mathbf{A}$. We suppose that a basis Σ is fixed for Λ. For any $k \geq 0$, denote by $\mathbf{A}[\Lambda_k]$ the free right $\mathbf{A}$-module generated by Λ_k, defined by

$$\mathbf{A}[\Lambda_k] = \bigoplus_{f\in\Sigma_k} \mathbf{A}(t_0(f), -).$$

The value of the functor $\mathbf{A}[\Lambda_k]$ in a 0-cell q is the space given by the linear combinations of $f \otimes u$, denoted by $[f]u$, where u is 1-cell in $\mathbf{A}$ and f is a k-cell in Σ_k, such that the 0-composition $s_1(f)u$ is defined in $\mathbf{A}(s_0(f), q)$. The elements of $\mathbf{A}[\Lambda_0]$ are identified to the 1-cells of $\mathbf{A}$.

The mapping of every 1-cell x in Σ_1 to the element $[x]$ in $\mathbf{A}[\Lambda_1](s_0(x), t_0(x))$ is extended to the algebroid Σ_1^ℓ by setting

$$[1_p] = 0, \quad [xy] = [x]y,$$

for any 0-cell p and 0-composable 1-cells x and y in Σ_1^ℓ. The mapping of every k-cell f of Σ_k to the element $[f]$ in $\mathbf{A}[\Lambda_k](t_0(f))$ is extended to any k-cell f of Λ_k^ℓ by induction on the size of f. For any $(k-1)$-cell u, any k-cells f and g and scalar λ, we set

$$[1_u] \;=\; 0, \qquad [f+g] \;=\; [f]+[g], \qquad [f \star_0 g] \;=\; [f]\overline{g}, \qquad [\lambda f] \;=\; \lambda[f].$$

Note that using relations in 2.1.6, we deduce that

$$[f^-] \;=\; -[f], \qquad [f \star_l g] \;=\; [f]+[g],$$

for any k-cells f and g and $1 \leq l \leq n-1$. The bracket map $[\cdot]$ is well-defined, because it is compatible with the exchange relations, for every $0 \leq l_1 < l_2 \leq k$:

$$[(f \star_{l_1} g) \star_{l_2} (h \star_{l_1} k)] \;=\; [(f \star_{l_2} h) \star_{l_1} (g \star_{l_2} k)] \;=\; [f]+[g]+[h]+[k],$$

and

$$[(f \star_0 g) \star_l (h \star_0 k)] \;=\; [f \star_0 g] + [h \star_0 k] \;=\; [f]u + [h]u,$$

where u denotes $\overline{g} = \overline{k}$. On the other hand, we have

$$[(f \star_l h) \star_0 (g \star_l k)] \;=\; [f \star_l h]\overline{g \star_1 k} \;=\; [f \star_l h]u \;=\; [f]u + [h]u.$$

To the linear ∞-polygraph Λ, we associate a complex of right $\mathbf{A}$-modules

$$0 \longleftarrow \mathbf{A}^{(0)} \overset{\varepsilon}{\longleftarrow} \mathbf{A}[\Lambda_0] \overset{\delta_0}{\longleftarrow} \mathbf{A}[\Lambda_1] \longleftarrow \;\ldots\; \longleftarrow \mathbf{A}[\Lambda_k] \overset{\delta_k}{\longleftarrow} \mathbf{A}[\Lambda_{k+1}] \longleftarrow \;\ldots$$

where the boundary maps are the functors defined as follows. The map ε is the augmentation defined by $\varepsilon(u) = u$ if u is an identity and $\varepsilon(u) = 0$ in the other cases. For any $[x]u$ in $\mathbf{A}[\Lambda_1]$, we define

$$\delta_0([x]u) = xu.$$

For $k \geq 1$, for any $[f]u$ in $\mathbf{A}[\Lambda_{k+1}]$, we define

$$\delta_k([f]u) = [s_k(f)]u - [t_k(f)]u.$$

By induction, we prove that $\delta_0([u]) = u$, for any 1-cell u in Λ_1^ℓ. We have $\varepsilon\delta_0 = 0$ and for any k-cell f, with $k \geq 2$, we have

$$\delta_{k-1}\delta_k[f] = [s_{k-1}s_k(f)] + [t_{k-1}s_k(f)] - [s_{k-1}t_k(f)] - [t_{k-1}t_k(f)].$$

It follows from the globular relations that $\delta_{k-1}\delta_k = 0$. As for bimodules, if the polygraph is acyclic, then the complex is acyclic.

5.1.5. Theorem. *If Λ is a (finite) ω-concentrated polygraphic resolution of an algebroid $\mathbf{A}$, then the complex $\mathbf{A}[\Lambda]$ is a (finite) ω-concentrated free resolution of the trivial right $\mathbf{A}$-module $\mathbf{A}^{(0)}$.*

Proof. Suppose that Λ is a polygraphic resolution of the algebroid $\mathbf{A}$, with a fixed section $\widehat{\cdot}$ of Λ. The polygraph Λ admits a right normalisation strategy σ from which we construct a contracting homotopy ι for the complex $\mathbf{A}^e[\Lambda]$ as in Theorem 5.1.3. We define the maps

$$\iota_0 : \mathbf{A}^{(0)} \to \mathbf{A}[\Lambda_0], \quad \iota_1 : \mathbf{A}[\Lambda_0] \to \mathbf{A}[\Lambda_1]$$

by

$$\iota_0(1_q) = 1_q \quad \text{and} \quad \iota_1(u) = [\widehat{u}].$$

We have $\iota_0\varepsilon(1_q) = 1_q$ and $\delta_0\iota_1(1_q) = 0$. And, if u is not an identity, we have $\iota_0\varepsilon(u) = 0$ and $\delta_0\iota_1(u) = u$. It follows that $\iota_0\varepsilon + \delta_0\iota_1 = \mathrm{Id}_{\mathbf{A}[\Lambda_0]}$. For $k \geq 2$, we define the map

$$\iota_k : \mathbf{A}[\Lambda_{k-1}] \to \mathbf{A}[\Lambda_k]$$

by

$$\iota_k([f]u) = [\sigma_{f\widehat{u}}].$$

As in the proof of Theorem 5.1.3, we prove by induction on the size of f, that for any $(k-1)$-cell f in Λ^{ℓ}_{k-1} that $\iota_k([f]u) = [\sigma_{f\widehat{u}}]$. For any $x[u]$ in $\mathbf{A}[\Lambda_1]$, we have

$$\begin{aligned}
(\iota_1\delta_0 + \delta_1\iota_2)([x]u) &= \iota_1(xu) + \delta_1([\sigma_{x\widehat{u}}]),\\
&= [\widehat{xu}] + [s_1(\sigma_{x\widehat{u}})] - [t_1(\sigma_{x\widehat{u}})],\\
&= [x\widehat{u}] = [x]u.
\end{aligned}$$

Thus $\iota_1\delta_0 + \delta_1\iota_2 = \mathrm{Id}_{\mathbf{A}[\Lambda_1]}$. Let $k \geq 2$ and let $[f]u$ in $\mathbf{A}[\Lambda_k]$, we have

$$\begin{aligned}
\delta_k\iota_{k+1}([f]u) = \delta_k([\sigma_{f\widehat{u}}]) &= [f\widehat{u}] - [\sigma_{s_{k-1}(f)\widehat{u}} \star_{k-1} \sigma^{-}_{t_{k-1}(f)\widehat{u}}],\\
&= [f\widehat{u}] - [\sigma_{s_{k-1}(f)\widehat{u}}] + [\sigma^{-}_{t_{k-1}(f)\widehat{u}}],\\
&= \iota_k([s_{k-1}(f)]u) - \iota_k([t_{k-1}(f)]u),\\
&= \iota_k\delta_{k-1}([f]u).
\end{aligned}$$

Thus $\iota_k\delta_{k-1} + \delta_k\iota_{k+1} = \mathrm{Id}_{\mathbf{A}[\Lambda_k]}$. □

This resolution and the associated theorem for right **A**-modules can be adapted to the setting of left **A**-modules.

5.2. Finiteness properties

Throughout this section, $n \geq 1$ denotes a natural number.

5.2.1. Algebroids of finite derivation type. An algebroid is of *finite n-derivation type*, FDT_n for short, when it admits a finite partial polygraphic resolution of length n. An algebroid is of *finite ∞-derivation type*, FDT_∞ for short, when it admits a finite polygraphic resolution, *i.e.*, when it is FDT_n for every $n \geq 1$.

In particular, an algebroid is FDT_1 when it is finitely generated, it is FDT_2 when it is finitely presented. The property FDT_3 corresponds to the finite derivation type condition originally defined by Squier for monoids in [31]. The property FDT_n, for $n \geq 3$ for higher-dimension categories were introduced in [20, 2.3.6.]. Let us note that for any $n \geq 1$, the property FDT_{n+1} implies the property FDT_n.

By Proposition 4.1.9, a linear p-polygraph is acyclic if and only if it is right normalising. It follows the following result.

5.2.2. Proposition. *Let $n \geq 1$ be a natural number. An algebroid* **A** *is* FDT_n *if and only if there exists a finite right normalising linear n-polygraph presenting* **A**.

In Section 4.2, we construct a polygraphic resolution for an algebroid from a convergent presentation. By Theorem 4.2.10, we have

5.2.3. Proposition. *An algebroid with a finite convergent presentation is* FDT_∞.

5.2.4. Finite homological type algebroid. An algebroid **A** is of *homological type left*-FP$_n$ (over $\mathbb{K}$) if there is an exact sequence of **A**-modules:

$$0 \longleftarrow \mathbb{K} \longleftarrow F_0 \longleftarrow F_1 \longleftarrow \ \ldots \ \longleftarrow F_{n-1} \longleftarrow F_n,$$

where the F_i are finitely generated free **A**-modules and $\mathbb{K}$ is the constant **A**-module. We say that **A** is of *homological type left*-FP$_\infty$ if it is left-FP$_n$, for all $n > 0$. We say that **A** is of *homological type right*-FP$_n$ if $\mathbf{A}^o$ is of homological type left-FP$_n$. An algebroid **A** is of *homological type bi*-FP$_n$ if there is an exact sequence of **A**-bimodules:

$$0 \longleftarrow \mathbf{A} \longleftarrow F_0 \longleftarrow F_1 \longleftarrow \ \ldots \ \longleftarrow F_{n-1} \longleftarrow F_n,$$

where the F_i are finitely generated free **A**-modules and were **A** denotes the **A**-bimodule sending functorialy each pair of 0-cells (p, q) on the morphism space $\mathbf{A}(p, q)$.

Suppose that **A** is an algebroid of type FDT$_n$, then **A** admits a finite polygraphic resolution Λ of length n. By Theorem 5.1.3, this induces a finite resolution of **A**-bimodules $\mathbf{A}^e[\Lambda]$. Thus we have the following implication.

5.2.5. Proposition. *For any natural number $n \geq 1$, for algebroids, the property* FDT$_n$ *implies the property* FP$_n$.

By Proposition 5.2.3, it follows.

5.2.6. Proposition. *An algebroid with a finite convergent presentation is of homological type bi*-FP$_\infty$*, and thus left and right* FP$_\infty$.

There is a more general notion of module over a category introduced by Baues, see [7]. The category of factorisations of **A** is the 1-category, denoted by $F\mathbf{A}$, whose 0-cells are the 1-cells of **A** and a 1-cell from w to w' are pairs (u, v) of 1-cells of **A** such that $vwu = w'$ holds in **A**. Composition is defined by concatenation. A *natural system* on **A** is a $F\mathbf{A}$-module. We denote by $\mathbb{K}$ the constant natural system, defined by $\mathbb{K}(w) = \mathbb{K}$ and $\mathbb{K}(u, v)$ is the identity, for any 0-cell w and 1-cell (u, v) in $F\mathbf{A}$. We say that **A** is *homological type* FP$_n$ if there is an exact sequence of natural systems:

$$0 \longleftarrow \mathbb{K} \longleftarrow F_0 \longleftarrow F_1 \longleftarrow \ \ldots \ \longleftarrow F_{n-1} \longleftarrow F_n,$$

where the F_i's are finitely generated free natural system on **A** and $\mathbb{K}$ is the constant natural system.

Obviously, the property FP$_n$ implies the property bi-FP$_n$ and the property bi-FP$_n$ implies both the properties left and right FP$_n$, [20, Proposition 5.2.4.].

Note that, finite homological type corresponds to the same notion for 1-categories, see [20, 5.2.]. A 1-category **C** is of homological type (resp. bi, resp left-) right-FP$_n$ over a field $\mathbb{K}$ if the free algebroid $\mathbb{K}\mathbf{C}$ is of homological type (resp. bi, resp. left-) right-FP$_n$.

5.2.7. Abelian finite derivation type. Recall from [20, 5.7.] that a 1-category is of *Abelian finite derivation type*, FDT$_{ab}$ for short, when it admits a presentation by a finite 2-polygraph Σ which is FDT$_{ab}$, that is the free Abelian $(2, 1)$-category on Σ admits a finite homotopy basis. A finite presented 1-category **C** is FDT$_{ab}$ if and only if **C** is of homological type FP$_3$, [20, Theorem 5.7.3.]). By definition, for an algebroid **A**, the properties FDT$_3$ and FDT$_{ab}$ are equivalent. It follows that for finitely presented algebroids, the properties FDT$_3$ and FP$_3$ are equivalent.

5.3. Convergence and Koszulity

5.3.1. Koszul algebroids. Let us recall the definition of (generalized) Koszul algebras, and state it in the case of algebroids. Let **A** be a graded N-homogeneous algebroid, with $N > 1$. The algebroid **A** is called *left-Koszul* if $\mathbf{A}^{(0)}$ considered as a graded left $\mathbf{A}$-module admits a graded projective resolution of the form

$$0 \longleftarrow \mathbf{A}^{(0)} \longleftarrow M_0 \longleftarrow M_1 \longleftarrow M_2 \longleftarrow \dots$$

such that every M_i is generated (as a graded left $\mathbf{A}$-module) by $M_i^{(\ell_N(i))}$. Similarly, one can define *right-Koszul* algebroids using right $\mathbf{A}$-modules. The algebroid $\mathbf{A}$ is called *bi-Koszul* if $\mathbf{A}$ considered as a graded left $\mathbf{A}^e$-module admits a graded projective resolution of the form

$$0 \longleftarrow \mathbf{A} \longleftarrow M_0 \longleftarrow M_1 \longleftarrow M_2 \longleftarrow \dots$$

such that every M_i is generated (as a graded $\mathbf{A}$-bimodule) by $M_i^{(\ell_N(i))}$.

5.3.2. Remark. The groups $\mathrm{Tor}^{\mathbf{A}}_{k,(i)}(\mathbf{A}^{(0)}, \mathbf{A}^{(0)})$ for a left-Koszul (or right-Koszul) algebroid $\mathbf{A}$ vanish for $i \neq \ell_N(k)$, where the first grading in the Tor refers to the homological degree and the second one to the internal grading of $\mathbf{A}$. When $\mathbf{A}$ is a graded algebra, this property of the Tor groups is an equivalent definition of Koszul algebras, as Berger proved in [8, Theorem 2.11.].

5.3.3. Remark. In the general case of a graded algebra $\mathbf{A}$ with N-homogeneous relations, it has been proven Berger and Marconnet ([9, Proposition 2.1]) that the groups $\mathrm{Tor}^{\mathbf{A}}_{k,(i)}(\mathbb{K}, \mathbb{K})$ always vanish for $i < \ell_N(k)$. This means that the Koszul property corresponds to a limit case.

We now relate the Koszul property with the properties of the polygraphic resolutions.

5.3.4. Theorem. *Let $\mathbf{A}$ be an N-homogeneous algebroid. If $\mathbf{A}$ has a ℓ_N-concentrated polygraphic resolution, then $\mathbf{A}$ is right-Koszul (resp. left-Koszul, resp. bi-Koszul).*

Proof. Suppose that Λ is a ℓ_N-concentrated polygraphic resolution of $\mathbf{A}$. By Theorem 5.1.5, the resolution $\mathbf{A}[\Lambda]$ is a ℓ_N-concentrated free resolution of the trivial right $\mathbf{A}$-module $\mathbf{A}^{(0)}$, hence $\mathbf{A}$ is right-Koszul. For the left-Koszul case, we can use the left version of Theorem 5.1.5 to show that $\mathbf{A}$ is left-Koszul. By Theorem 5.1.3, the resolution $\mathbf{A}^e[\Lambda]$ is a ℓ_N-concentrated free resolution of the trivial $\mathbf{A}$-bimodule $\mathbf{A}$, hence $\mathbf{A}$ is bi-Koszul. $\square$

5.3.5. Proposition. *Let Λ be a polygraphic resolution with a basis Σ of an algebroid $\mathbf{A}$ such that $(\Sigma_0, \Sigma_1, \Lambda_2 \dots, \Lambda_{k-1})$ is ℓ_N-concentrated, for some $k \geq 3$. If $\dim\left(\Lambda_{k+1}^{(i)}\right) < \dim\left(\Lambda_k^{(i)}\right)$ for some $i > \ell_N(k)$, then $\mathbf{A}$ is not Koszul.*

Proof. Using Theorem 5.1.3, we obtain a resolution of $\mathbf{A}^{(0)}$ by right $\mathbf{A}$-modules, which can be used to compute the groups $\mathrm{Tor}^{\mathbf{A}}(\mathbf{A}^{(0)}, \mathbf{A}^{(0)})$. Then, using that Λ_{k-1} is concentrated in degree $\ell_N(k-1)$, we obtain that for any $i > \ell_N(k)$, $\mathrm{Tor}^{\mathbf{A}}_{k,(i)}(\mathbf{A}^{(0)}, \mathbf{A}^{(0)})$ is the quotient of a space of dimension $\dim\left(\Lambda_k^{(i)}\right)$ by the image of a space of dimension at most $\dim\left(\Lambda_{k+1}^{(i)}\right)$. Thus, using that $\dim\left(\Lambda_{k+1}^{(i)}\right) < \dim\left(\Lambda_k^{(i)}\right)$ for some $i > \ell_N(k)$, we obtain that $\mathrm{Tor}^{\mathbf{A}}_{k,(i)}(\mathbf{A}^{(0)}, \mathbf{A}^{(0)})$ does not vanish, which contradicts the Koszulity of $\mathbf{A}$. $\square$

We will use this condition with a polygraphic resolution obtained by completion in Example 5.3.14.

As a consequence of Theorem 4.2.10 and Theorem 5.3.4, we have

5.3.6. Theorem. *Let $\mathbf{A}$ be an algebra presented by a quadratic convergent linear 2-polygraph Λ. Then Λ can be extended into a ℓ_2-concentrated polygraphic resolution. In particular, any algebra having a presentation by a quadratic convergent linear 2-polygraph is Koszul.*

5.3.7. Linear coherent presentations. Let $\mathbf{A}$ be an algebroid. A *linear coherent presentation* of $\mathbf{A}$ is a linear 3-polygraph Λ, whose presented algebroid is isomorphic to $\mathbf{A}$ and such that the linear cellular extension Λ_3 of Λ_2^ℓ is a homotopy basis.

Given a N-homogeneous algebra $\mathbf{A}$, a coherent ℓ_N-concentrated presentation of $\mathbf{A}$ having an empty homotopy basis can be extended into a polygraphic resolution with an empty set of k-cells for $k \geq 3$, thus a ℓ_N-concentrated polygraphic resolution. By Theorem 5.3.4, we deduce the following result:

5.3.8. Proposition. *If a N-homogeneous algebroid $\mathbf{A}$ has a coherent ℓ_N-concentrated presentation with an empty homotopy basis, then it is Koszul.*

By Proposition 4.2.6, if Λ is a convergent monic linear 2-polygraph, then it can be extended into a coherent presentation whose homotopy basis is made of generating confluences. In particular, when the polygraph Λ has no critical branching, this homotopy basis is empty, and thus trivially ℓ_N-concentrated. Thus as a consequence of Theorem 5.3.8, we have the following property:

5.3.9. Corollary. *An algebroid $\mathbf{A}$ having a terminating presentation by a N-homogeneous polygraph without any critical branching is Koszul.*

5.3.10. Example. Consider the cubical algebra

$$\mathbf{A}\langle\, x, y, z \mid xyz = x^3 + y^3 + z^3 \,\rangle$$

given in Example 3.4.7. The polygraph Λ defined by the basis

$$\Sigma_1 = \{x, y, z\}, \quad \Sigma_2 = \{xyz \Rightarrow x^3 + y^3 + z^3\}$$

is convergent, without any critical branching. It follows that Λ_2^ℓ admits an empty homotopy basis, hence by Proposition 5.3.8, $\mathbf{A}$ is Koszul. Moreover, we obtain that $\mathrm{Tor}^{\mathbf{A}}_{0,(0)}(\mathbb{K},\mathbb{K}) \simeq \mathbb{K}$, $\mathrm{Tor}^{\mathbf{A}}_{1,(1)}(\mathbb{K},\mathbb{K}) \simeq \mathbb{K}^3$, $\mathrm{Tor}^{\mathbf{A}}_{2,(3)}(\mathbb{K},\mathbb{K}) \simeq \mathbb{K}$ and $\mathrm{Tor}^{\mathbf{A}}_{k,(i)}(\mathbb{K},\mathbb{K})$ vanishes for other values of k and i.

5.3.11. Homotopical reduction at the polygraphic level. It is sometimes possible to reduce the size of a polygraphic resolution, using the same method as in [17, Section 2.3]. This can allow us to obtain ℓ_N-concentrated coherent presentations (and thus prove Koszulity) even when the original presentation was not ℓ_N-concentrated.

The idea is the following: it is sometimes possible to remove a $k+1$-cell simultaneously with a k-cell appearing only once in its source. It is similar to algebraic discrete Morse theory. The new resolution is slightly changed in the higher dimensions and is smaller than the original one.

5.3.12. Proposition. *Let Λ be an acyclic linear n-polygraph such that Λ_{k-1} and Λ_k are generated respectively by Σ_{k-1} and Σ_k for some $1 < k < n$. Suppose that there exist $\gamma \in \Sigma_k$ and $A \in \Sigma_{k+1}$ such that $\gamma = s(A)$ and that $t(A) \in \mathbb{K}(\Sigma_k \setminus \{\gamma\})$ (that is γ does not appear in the target of A).*

Then the linear n-polygraph $\Lambda' = (\Sigma_0, \Sigma_1, \Lambda_2, \ldots, \mathbb{K}(\Sigma_k \setminus \{\gamma\}), \mathbb{K}(\Sigma_{k+1} \setminus \{A\}, \Lambda_{k+2}, \ldots)$ is acyclic and Tietze-equivalent to Λ.

Proof. We need to show that $(\Sigma_0, \Sigma_1, \Lambda_2, \ldots, \Lambda'_k, \Lambda'_{k+1}, \Lambda_{k+2}, \ldots)$ actually defines a polygraphic resolution, so we need to define maps s' and t' in every dimension, satisfying the globular relations, and to prove acyclity.

We define s'_j and t'_j respectively as s_j and t_j from Λ_{j+1} to Λ_j^ℓ for any $j < k-1$.

We define s'_{k-1} and t'_{k-1} from Λ'_k to Λ_{k-1}^ℓ respectively as restrictions of s'_k and t'_k from Λ_k to Λ_{k-1}^ℓ, as Λ'_k is defined by a basis included in the basis of Λ_k.

To define s'_k and t'_k from Λ'_{k+1} to ${\Lambda'}_k^\ell$, let us first notice that $\Lambda'_{k+1} \oplus \mathbb{K}A$ and that there is a canonical projection p_k from Λ_k^ℓ to ${\Lambda'}_k^\ell = \Lambda_k^\ell/(\gamma - t(A))$ sending γ to $t(A)$ and the other elements of Σ_k to themselves. For any $B \in \Sigma_{k+1} \setminus \{A\}$, we can now define $s'_k(B)$ by $p_k(s_k(B))$. We define t'_k similarly.

Note that ${\Lambda'}^{\ell}_{j} = \Lambda^{\ell}_{j}/(A, \gamma - t(A))$ for any $j > k$. We define $s'_j = p_j \circ s_j$ and $t'_j = p_j \circ t_j$ where p_j denotes the projection $\Lambda^{\ell}_{j} \to {\Lambda'}^{\ell}_{j}$.

The globular relations still hold.

To prove the acyclicity of Λ', note first that Λ and Λ' coincide in dimensions smaller than k. Thus we only have to prove acyclicity in dimension j when $j \geq k$.

The space of j-spheres of Λ' is a quotient of the space of j-spheres of Λ. Given a j-sphere S' of Λ', we take a j-sphere S of Λ lifting S'. There exists a $j+1$-cell B of Λ^{ℓ}_{j+1} such that S is the boundary of B. Then $p_{j+1}(B)$ is a $j+1$-cell of ${\Lambda'}^{\ell}_{j+1}$ whose boundary is S'. □

5.3.13. Example. Consider the following quadratic algebra

$$\mathbf{A}\langle\, x, y, z \mid x^2 + yz = 0,\ x^2 + azy = 0 \,\rangle,$$

with $a \neq 0$ and $a \neq 1$, see [29, Section 4.3]. Consider the linear 2-polygraph Λ with $\Sigma_1 = \{x, y, z\}$ and Λ_2 is generated by the following quadratic 2-cells:

$$yz \overset{\alpha}{\Longrightarrow} -x^2 \qquad zy \overset{\beta}{\Longrightarrow} bx^2$$

where $b = -1/a$. There are two cubical critical branchings

$$\begin{tikzcd}[column sep=small]
 & byx^2 \\
yzy \arrow[ur, Rightarrow, bend left, "y\beta"] \arrow[dr, Rightarrow, bend right, "\alpha z"'] & \\
 & -x^2y
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & bx^2z \\
zyz \arrow[ur, Rightarrow, bend left, "\beta z"] \arrow[dr, Rightarrow, bend right, "z\alpha"'] & \\
 & -zx^2
\end{tikzcd}$$

We complete the linear polygraph Λ into a convergent linear polygraph, denoted by $\widetilde{\Lambda}$, by adding the cubical rules

$$yx^2 \overset{\gamma}{\Longrightarrow} ax^2y \qquad zx^2 \overset{\delta}{\Longrightarrow} -bx^2z$$

The linear polygraph $\widetilde{\Lambda}$ has the following four confluent critical branchings:

$$\begin{tikzcd}[column sep=small]
 & byx^2 \arrow[dd, Rightarrow, "b\gamma"] \\
yzy \arrow[ur, Rightarrow, bend left, "y\beta"] \arrow[dr, Rightarrow, bend right, "\alpha y"'] \arrow[r, phantom, "A_1"] & \\
 & -x^2y
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & bx^2z \\
zyz \arrow[ur, Rightarrow, bend left, "\beta z"] \arrow[dr, Rightarrow, bend right, "z\alpha"'] \arrow[r, phantom, "B_1"] & \\
 & -zx^2 \arrow[uu, Rightarrow, "-\delta"']
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
 & bx^4 & \\
zyx^2 \arrow[ur, Rightarrow, bend left, "\beta x^2"] \arrow[dr, Rightarrow, bend right, "z\gamma"'] \arrow[rr, phantom, "C_1"] & & x^2zy \arrow[ul, Rightarrow, bend right, "x^2\beta"'] \\
 & azx^2y \arrow[ur, Rightarrow, bend right, "a\delta y"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & -x^4 & \\
yzx^2 \arrow[ur, Rightarrow, bend left, "\alpha x^2"] \arrow[dr, Rightarrow, bend right, "y\delta"'] \arrow[rr, phantom, "D_1"] & & x^2yz \arrow[ul, Rightarrow, bend right, "x^2\alpha"'] \\
 & -byx^2z \arrow[ur, Rightarrow, bend right, "-b\gamma z"'] &
\end{tikzcd}$$

By Proposition 4.2.6, these four 3-cells extend the polygraph $\widetilde{\Lambda}$ into a coherent presentation of the algebra **A**.

Note that the 3-cells C_1 and D_1 are in weight 4, thus the linear 3-polygraph $\widetilde{\Lambda}$ is not ℓ_2-concentrated. We now investigate higher critical branchings, to understand the polygraphic resolution of **A**. There are four critical triple branchings, whose 1-sources are respectively

$$yzyz,\ yzyx^2,\ zyzy,\ zyzx^2.$$

The critical branching on $yzyz$ (resp. $zyzy$) is denoted A_2 (resp. B_2):

$$
\begin{tikzcd}
 & -x^2yz \arrow[dr, "-x^2\alpha"] \arrow[dd, phantom, "-D_1"] & & & & -x^2yz \arrow[dr, "-x^2\alpha"] & \\
yzyz \arrow[ur, "\alpha yz"] \arrow[r, "y\beta z"] \arrow[dr, "yz\alpha"'] \arrow[r, phantom, shift left=2, "A_1z"] \arrow[r, phantom, shift right=3, "yB_1"'] & byx^2z \arrow[u, Rightarrow, "b\gamma z"] \arrow[d, Rightarrow, "-y\delta"] & x^4 & \overset{A_2}{\Rrightarrow} & yzyz \arrow[ur, "\alpha yz"] \arrow[dr, "yz\alpha"'] & \equiv & x^4 \\
 & -yzx^2 \arrow[ur, "-\alpha x^2"'] & & & & -yzx^2 \arrow[ur, "-\alpha x^2"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & bx^2zy \arrow[dr, "bx^2\beta"] \arrow[dd, phantom, "bC_1"] & & & & bx^2zy \arrow[dr, "bx^2\beta"] & \\
zyzy \arrow[ur, "\beta zy"] \arrow[r, "z\alpha y"] \arrow[dr, "zy\beta"'] \arrow[r, phantom, shift left=2, "B_1y"] \arrow[r, phantom, shift right=3, "zA_1"'] & -zx^2y \arrow[u, Rightarrow, "-\delta y"] \arrow[d, Rightarrow, "bz\gamma"] & b^2x^4 & \overset{B_2}{\Rrightarrow} & zyzy \arrow[ur, "\beta zy"] \arrow[dr, "zy\beta"'] & \equiv & b^2x^4 \\
 & bzyx^2 \arrow[ur, "b\beta x^2"'] & & & & bzyx^2 \arrow[ur, "b\beta x^2"'] &
\end{tikzcd}
$$

The 4-cell A_2 relates D_1 with A_1 and B_1. The 4-cell B_2 relates C_1 with A_1 and B_1. Note that the 4-cells A_2 and B_2 are in weight 4.

Using the homotopical reduction procedure explained in 5.3.11, we can first eliminate the 4-cell A_2 together with the 3-cell D_1, and the 4-cell B_2 together with the 3-cell C_1 (the order between these two eliminations is not important). Using the same process, we can then eliminate the 3-cell A_1 with the 2-cell γ and the 3-cell B_1 with the 2-cell δ. After these steps, we obtain a partial polygraphic resolution of length 3 with an empty homotopy basis. By Proposition 5.3.8, it follows that the algebra **A** is Koszul. Moreover we obtain that $\mathrm{Tor}^{\mathbf{A}}_{0,(0)}(\mathbb{K},\mathbb{K}) \simeq \mathbb{K}$, $\mathrm{Tor}^{\mathbf{A}}_{1,(1)}(\mathbb{K},\mathbb{K}) \simeq \mathbb{K}^3$, $\mathrm{Tor}^{\mathbf{A}}_{2,(2)}(\mathbb{K},\mathbb{K}) \simeq \mathbb{K}^2$ and $\mathrm{Tor}^{\mathbf{A}}_{k,(i)}(\mathbb{K},\mathbb{K})$ vanishes otherwise.

5.3.14. Example. Consider the following quadratic algebra

$$\mathbf{A}\langle\, x, y \mid x^2 = y^2 = xy \,\rangle,$$

and consider the presentation of **A** by the rules

$$xy \overset{\alpha}{\Longrightarrow} x^2 \qquad y^2 \overset{\beta}{\Longrightarrow} x^2$$

There are two critical pairs

$$
\begin{tikzcd}
 & yx^2 \arrow[dr, phantom] & \\
y^3 \arrow[ur, Rightarrow, "y\beta"] \arrow[dr, Rightarrow, "\beta y"'] & & x^3 \\
 & x^2y \arrow[ur, Rightarrow, "x\alpha"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & x^2y \arrow[dr, Rightarrow, "x\alpha"] & \\
xy^2 \arrow[ur, Rightarrow, "\alpha y"] \arrow[rr, Rightarrow, "x\beta"'] & & x^3
\end{tikzcd}
$$

By adding the rule

$$yx^2 \overset{\gamma}{\Longrightarrow} x^3$$

we obtain a convergent polygraph Λ (the termination is obvious, as the number of occurences of y decreases), and the two new ambiguities are solvable. By Proposition 4.2.6, the following 3-cells form an homotopy basis of Λ_2^ℓ:

$$
\begin{tikzcd}
& yx^2 \arrow[dr, Rightarrow, "\gamma"] \arrow[dd, phantom, "\Downarrow A"] & \\
y^3 \arrow[ur, Rightarrow, "y\beta"] \arrow[dr, Rightarrow, "\beta y"'] & & x^3 \\
& x^2y \arrow[ur, Rightarrow, "x\alpha"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& x^2y \arrow[dr, Rightarrow, "x\alpha"] & \\
xy^2 \arrow[ur, Rightarrow, "\alpha y"] \arrow[rr, Rightarrow, bend right, "x\beta"'] \arrow[rr, phantom, "\Downarrow B"] & & x^3
\end{tikzcd}
$$

$$
\begin{tikzcd}
xyx^2 \arrow[r, Rightarrow, bend left, "\alpha x^2"] \arrow[r, Rightarrow, bend right, "x\gamma"'] \arrow[r, phantom, "\Downarrow C"] & x^4
\end{tikzcd}
\qquad
\begin{tikzcd}
& yx^3 \arrow[d, Rightarrow, "\gamma x"] \\
y^2x^2 \arrow[ur, Rightarrow, "y\gamma"] \arrow[r, Rightarrow, "\beta x^2"'] \arrow[r, phantom, shift left=3, "\Downarrow D"] & x^4
\end{tikzcd}
\qquad
\begin{tikzcd}
& yx^3 \arrow[dr, Rightarrow, "\gamma x"] \arrow[dd, phantom, "\Downarrow E"] & \\
yx^2y \arrow[ur, Rightarrow, "yx\alpha"] \arrow[dr, Rightarrow, "\gamma y"'] & & x^4 \\
& x^3y \arrow[ur, Rightarrow, "x^2\alpha"'] &
\end{tikzcd}
$$

There are seven critical triple branchings on the following words:

$$
xyx^2y,, \quad xy^2x^2, \quad xy^3, \quad yx^2yy, \quad y^2x^2y, \quad y^3x^2, \quad y^4,
$$

and two of them are in weight 4. Therefore in weight 4, there are three generating 3-cells and only two generating 4-cells. Thus we can use Proposition 5.3.5 and we obtain that the algebra **A** is not Koszul.

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